

1 (12 points)

Answer the following questions.

- (1) A central force $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$ acts on a particle of mass m , whose position is identified by a vector \mathbf{r} originating in the origin. Here, \mathbf{e}_r is a unit vector in the direction of \mathbf{r} , while r is the magnitude of vector \mathbf{r} . Write the angular momentum vector of the particle in this situation, and show that it is conserved.
- (2) The Lagrangian L of a particle of mass m , which is in motion under the influence of a central-force potential $U(r)$ in a 2-dimensional plane, is given as follows.

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

Here, (r, φ) are the polar coordinates of the particle's position, while \dot{r} and $\dot{\varphi}$ represent the time derivatives of r and φ , respectively. By writing the Lagrange equation of motion for cyclic coordinates, show that the angular momentum is conserved.

2 (3 points)

The escape velocity of a particle on Earth is the minimum velocity required at Earth's surface in order that that particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy, show that the escape velocity for Earth, ignoring the presence of the Moon, is 11.2 Km/s.

3 (10 points)

Rockets are propelled by the momentum reaction of the exhaust gas expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is:

$$m \frac{dv}{dt} = -v' \frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m , assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.07 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300.

4 (30 points)

Answer the following questions, after specifying which system of units you will be using.

- (1) When a point charge is positioned at the origin ($\mathbf{r} = 0$) in an isotropic material, a separation of charge occurs around it, the Coulomb field of the point charge is screened, and the electrostatic potential $\phi(\mathbf{r})$ takes the form

$$\phi(\mathbf{r}) = \frac{A}{r} \cdot \exp\left(-\frac{r}{\lambda}\right) \quad (1)$$

Here, r is the distance from the origin, and A and λ are constants. In this case, determine the charge density distribution $\rho(\mathbf{r})$ in the space surrounding the origin. Use the fact that in spherical symmetry the Laplacian is given by

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right)$$

- (2) In formula (1) above, the potential becomes infinitely large at the origin. This shows that the point charge is located at the origin. Using Gauss's Law, find the charge of the point particle.
- (3) Write Maxwell's equations for the case when a current density $\mathbf{i}(\mathbf{r}, t)$ and charge density $\rho(\mathbf{r}, t)$ exist within a material which has a constant electric permittivity ϵ and magnetic permeability μ , and by doing so, show that the formula for the conservation of the energy of the electromagnetic field takes the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = Q$$

If \mathbf{H} is the magnetic field strength and \mathbf{E} the electric field, then in the above formula

$$u = \frac{1}{2\mu} \mathbf{B}^2 + \frac{1}{2} \epsilon \mathbf{E}^2$$

Explain the physical meanings of u , \mathbf{S} and Q . When performing the derivation, you may use the below formula relating vectors \mathbf{A} and \mathbf{B} ,

$$\mathbf{a} \cdot (\nabla \times \mathbf{b}) - \mathbf{b} \cdot (\nabla \times \mathbf{a}) = \nabla \cdot (\mathbf{b} \times \mathbf{a})$$

5 (9 points)

A wave function is given as follows.

$$\Psi(x) = Axe^{-\alpha x^2}$$

Assume that it describes a state of a simple harmonic oscillator. Here α and A are constants that may depend on the mass m of the oscillator and of its classical frequency of vibration ω .

Determine:

- (1) α (Hint: use the Schrödinger equation.)
- (2) the energy of this state and the normalization constant of the wave function.

6 (10 points)

When studying for example the quantum behavior of a particle on a potential barrier, the transmission coefficient T is defined as the ratio of the flux of the transmitted wave to that of the incident wave.

Examine the case of a particle coming from $x = -\infty$ in the potential V :

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \text{ and } x > l \\ V_0 > 0 & \text{for } 0 < x < l \end{cases}$$

Find the transmission coefficient T in this case.

7 (6 points)

Consider a spin $\frac{1}{2}$ particle. Compute

$$\langle (\Delta S_x)^2 \rangle \equiv \langle S_x^2 \rangle - \langle S_x \rangle^2,$$

where S_x is the spin operator in the x direction and the expectation value $\langle \rangle$ is taken for the $S_z +$ state, namely the eigenstate of the spin in the z direction with eigenvalue $+\frac{\hbar}{2}$.

Using your result, check the validity of the uncertainty relation

$$\langle (\Delta S_x)^2 (\Delta S_y)^2 \rangle \geq \frac{1}{4} \left| \langle [S_x, S_y] \rangle \right|^2.$$

8 (20 points)

A cylindrical container with a freely moving piston is filled with an ideal gas consisting of 1 mole of monatomic molecules. The region outside the container is vacuum. Initially, an external pressure is applied to the piston in order to maintain the state of the gas at an absolute temperature T_1 and a volume V_1 . This state is denoted State 1. The gas volume is then expanded to V_2 ($V_2 > V_1$), by each of the following methods.

Method A (isothermal expansion): From State 1, the external pressure on the piston is controlled so as to allow the gas volume to slowly expand to volume V_2 while the gas temperature remains constant.

Method B (adiabatic expansion): From State 1, the external pressure on the piston is controlled so as to allow the gas to slowly expand to volume V_2 with no heat transfer between the gas and the outside.

Answer the following questions, denoting the gas constant by R and using the fact that the internal energy of an ideal gas depends only on its temperature.

- (1) In the isothermal expansion process of Method A, what is the amount of work done by the gas (ΔW), the quantity of heat acquired by the gas (ΔQ), and the change in entropy of the gas (ΔS)? Include signs in your answers.
- (2) In the adiabatic expansion process of Method B, what is the amount of work done by the gas (ΔW), the quantity of heat acquired by the gas (ΔQ), and the change in entropy of the gas (ΔS)? Include signs in your answers. Note that the relationship between the pressure P and the volume V represented by $PV^{5/3} = \text{Constant}$ is valid in adiabatic processes involving ideal gases consisting of monatomic molecules.