International Physics Course
Entrance Examination Questions

(December 2011)

Please answer the four questions from Problem 1 to Problem 4. You can use as many answer sheets as you need. Your name, question numbers, and page numbers are to be written at the top of EACH answer sheet.
EXAMPLE: Albert Einstein, Problem 1, page 1.

Problem 1

Consider the dynamics of a system consisting of a sphere of mass M and radius R and a point mass m under mutual gravitational attraction. Let G be the gravitational constant. The internal degrees of freedom of the sphere need not to be considered.

(1) How can the reduced mass μ of this system be expressed?

(2) Taking r as the distance between the center of the sphere and the point mass, considering only planar motion, and using polar coordinates (r, φ), write the Lagrangian describing the relative motion. Indicate the derivative with respect to time by an over-dot ($\dot{}$).

(3) The form of the Lagrangian shows that there are two conserved quantities. Express these in terms of whichever of the following may be necessary: G, M, m, μ, the coordinates, and the derivatives of the coordinates.

(4) Consider the scattering of the point mass on the sphere. Take v_∞ as the magnitude of the relative velocity as $r \to \infty$ and b as the impact parameter. Taking into account the relationship between v_∞, b, and the conserved quantities derived for point (3) above, derive a formula for determining r_m, the distance at the time of nearest approach of the point mass to the sphere. Neglect the size of the sphere.

Problem 2

Using a pressure cooker, it is possible to prepare food by maintaining an internal temperature higher than 100ºC. Answer each of the following questions in order to understand the underlying principle.

(1) The gas-liquid phase boundary is given by the Clausius–Clapeyron formula

$$\frac{dP}{dT} = \frac{L}{T\Delta V}$$

where P and T are the pressure and temperature at the boiling point, respectively, L is the latent heat per mole at the boiling point, and ΔV is the change in volume per mole between the gas and liquid phases. Derive this formula.

(2) For water, find (in units of [atmospheres/K] and with two significant digits) the value of $a = (dP/dT)_{T=T_0}$ at the boiling point T_0 when the pressure P_0 is equal to 1 atmosphere. In this case, the latent heat of water can be taken to be $L = 540$ cal/g and the gas constant as $R = 2.0$ cal mol^{-1} K^{-1}.

(3) If the temperature of the air and water inside the pressure cooker prior to heating is $T_1 (< T_0)$, then how many atmospheres does the internal pressure of the cooker P reach when the temperature due to heating is increased to T? Assume that the water does not boil.

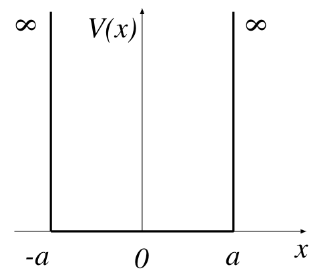
(4) Assume that the vapor pressure curve of water can be approximated by a straight line of gradient a passing through a pressure P_0 and a boiling point T_0, and that T_B is the temperature just below the boiling temperature of the water inside the cooker. Express $\Delta T = T_B - T_0$ in terms of a, T_0, T_1, and P_0.

(5) When $T_1 = 27$ºC, what are the values of $\Delta T = T_B - T_0$ and of the pressure P_B inside the cooker at the temperature T_B? Give your answer with two significant digits.

Problem 3

Let us consider the quantum mechanical motion of a particle of mass m that is confined in a one-dimensional potential $V(x)$:

$$V(x) = \begin{cases} \infty & (|x| \geq a) \\ 0 & (|x| < a) \end{cases}$$



(1) Obtain the normalized wave functions of the stationary states $\phi_n(x)$ and the corresponding eigenenergies E_n.

(2) A wave function $\psi(x,t)$ at a time t can be expanded in terms of $\phi_n(x)$ as

$$\psi(x,t) = \sum_n c_n(t)\phi_n(x) .$$

Derive the differential equation that the $c_n(t)$ satisfy.

(3) Let us consider the wave function $\psi(x,t)$ that at time t=0 satisfies the initial condition:

$$\psi(x,t=0) = \frac{1}{\sqrt{2}}[\phi_1(x) + \phi_2(x)] ,$$

where $\phi_1(x)$ and $\phi_2(x)$ are the ground and the first excited states, respectively.

(3.1) Find the wave function $\psi(x,t)$ at an arbitrary time t.

(3.2) Find the expectation values of the coordinate and the Hamiltonian, $\langle x \rangle$ and $\langle H \rangle$, in the state $\psi(x,t)$.

Problem 4

It is no exaggeration to say that the whole idea of electromagnetism can be condensed into the Maxwell equations:

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad\qquad \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Here, ε_0 and μ_0 are the vacuum permittivity and permeability, respectively, ρ is the charge density, and $\mathbf{j}$ is the current density. Maxwell predicted the existence of electromagnetic waves through the above equations. We will begin with a confirmation of this.

(1) In order to verify the existence of electromagnetic waves that travel in vacuum ($\rho = 0$, $\mathbf{j} = 0$) at the speed of light $c = (\varepsilon_0 \mu_0)^{-1/2}$, derive the wave equations from the Maxwell equations. When doing so, consider for simplicity a plane wave of the following kind. The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are vectors along the *x*-axis and *y*-axis respectively, with $\mathbf{E} = (E,0,0)$ and $\mathbf{B} = (0,B,0)$. The *x*-component of the electric field E and the *y*-component of the magnetic field B are functions of *z* and *t* alone. The *z*-axis is the direction of propagation of the electromagnetic wave. The wave equation should be derived for either *E* or *B*.

Hertz made skillful use of the reflection of electromagnetic waves and showed that the speed at which they travel is equal to the speed of light, *c*. Let us think about this experiment.

(2) Consider the case in which the plane wave in vacuum considered in (1) propagates in the positive direction along the *z*-axis. For simplicity, *E* and *B* can be written as follows.

$$E = E_0 \cos(kz - \omega t),\ B = B_0 \cos(kz - \omega t),\ E_0 = cB_0.$$

Here E_0 and B_0 are constants that indicate the amplitude of the electric and magnetic field respectively, and $k(>0)$ and $\omega(>0)$ are also constants (the wave number and angular frequency, respectively).

This electromagnetic wave is 100% reflected by a metal plate placed in the *xy*-plane, which is perpendicular to the direction of propagation. The boundary of this metal plate is at $z=0$ and its electrical resistance can be ignored. Because of the reflection, in the vacuum at $z<0$ a standing wave is generated by the overlapping of the original wave and the reflected wave. The electric field does not penetrate into the metal plate and so its magnitude at the surface of the metal plate ($z=0$) is zero.

Perform the following tasks.

(2-1) Write the electric and magnetic fields of the standing wave as formulae involving *z* and *t*.

(2-2) Show with graphs the general shape of the electric field of the standing wave when its amplitude reaches the maximum and when the amplitude of the magnetic wave reaches its maximum.

(2-3) Use these results to simply describe a method for verifying that the speed at which electromagnetic waves propagate is *c*. Assume the angular frequency of the electromagnetic waves ω to be a known quantity.

(3) The amplitude of the magnetic field of the standing wave considered in (2) is spatially discontinuous at $z=0$. This is due to the current density $\mathbf{j}$, which is generated by the electromagnetic wave and which flows through the metal plate. Find the expression of $\mathbf{j}$, using the Dirac delta function $\delta(z)$ if desired.