Linear Σ model in the Gaussian functional approximation

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The Gell-Mann–Levy [GML] linear sigma model has long been a subject of nonperturbative studies, both for its particle physics and statistical mechanics applications [1]. In this paper we apply a new chirally invariant version of the Lorentz invariant self-consistent mean-field, variational approximation that goes by many names, *inter alia* the Gaussian functional approximation [2] to the linear sigma model. The improvement that we bring to this paper is the correct implementation of the chiral symmetry in this approximation. We prove the chiral Ward-takahashi identities, among them the Nambu-Goldstone theorem, the Dashen relation and the axial current (partial) conservation (PCAC) in this approximation. Then we solve numerically the gap and Bethe-Salpeter equations and discuss the particle content of the theory in this approximation.

We confine ourselves to the $O(N=4)$ symmetric linear σ model. The Lagrangian density of this theory is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}\right)^2 + \varepsilon\sigma + \frac{1}{2}m_0^2\boldsymbol{\phi}^2 - \frac{\lambda_0}{4}\left(\boldsymbol{\phi}^2\right)^2 . \qquad (1)$$

where

$$\boldsymbol{\phi} = (\phi_0, \phi_1, \phi_2, \phi_3) = (\sigma, \boldsymbol{\phi}),$$

is an O(4) vector. We assume here that λ_0 and μ_0^2 are not only positive, but such that spontaneous symmetry breakdown (SSB) occurs in the mean-field approximation [MFA] to be introduced later. We use the Gaussian ground state functional Ansatz

$$\Psi_0[\vec{\phi}] = \mathcal{N}\exp\left(-\frac{1}{4\hbar}\int d\mathbf{x}\int d\mathbf{y}\,[\phi_i(\mathbf{x}) - \langle\phi_i(\mathbf{x})\rangle]\,G_{ij}^{-1}(\mathbf{x},\mathbf{y})\,[\phi_j(\mathbf{y}) - \langle\phi_j(\mathbf{y})\rangle)]\right) , \qquad (2)$$

where $\mathcal{N}$ is the normalization constant, $\langle\phi_i(\mathbf{x})\rangle$ is the vacuum expectation value (v.e.v.) of the i-th spinless field which henceforth we will assume to be translationally invariant $\langle\phi_i(\mathbf{x})\rangle = \langle\phi_i(0)\rangle \equiv \langle\phi_i\rangle$ and $G_{ij}(\mathbf{x},\mathbf{y}) = \delta_{ij}G(\mathbf{x},\mathbf{y},m_i)$. We vary the energy density with respect to the field vacuum expectation values $\langle\phi_i\rangle$ and the "dressed" masses m_i:

$$\left(\frac{\partial\mathcal{E}(m_i,\langle\phi_i\rangle)}{\partial\langle\phi_i\rangle}\right)_{min} = \left(\frac{\partial\mathcal{E}(m_i,\langle\phi_i\rangle)}{\partial m_i}\right)_{min} = 0; \quad i = 0,\dots 3 ; \qquad (3)$$

These equations can be identified with truncated Schwinger-Dyson (SD) equations for the one- and two-point Green functions, see Figs. 1,2.

Having determined the value of the parameter $\varepsilon = v m_\pi^2$ in terms of observables, one can solve the gap equations. We fix the v.e.v. v at the pion decay constant f_π value of 93 MeV and the physical pion mass at $m_\pi = 140$ MeV. Thus the gap Eqs. (3) turn into a single equation that is solved numerically. Thence we see that with increasing cutoff Λ all solutions to the gap equation approach the symmetry restoration limit $M \to \mu$ for large values of M, or equivalently large λ_0. This means that the large boson loop effects lead to symmetry restoration, in contrast to the fermion loops which lead to symmetry breaking. Solutions that lie above the 2μ threshold require rather small values of (either kind of) cutoff Λ, however, in

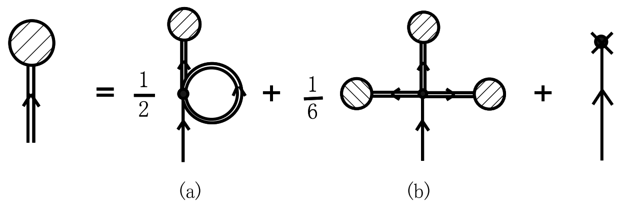

Figure 1: One-point Green function Schwinger-Dyson equation determining the dynamics of the σ model in the Hartree approximation.

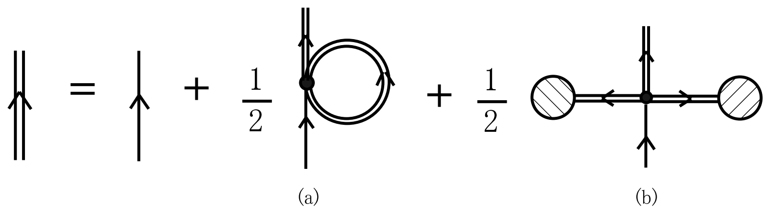

Figure 2: Two-point Green function Schwinger-Dyson equation in the Hartree approximation.

agreement with the small (second) meson-loop cutoff found in $1/N_c$ studies of the NJL chiral quark model [3].

We reduced the Bethe-Salpeter equation in the isoscalar channel to a single algebraic equation involving transcedental analytic functions with branch cuts at and imaginary parts above the corresponding thresholds. This equation has in general both real and imaginary parts. From numerical solutions to the real part of the BS Eq. one can see that the σ mass is always shifted *downward* from the "elementary" sigma field's (ϕ_0) mass M, in agreement with the variational property of the mean-field approximation. For low values of $M(\leq 2\mu)$ the σ meson mass position drops below the 2μ threshold and the σ meson consists predominantly of the bare ϕ_0 state with some 2π and 2σ "cloud" admixture. With increasing coupling constant λ_0 the physical σ's mass increases above the 2μ threshold and the "bare" and "dressed" components of the wave function cannot be separated any more. For weak couplings only one state has been found. The σ mass changes continuously with decreasing coupling λ_0 and connects smoothly to the perturbative σ mass in the weak coupling limit. For many values of the cutoff Λ and above some critical value of M, there is a second root. With increasing $M \sim \sqrt{\lambda_0}$ the two roots sometimes merge into one and then immediately disappear. For other values of parameters the two roots diverge, the lower one moving to zero, while the heavier one moves back up. In either case the lower zero, which is connected to the perturbative solution, has an upper limit generally lower than 1 GeV. That is perhaps the most interesting result of this paper.

References

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