Question 4: Statistical mechanics

I. Consider a system which can take the energies $E_0, E_1, E_2 \cdots$, and assume the system is in equilibrium at temperature T.

(Q1) What is the probability that the system takes the energy E_n?

(Q2) Let us define the partition function

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n}, \tag{1}$$

where $\beta = 1/(k_B T)$ and k_B is the Boltzmann constant. Write the expectation value of the energy in terms of Z.

II. Consider a system of N independent spins $i = 1, 2, \cdots N$ under the external magnetic field H, and in equilibrium at temperature T. The magnetic moment of i-th spin, μ_i, can take either μ_0 or $-\mu_0$ depending on the spin direction along the axis of H, and its energy is given by $-\mu_i H$. Here μ_0 is the absolute value of the magnetic moment. We neglect the interaction between different spins. In the following, $\langle X \rangle$ represents the expectation value of X.

(Q3) Calculate the partition function Z.

(Q4) The magnetization is defined by $m = (1/N)\sum_i \mu_i$. Derive its expectation value $\langle m \rangle$, and plot $\langle m \rangle$ as a function of H at the fixed temperature.

(Q5) Calculate the magnetic susceptibility $\chi = \lim_{H\to 0} \partial \langle m \rangle / \partial H$ as function of T.

(Q6) Fluctuation of m is calculated by $\delta m = \sqrt{\langle m^2 \rangle - \langle m \rangle^2}$. Find δm, and show that δm vanishes in the limit of $N \to \infty$.

III. Consider a system of N independent spins under the external magnetic field H and in temperature T. Now the magnetic moment μ_i can take three states, μ_0, 0 or $-\mu_0$.

(Q7) Derive the magnetic susceptibility χ.

(END)