International Physics Course
Entrance Examination Questions

(May 2010)

Please answer the four questions from Problem 1 to Problem 4. You can use as many answer sheets you need. Your name, question numbers and page numbers are to be written at the top of EACH answer sheet.
EXAMPLE: Albert Einstein, Problem 1, page 1.

Problem 1

A system comprising a pair of point charges $+q$ and $-q$ $(q > 0)$ separated by a distance of *s* is called an electric dipole, and is characterized by an electric dipole moment *p*, defined as $p \equiv qs$. Here, the vector *s* takes the direction from the $-q$ point charge to the $+q$ point charge. Furthermore, *s* is assumed to be constant.

The force acting between two electric dipoles, p_1 and p_2, will be determined by using the following procedure. First, if the distance *r* between the dipoles is sufficiently large compared to the spatial extent of the electric dipoles themselves (the distance *s* between the point charges), an approximation can be made by considering only the first-order term (*s*/*r*), and ignoring second-order and higher terms.

(1) Consider a situation in which the electric dipole p_2 is placed in a static electric field. Use the fact that when the static electric field potential is $\phi(x)$, the potential energy $V(r)$ of an electric dipole p placed at a point P with a position vector *r* is given by

$$V(r) = +q\phi(r + s/2) - q\phi(r - s/2)$$

to express the potential energy $V_2(r)$ possessed by the electric dipole p_2, in terms of p_2 and the static electric field $E(r)$.

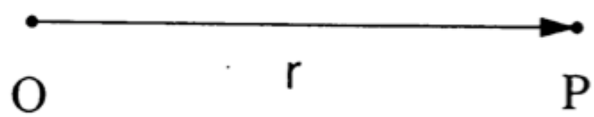
If the above electric field is thought of as arising due to the electric dipole p_1 being positioned at the origin *O*, it is possible to find the interaction energy W_{12} between the dipoles.

(2) Determine the static electric field $E_1(r)$ created at position *r* by the electric dipole p_1 positioned at the origin *O*.

(3) Find the energy of interaction W_{12} between the electric dipoles p_1 and p_2.

The force acting between the electric dipoles p_1 and p_2, positioned at points *O* and *P*, respectively, differs depending on their orientation.

(4) With what arrangement of the electric dipoles would the energy of interaction between them, W_{12}, be the lowest? Draw a diagram in your answer (copy the diagram onto the answer sheet and draw p_1 and p_2 as small arrows). Determine the force acting between the dipoles in this situation.



Problem 2

An electromagnetic field (radiation field) with an angular frequency ω exists in a cavity of volume V enclosed by walls at a temperature T. Because of the wave-particle duality, the radiation field can be thought of as a gas comprised of many non-interacting particles (photons), which have energy $\hbar\omega$ and momentum $\frac{\hbar\omega}{c}$. When the internal energy of this photon gas in a state of thermal equilibrium is taken to be U, the energy density $u = \frac{U}{V}$ can be written as follows using the energy density spectrum $u(\omega, T)$:

$$u(T) = \int_0^\infty u(\omega, T)\, d\omega$$

Here, the energy density spectrum is given by the following formula, where k is Boltzmann's constant:

$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp\left[\frac{\hbar\omega}{kT}\right] - 1}$$

It is known from classical electromagnetism, that the relationship $p = \frac{1}{3} u(T)$ exists between the radiation pressure (pressure exerted on the walls by the photon gas) p and the energy density $u(T)$. Considering the above, answer the questions below.

(1) From the first law of thermodynamics $dU = \delta Q + \delta W$ (or $dU = d'Q + d'W$), show that when the system undergoes an infinitesimal quasi-static change, the relationship

$$\delta Q = C_V dT + \left[p + \left(\frac{\partial U}{\partial V}\right)_T\right] dV$$

holds. Here, δQ is the quantity of heat absorbed by the system, and C_V is the heat capacity at constant volume, which is generally a function of V and T.

(2) Show that the internal energy U of the photon gas can be written in the form $U(V, T) = aVT^4$, where a is a constant (you need not determine an explicit expression for a).

<u>Solve the questions below, also using a if necessary.</u>

(3) For the photon gas, heat capacity at constant pressure does not exist. Why is this?

(4) After finding the heat capacity at constant volume C_V, use the relationship shown in (1) to determine the entropy of the photon gas $S(V,T)$. From the third law of thermodynamics, $S=0$ at absolute zero.

(5) Determine the relationship between V and T, when the photon gas undergoes a quasi-static adiabatic transition. It is possible to depict this relation in the form $pV^{\gamma}=$ constant. What is the value of γ?

(6) Consider the Carnot cycle that takes the photon gas as the working material:

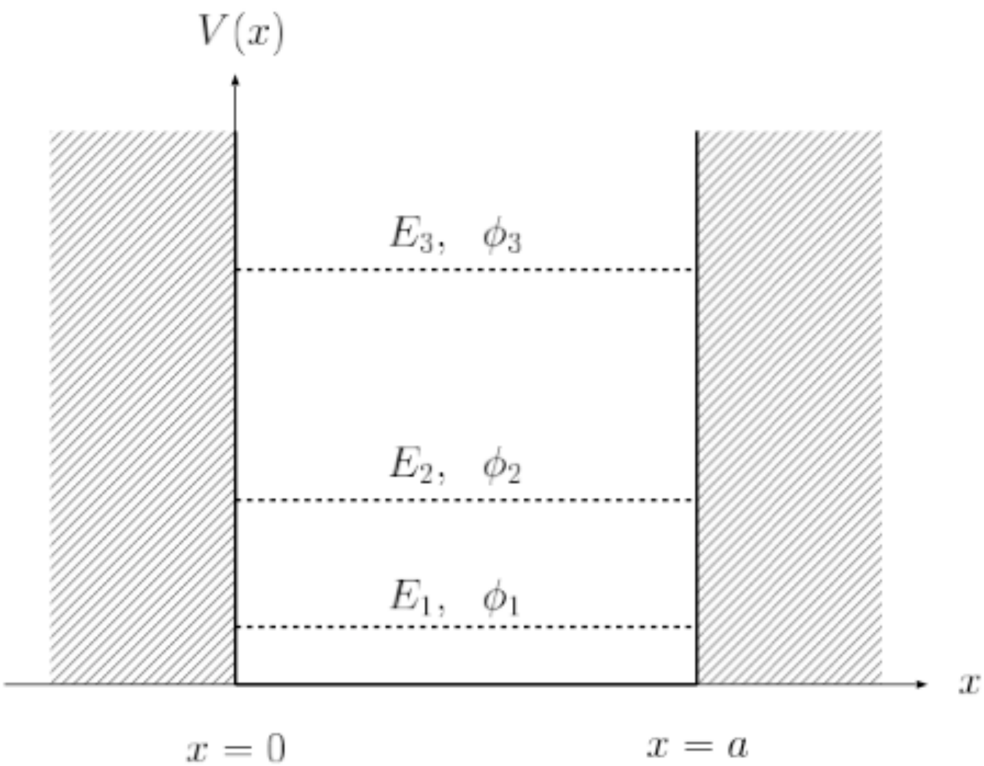
A → B: quasi-static isothermal expansion
B → C: quasi-static adiabatic expansion
C → D: quasi-static isothermal compression
D → A: quasi-static adiabatic compression

Draw the course of these state transitions on the *p*-*V* plane. Draw the *p*-*V* plane on your answer sheet with *p* on the vertical axis and *V* on the horizontal axis. Points should be noted to indicate the states A, B, C and D (an outline is sufficient), and arrows should be used to show the direction of the transitions.

Problem 3

In 1-dimensional space, consider the quantum mechanics of a particle trapped in a potential well of infinite height

$$V(x) = \begin{cases} 0 & (0 < x < a) \\ +\infty & (x \leq 0, a \leq x) \end{cases}$$



First, a single particle of mass *m* is trapped in the well.

(1) Determine the energy of the *n*-th $(n = 1,2,3 \ldots)$ eigenstate E_n (as counted from the lowest energy) of this system and the normalized eigenfunction $\phi_n(x)$ $(0 < x < a)$.

Next, let us assume that two particles of masses m_1 and m_2, which are distinguishable from each other, have been trapped in the same well. Here, the spin degree of freedom of the particles is to be ignored and there is no interaction between them.

(2) Let the coordinates of the particles be x_1 and x_2 respectively. Using $V(x_1)$ and $V(x_2)$, write the time-independent Schrödinger equation which the wave function for this two-particle system $\psi(x_1, x_2)$ satisfies. Here, the energy eigenvalue is *E*.

(3) $\psi(x_1, x_2)$ can be written as $\psi(x_1, x_2) = f(x_1)g(x_2)$. Write the Schrödinger equations satisfied by $f(x_1)$ and $g(x_2)$. Here, the energy eigenvalues are ε_f and ε_g respectively. In addition, show the relation between *E* from question (2), and ε_f and ε_g.

(4) Find the ground-state energy E_{gs} of this two-particle system E_{gs}. Express the corresponding normalized eigenfunction $\psi_{gs}(x_1, x_2)$, using the necessary ϕ_1, ϕ_2, ϕ_3 ... determined in question (1).

(5) Find the energy of the first and second excited states for the same system, E_{1st} and E_{2nd}. Express the corresponding eigenfunctions $\psi_{1st}(x_1, x_2)$ and $\psi_{2nd}(x_1, x_2)$, using the necessary ϕ_1, ϕ_2, ϕ_3 ... determined in question (1). Assume that $m_1 < m_2 < \frac{8}{3}m_1$.

Next, two fermions of the same mass *m* and spin ½ (indistinguishable from each other) are trapped in the same well. Again, assume that there is no interaction between the two particles. We will represent the state in which the spin is up in relation to the relevant quantization axis by α, and the state in which the spin is down by β.

(6) At the lowest energy level of the well, the eigenfunction for the ground state which contains a spin-up and spin-down particle can be written as

$$\psi = \phi_1(x_1)\alpha(1)\phi_1(x_2)\beta(2)$$

However, this does not take into account the fact that the two fermions are indistinguishable from each other. Since fermions obey Fermi-Dirac statistics, the eigenfunction of this two-particle system must be antisymmetric with respect to particle exchange (simultaneous exchange of the coordinates of the particles x_1 and x_2, and of the spin coordinates). Bearing this in mind, write the correct eigenfunction for the ground state. Determine the magnitude of the total spin *S* for this state.

(7) Using the antisymmetry of the eigenfunction discussed in question (6), show that it is not possible for two fermions with the same spin to occupy the same energy level.

Problem 4

Two atoms with masses m_1 and m_2 interact with each other via the potential,

$$V(r) = \frac{a}{r^6} - \frac{b}{r^4}$$

where *a* and *b* are positive constants, and r is the distance between the two atoms. Let us consider the relative motion of the system consisting of these two atoms, within the scope of classical mechanics.

(1) What is the expression for the reduced mass μ? In answering the questions below, you may use μ.

(2) With the above potential, the two atoms form a molecule, which is at rest. Determine the relative distance r_0 between the atoms and the energy of the molecule.

(3) When the molecule is not rotating, determine the equation of motion for infinitesimal vibrations around the distance r_0 found in (2), and the angular frequency.

(4) The molecule rotates about an axis that passes through the center of mass and is perpendicular to the straight line linking the two atoms. Assume that the distance between the atoms remains constant. Denoting the rotation angle with θ, express the angular momentum of the relative motion. Find the largest value of angular momentum for which the molecule can maintain its bound state without dissociating (so that the two atoms do not infinitely separate). Find the relative distance r_1 at this time.