The Physics of Complexity
nonlinear science

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keywords

- Linear / Nonlinear
- Topology
- Singularity

Z. Yoshida, *Nonlinear Science ---the challenge of complex systems* (Springer, 2009)

Galileo’s Natural Philosophy

- “Philosophy is written in this grand book, the universe ... It is written in the language of mathematics”

- Observation=Measurement:
  object → vector

Galileo Galilei
(1565-1642)
(portrait by J. Sustermans)

The realm of science: linear space

Descartes' discourse on methods

- Clarity and distinctness by *Reductionism*

- *Dualism , Parallelism*

- *Objectivity / Subjectivity*

René Descartes
(1596-1650)
(Louvre Museum)

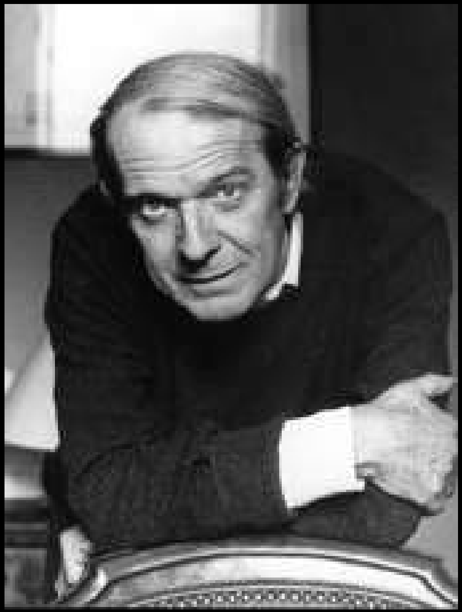
Phenomenologists’ criticism

- Galileo’s natural philosophy replaced the universe by a “fiction” written in mathematical language.
- “Theories” have *hidden* the diversity and complexity of the real world.

Edmund Husserl (1859-1938)

Structuralists' criticism

What is the *a priori* of recognition/description analysis?



Gilles Deleuze
(1925-1995)

rhizome

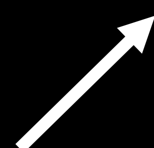
Jacques Derrida
(1930-2004)

déconstruction

“Vector” composition/ decomposition

- “state” = “vector” (in a Hilbert space)
- “measurement” = “parameterization”

=



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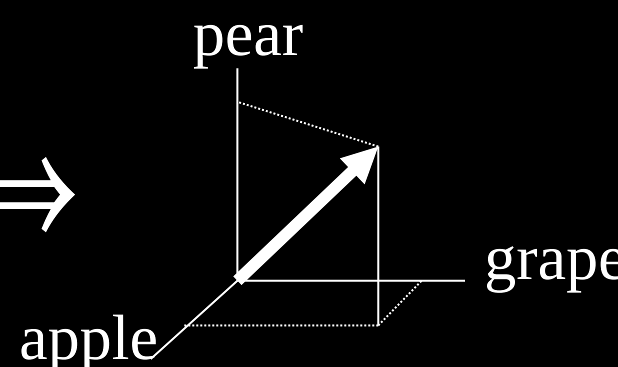


measurement
= decomposition

Linear space (vector space)
= the realm of theoretical fiction

- Proportionality law:

  1 apple : ¥100 → 3 apples: ¥300

- Nonlinearity = "scale-consciousness"

  1 apple : ¥100 → 30000 apples: ¥3000000 ?

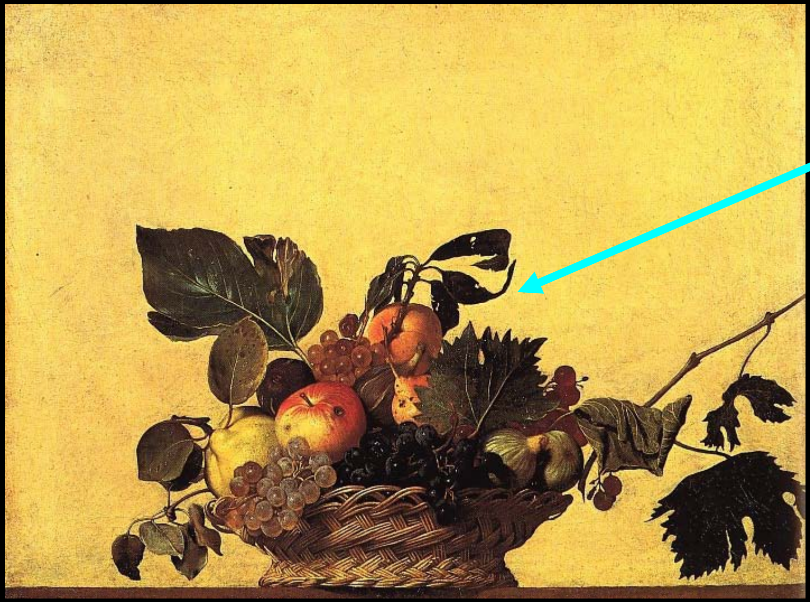
  How is the unit price determined?

  The connection with the real world is recovered.

  Nonlinearity = Complexity = Reality

Basis = Scale

Measurement (parameterization)



apples : 3
grapes : 2
pears : 2

$$= \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

Scale = Unit = Basis
= {1 apple, 1 grape, 1 pear}

Topology = Perspective (of scale)

- Object is subject to the scale of "interest"

- Connection of multi-scales: scale hierarchy

Ding an sich (thing in itself)

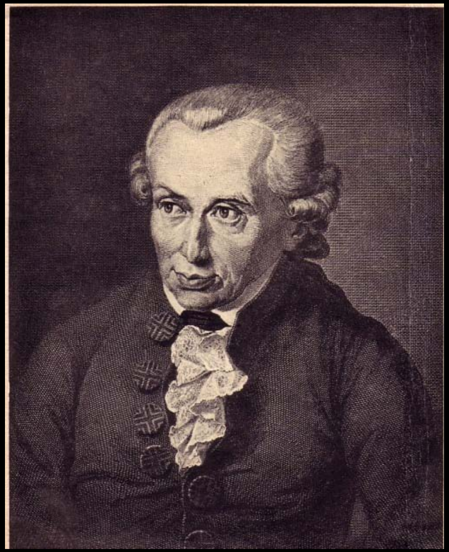
Observation → "phenomenon"

What we can measure = "change"

"mode" of phenomenon → thing

quantization

"singularity"



Immanuel Kant
(1724-1804)
(Steel engraving by J. L. Raab,
after a painting by Döbler)

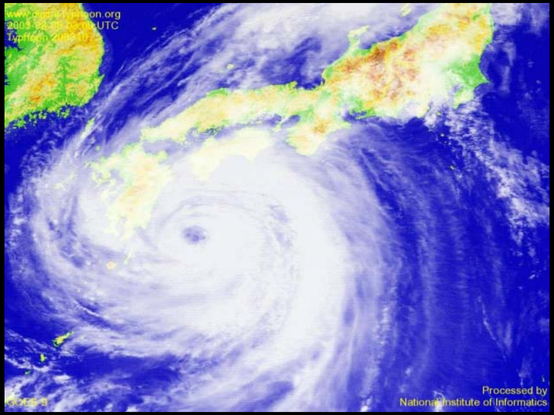
What is VORTEX ?

- A universal phenomenon, ranging from nature to society

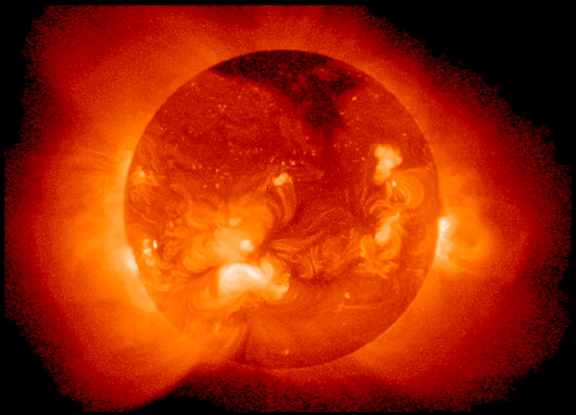
- topological charge → quantized structures

- plasma = matter (flow) vortex + EM vortex

vortexes in nature

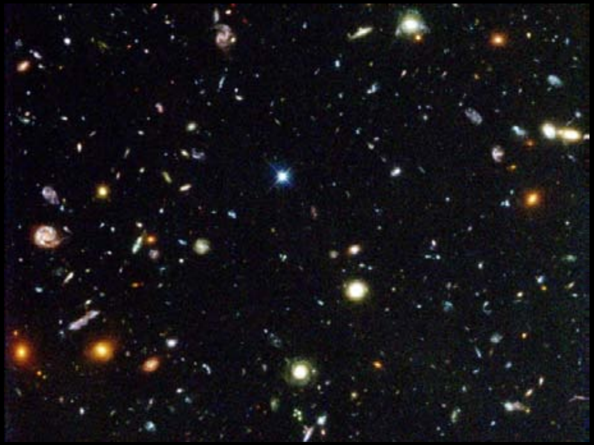


http://agora.ex.nii.ac.jp/



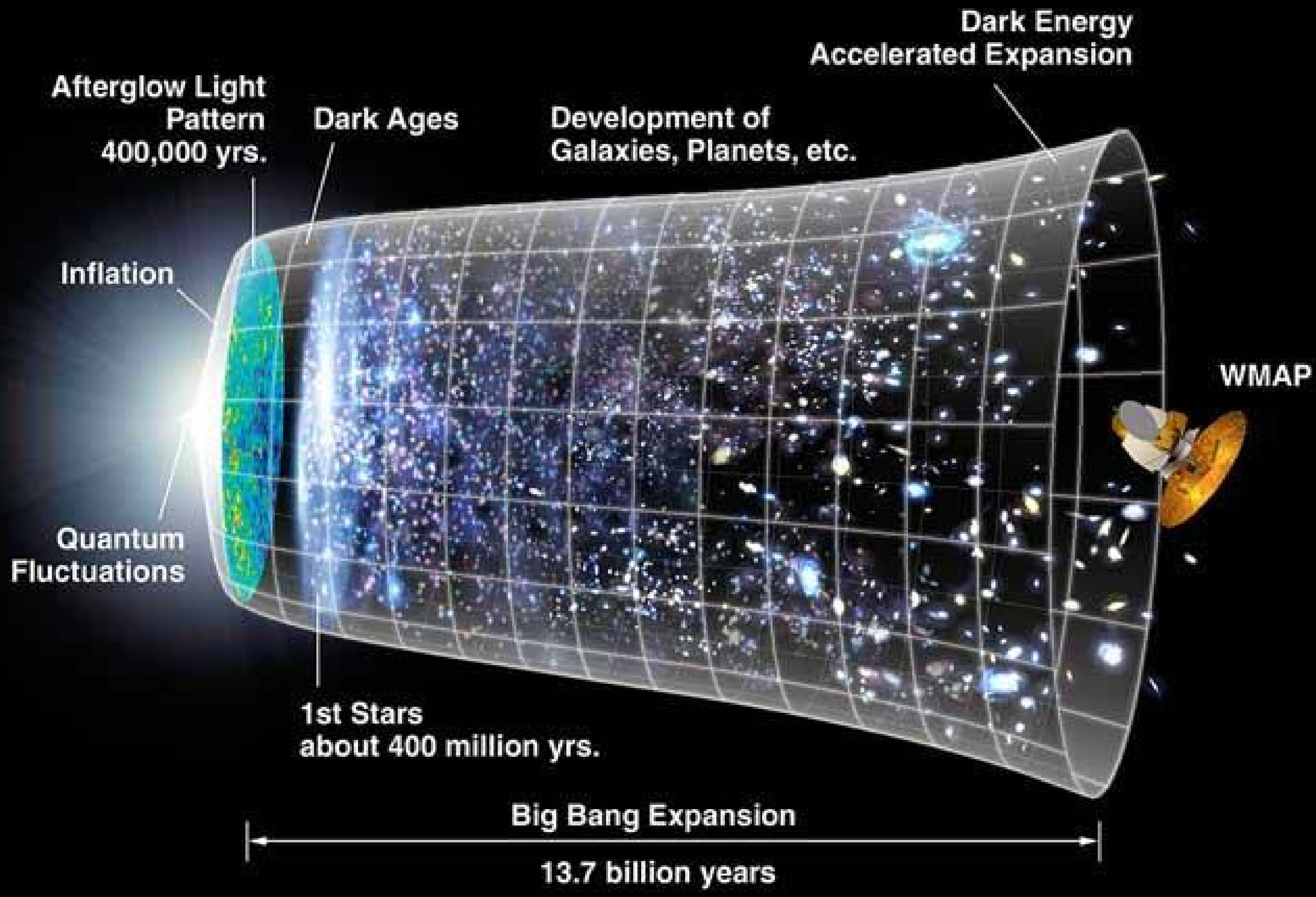
http://solar.physics.montana.edu/

Photo by K. Okano



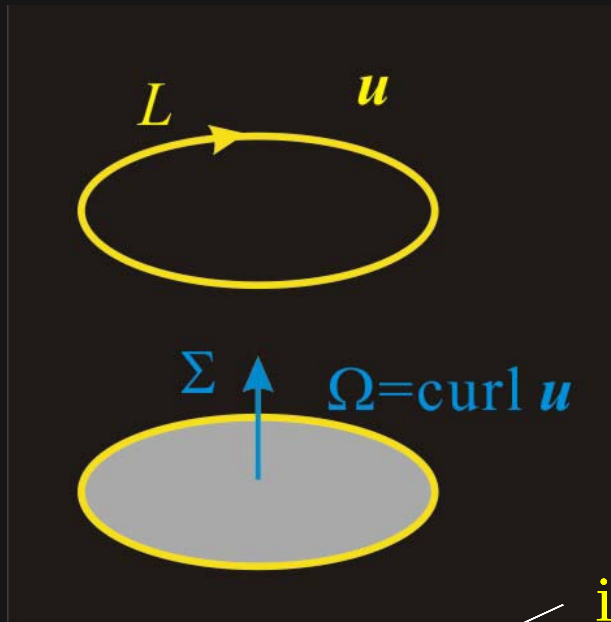
http://apod.nasa.gov/

Heterogeneity



Cited from http://www.astronomynotes.com/cosmolgy/s12.htm

Mathematical identity of VORTEX



circulation of flow

flux of vorticity

identity ← singularity

conservation of circulation/flux

$$\partial_t \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla \theta \quad \left(\boldsymbol{\Omega} = \nabla \times \boldsymbol{u} \right)$$

$$\partial_t \boldsymbol{\Omega} + \nabla \times \left(\boldsymbol{\Omega} \times \boldsymbol{u} \right) = 0$$

How vortex is “singular”

- Remember the Hamilton-Jacobi equations:

$$\partial_t S = -H, \quad \nabla S = \boldsymbol{P}$$

← momentum is irrorational

- Fields constructed from the action cannot have a vorticity of the phase (ex. super-fluid).
- How can a fluid (plasma) create vorticity?

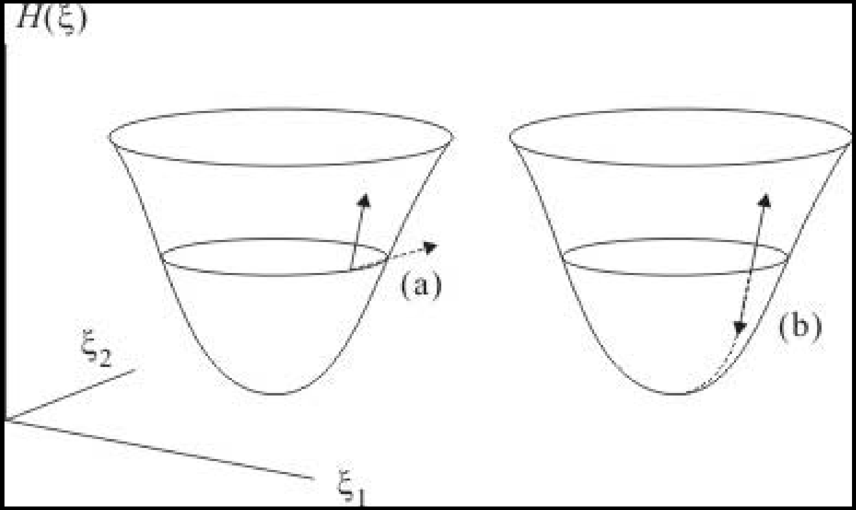
Two distinct dynamics

- Hamilton's principle

$$\delta \int L dt = 0 \quad \Rightarrow \quad \partial_t u = \{H, u\} \quad or \quad \partial_t u = A \partial_u H$$

- Dirichlet's principle

$$\partial_t u = -\partial_u H$$



Abstract Hamiltonian system

- Hamiltonian mechanics is dictated by
  A (symplectic operator) and *H* (Hamiltonian)

$$\frac{d}{dt}u = A\partial_u H(u)$$

Poisson bracket: $\left[G, F\right] = (A\partial_u G(u), \partial_u F(u))$

$$\frac{d}{dt}F(u) = \left[H, F\right]$$

$$[[F,G],H]+[[G,H],F]+[[H,F],G]=0.$$

Examples of Hamiltonian systems

classical mechanics:

$$\frac{d}{dt}u = \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = A\partial_u H,$$

quantum mechanics:

$$\partial_t u = -i\partial_u \langle Hu, u \rangle / 2$$

These examples are "canonical" because *A* is a regular operator.

non-canonical Hamiltonian mechanics

- Ker(A) = Coker(A) → "topological constraint"

- Ker(A) → Casimirs

$$\exists C \ s.t. \left[G, C\right] = 0 \ (\forall G)$$
$$i.e. \quad \partial_u C \in \mathrm{Ker}(A)$$

- H → $H+C$ does not change the dynamics.

Equilibrium (stationary points)

- Simple system: H is typically quadratic.
- Critical point: $\partial_u H = 0$ (vacuum)

- Topological constraint = Casimir-invariant may yield non-trivial class of equilibria characterized by

$$\partial_u H_\mu(u) = 0 \quad \left(H_\mu = H + \sum_j \mu_j C_j\right).$$

Helicity (a *Casimir*)

- Vortex dynamics system:

$$\partial_t \boldsymbol{u} = -\boldsymbol{\Omega} \times \boldsymbol{u} - \nabla \theta \quad \left(\boldsymbol{\Omega} = \nabla \times \boldsymbol{u}\right)$$
$$\Leftrightarrow \quad \dot{\boldsymbol{u}} = A \partial_{\boldsymbol{u}} H \quad \left(H = \tfrac{1}{2}\left\|\boldsymbol{u}\right\|^2,\ A\boldsymbol{u} = -P\boldsymbol{\Omega} \times \boldsymbol{u}\right)$$

(We assume incompressible $\boldsymbol{u}$. P is the "projection" onto the function space of incompressible fields.)

- Helicity:

$$C = \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{u}\, dx \quad \Rightarrow \quad A \partial_{\boldsymbol{u}} C = 0$$

MHD case

- Canonical form of MHD

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{B} \end{pmatrix} = A \begin{pmatrix} \partial_{\boldsymbol{v}} H \\ \partial_{\boldsymbol{B}} H \end{pmatrix}$$

$$H = \frac{1}{2}\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 \right), \quad A = \begin{pmatrix} -P\boldsymbol{\Omega} \times \circ & P[(\nabla \times \circ) \times \boldsymbol{B}] \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 \end{pmatrix}$$

- Casimirs (helicities)

$$C_1 = (\boldsymbol{A}, \boldsymbol{B}), \quad C_2 = (\boldsymbol{v}, \boldsymbol{B})$$

Structured equilibria

- Parameterized Hamiltonian (Lyapunov function):

$$\partial_u \tilde{H}_\mu(\boldsymbol{u}) = \partial\left(H(\boldsymbol{u}) + \sum \mu_j C_j(\boldsymbol{u})\right) = 0$$

- “Quantized” stationary points:

$$(\mathrm{curl} - \lambda_1)\cdots(\mathrm{curl} - \lambda_N)\boldsymbol{u} = 0$$

- Beltrami-class of equilibria (multi-scale):

$$\boldsymbol{u} = \sum a_j \boldsymbol{G}_j \quad \left(\nabla \times \boldsymbol{G}_j = \lambda_j \boldsymbol{G}_j\right)$$

Summary

- Perspective = subjectivity : *a priori of objectivity*
  - → "decomposition" → linear space
  - → scale → nonlinearity
- Phenomenon → "Thing"
  - → singularity (quatization)
  - → Beltrami fields (quantized vortex)