Cubic Casimir operator of $\mathbf{SU}_C(3)$ and confinement in the nonrelativistic quark model

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In spite of the wide-spread concensus on the validity of QCD as the theory of strong interactions, QCD has proven essentially intractable, except in perturbative approximations. In particular, the existence and energetics of coloured states remain unexplored. Moreover, multiple colour-singlet multi-quark states appear in the theory and it is unclear just which one lies lowest. The spin-statistics problem of the simple quark model led to the introduction of the colour degrees of freedom that obey the SU(3) Lie algebra. This "colour SU(3)" is exactly conserved, hence the quark model spectrum must fall into irreducible representations of this group, and the quark Hamiltonian must be expressible in terms of SU(3) invariant operators. There are two such independent "invariants" for SU(3), the so-called Casimir operators [1, 2]. This note will show that the introduction of a three-quark force proportional to the second ("cubic") Casimir operator can fix (at least some of) these shortcomings. The second Casimir operator $C^{(2)}$ of SU(3) is tri-linear ("cubic") in the group generators F^a, *viz.* $C^{(2)} = d^{abc}\mathrm{F}^a\mathrm{F}^b\mathrm{F}^c$, so it can only appear in three- or more-quark potentials.

The three-quark potential can be factored into a colour part $\mathcal{C}_{123}$ and the spin-spatial part $\mathcal{V}_{123}$:

$$V_{123} = \mathcal{C}_{123}\mathcal{V}_{123}. \tag{1}$$

The following 3-body colour factors are allowed:

$$\mathcal{C}_{123} = \begin{cases} \sum_{i<j}^{3} \mathrm{F}_i \cdot \mathrm{F}_j = \frac{1}{2}C^{(1)}_{i+j+k} - 2 \\ d^{abc}\mathrm{F}_1^a\mathrm{F}_2^b\mathrm{F}_3^c = \frac{1}{6}\left[C^{(2)}_{i+j+k} - \frac{5}{2}C^{(1)}_{i+j+k} + \frac{20}{3}\right] \\ if^{abc}\mathrm{F}_1^a\mathrm{F}_2^b\mathrm{F}_3^c, \end{cases} \tag{2}$$

where $\mathrm{F}^a = \frac{1}{2}\lambda^a$ is the quark colour charge, the lower index indicates the number of the quark, λ^a are the Gell-Mann matrices, and f^{abc}, d^{abc} are the SU(3) structure constants and $i+j+k$ stands for the three-quark colour state. Only the first two factors, Eqs. (2,2) are SU(3) invariants, however, i.e., only they can be expressed in terms of Casimir operators, and only the second factor, Eq. (2), depends on the cubic Casimir operator. The third colour factor, Eq. (2), is an off-diagonal operator that annihilates the two SU(3) eigenstates with definite exchange symmetry, i.e. the **1** and **10** and converts one **8** state into another. Therefore, it is not allowed in the quark model Hamiltonian

Keeping with the tradition of the quark model, we take the harmonic oscillator for both the two- and three-quark spatial parts of potentials:

$$\mathcal{V}_{12} = \frac{1}{2}m\omega^2(\mathbf{r}_1 - \mathbf{r}_2)^2 \tag{3}$$

$$\mathcal{V}_{123} = c\frac{1}{2}m\omega^2\left[(\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_3 - \mathbf{r}_2)^2 + (\mathbf{r}_1 - \mathbf{r}_3)^2\right]; \tag{4}$$

with an as yet undetermined strength c for the latter. With the harmonic oscillator assumption we find that the $\mathrm{F}_i \cdot \mathrm{F}_j$ model two-body interaction leads to the same form of the effective potential in the q^3 system as the three-body force Eq. (2). With these results we can write down the Hamiltonians for few-quark systems and then solve for their spectra from which we can read off the stability conditions. This leads us to conclude that the cubic Casimir three-body force cannot stabilize the 3q system with the $\mathrm{F}_i \cdot \mathrm{F}_j$ model two-quark interaction. A sufficient condition for the stabilization of the **8** $q\bar{q}$ state is to have as the two-body potential

$$V_{12} \quad = \quad \left[c_1 + \frac{4}{3} + \mathrm{F}_i \cdot \mathrm{F}_j\right] \frac{1}{2} m\omega^2 \left(\mathbf{r}_1 - \mathbf{r}_2\right)^2 , \tag{5}$$

with $c_1 > 0$. For simplicity we take $c_1 = 1$. Note that with this new two-body interaction and *no* three-body force ($c = 0$), all three q^3 colour states are stable, but the octet **8** is lighter than the singlet **1**, again in contrast with the experiment! Turning on the three-body force, $c \neq 0$, we find the following stability condition

$$-\frac{3}{2} < c < \frac{78}{5}. \tag{6}$$

For values of $c < \frac{2}{5}$ we find the anticipated ordering of colour states: singlet **1** is the lowest lying, the next lowest is the octet **8**, and then the decimet **10**. The ratio of their ground state energies can be made arbitrarily large by choosing c sufficiently close to - 1.5. For example, with $c =$ - 1.43, the **8** and **10** states are lying above 4 GeV. We conclude that the colour-independent two-body force stabilizes the q^3 system, whereas the cubic Casimir three-body force makes it well ordered in colour, i.e. properly confined.

Having found a confining potential that predicts the presumed ordering of the q^3 colour spectrum, we turn to its application to the $q^2\bar{q}^2$ system. We must be careful about the definition of the colour factors in the nonrelativistic three-body potential involving antiquarks as they are sensitive to the C-conjugation properties of the relativistic interaction from which the potential was derived (the latter's properties carry over into the nonrelativistic limit for odd number of quarks). We conclude that the "cubic Casimir" three-body interaction must have the following colour factor when antiquarks are involved

$$\bar{\mathcal{C}}_{123} = \begin{cases} -d^{abc}\mathrm{F}_1^a\mathrm{F}_2^b\bar{\mathrm{F}}_3^c = \frac{-1}{6}\left[C^{(2)}_{i+j+k} - \frac{5}{2}C^{(1)}_{i+j} + \frac{50}{9}\right] \\ \;\; d^{abc}\mathrm{F}_1^a\bar{\mathrm{F}}_2^b\bar{\mathrm{F}}_3^c \\ -d^{abc}\bar{\mathrm{F}}_1^a\bar{\mathrm{F}}_2^b\bar{\mathrm{F}}_3^c \end{cases} \tag{7}$$

Note that the first (quadratic) Casimir is evaluated between the two-quark (sub-)state $i + j$, which leads to a distinction between the two overall colour triplets (which are symmetric and antisymmetric in the quark indices). From the signs of the two interaction potentials we see that the mass/energy of the "octet" state is enhanced for $c \leq 0$ and vice versa for the "singlet" state. We conclude that the 3-body interaction can elevate the mass of the unobserved "octet" states above the conventional/ordinary two-meson states and thus make them less stable and less likely to be detected. These results have been confirmed and extended to the $6q$ system in Ref. [4].

References

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