Coupled channel hadronic interactions from Lattice QCD

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for HAL QCD collaboration

HAL (Hadrons to Atomic nuclei from Lattice) QCD Collaboration

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Introduction
Once we obtain a “proper” nuclear potential, we apply them to the structure of (hyper-) nucleus and neutron star calculation.

Can we derive hadronic interactions from QCD?
Derivation of hadronic interaction from QCD

Start with the fundamental theory, QCD, to obtain a “proper” interaction

Lattice QCD simulation

Lüscher's finite volume method

M. Lüscher, NPB354(1991)531

1. Measure the discrete energy spectrum, E
2. Put the E into the formula which connects E and δ

HAL QCD method

Ishii, Aoki, Hatsuda, PRL99 (2007) 022001

1. Measure the NBS wave function, Ψ
2. Calculate potential, V, through Schrödinger eq.
3. Calculate observables by scattering theory

Scattering phase shift
Lüscher's method
Lüscher's finite volume formula

Lattice calculation is performed in a finite volume.  
→ we have discrete energy spectrum even in the scattering state.

Lüscher's formula describes a fixed relation  
between the scattering phase shift in the continuum  
and an energy level in a finite volume \( L^3 \)

Assumptions

- Two particles scatter from a finite range interaction.
- The Helmholtz wave function is in the asymptotic region: \( R < r < L/2 \),  
i.e., two particles are well separated

\[
\frac{1}{\tan \delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{L^3} \sum_p \frac{1}{p^2 - k^2}
\]

S-matrix has only one parameter, \( \delta \),  
and it is related to an energy level in a one-to-one correspondence.
Extension of Lüscher's method to the multi-channel situation

Parametrization of 2x2 S-matrix

\[
S(E) = \begin{pmatrix}
\eta e^{2i\delta_1} & i\sqrt{1-\eta^2} e^{i(\delta_1+\delta_2)} \\
i\sqrt{1-\eta^2} e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2}
\end{pmatrix}
\]

Two-channel S-matrix has 3-parameters

\[\delta_1(E), \delta_2(E), \eta(E)\]

These are related to the energy E by an eigenvalue equation (s-wave)

\[
\cos(\Delta_1 + \Delta_2 - \delta_1^0 - \delta_2^0) = \eta \cos(\Delta_1 - \Delta_2 - \delta_1^0 + \delta_2^0)
\]

\[
\frac{1}{\tan \Delta_i} = \frac{4\pi}{k_i} \cdot \frac{1}{L^3} \sum_p \frac{1}{p^2 - k_i^2}
\]

Unlike the single channel case,

the number of equations is less than the number of parameters in S-matrix.

Several prescriptions are proposed.

V. Bernard et al., JHEP1101 (2011) 019  M. Doring et al., EPJA 48 (2012) 114
Jia-Jun Wu et al., PRC90 (2014) 055206
HAL QCD method
**Nambu-Bethe-Salpeter wave function**

**Definition**: equal time NBS w.f.

\[ \Psi^\alpha(E, \vec{r}) e^{-Et} = \sum_{\vec{x}} \langle 0 | H_1^\alpha(t, \vec{x} + \vec{r}) H_2^\alpha(t, \vec{x}) | E \rangle \]

\( E \) : Total energy of the system

Local composite interpolating operators

\[ B_\alpha = \epsilon^{abc} (q_a^T C \gamma_5 q_b) q_c \alpha \]
\[ D_{\mu \alpha} = \epsilon^{abc} (q_a^T C \gamma_\mu q_b) q_c \alpha \]
\[ M = (\bar{q}_a \gamma_5 q_a) \]

- It satisfies the Helmholtz eq. in asymptotic region: \( (p^2 + \nabla^2) \Psi(E, \vec{r}) = 0 \)

- Using the reduction formula,

\[ \Psi^\alpha(E, \vec{r}) = \sqrt{Z_{H_1}} \sqrt{Z_{H_2}} \left( e^{i\vec{p} \cdot \vec{r}} + \int \frac{d^3q}{2E_q} \frac{T(q, p)}{4E_p(E_q - E_p - i\epsilon)} e^{i\vec{q} \cdot \vec{r}} \right) \]

C.-J.D. Lin et al., NPB619 (2001) 467.

Phase shift is defined as
\[ S \equiv e^{i\delta} \]

NBS wave function has a same asymptotic form with quantum mechanics. (NBS wave function is characterized from phase shift)
Potential in HAL QCD method

We define potentials which satisfy Schrödinger equation

\[
\left( p^2 + \nabla^2 \right) \Psi^\alpha(E, \vec{r}) \equiv \int d^3y \, U^\alpha_\alpha(\vec{x}, \vec{y}) \Psi^\alpha(E, \vec{y})
\]

Energy independent potential

\[
\left( p^2 + \nabla^2 \right) \Psi^\alpha(E, \vec{r}) = K^\alpha(E, \vec{r})
\]

\[
K^\alpha(E, \vec{r}) = \int dE' \, K^\alpha(E', \vec{x}) \int d^3y \, \tilde{\Psi}^\alpha(E', \vec{y}) \Psi^\alpha(E, \vec{y})
\]

\[
= \int d^3y \left[ \int dE' \, K^\alpha(E', \vec{x}) \tilde{\Psi}^\alpha(E', \vec{y}) \right] \Psi^\alpha(E, \vec{y})
\]

\[
= \int d^3y \, U^\alpha_\alpha(\vec{x}, \vec{y}) \Psi^\alpha(E, \vec{y})
\]

We can define an energy independent potential but it is fully non-local.

This potential automatically reproduce the scattering phase shift

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Time-dependent method

Let's start with the normalized four-point correlator.

\[ R_{I}^{B_1 B_2}(t, \vec{r}) = F_{B_1 B_2}(t, \vec{r}) e^{(m_1 + m_2)t} \]

\[ = A_0 \Psi(\vec{r}, E_0) e^{-(E_0 - m_1 - m_2)t} + A_1 \Psi(\vec{r}, E_1) e^{-(E_1 - m_1 - m_2)t} + \cdots \]

Each wave functions satisfy Schrödinger eq. with proper energy

\[ \left( \frac{p_0^2}{2\mu} + \nabla^2 \right) \Psi(\vec{r}, E_0) = \int U(\vec{r}, \vec{r}') \Psi(\vec{r}', E_0) d^3 r' \]

\[ \left( \frac{p_1^2}{2\mu} + \nabla^2 \right) \Psi(\vec{r}, E_1) = \int U(\vec{r}, \vec{r}') \Psi(\vec{r}', E_1) d^3 r' \]

\[ E_n - m_1 - m_2 \approx \frac{p_n^2}{2\mu} \]

A single state saturation is not required!!
BB interaction from NBS wave function

\[
\left( -\frac{\partial}{\partial t} + \frac{\nabla^2}{2\mu} \right) R_{I}^{B_{1}B_{2}}(t,\vec{r}) = \int U(\vec{r},\vec{r}') R_{I}^{B_{1}B_{2}}(t,\vec{r}) d^3r'
\]

Derivative (velocity) expansion of \( U \) is performed to deal with its non-locality.

- For the case of oct-oct system,
  \[
  U(\vec{r},\vec{r}') = \left[ V_C(r) + S_{12} V_{T_1}(r) \right] + \left[ \vec{L} \cdot \vec{S} V_{LS}(r) + \vec{L} \cdot \vec{S}\vec{a} V_{ALS}(r) \right] + O(\nabla^2)
  \]
  Leading order part

- For the case of dec-oct and dec-dec system,
  \[
  U(\vec{r},\vec{r}') = \left[ V_C(r) + S_{12} V_{T_1}(r) + S_{ii} V_{T_2}(r) \right] + O\left( Spin \ op^3 \right) + O(\nabla)
  \]
  Leading order part

- For the case of ps-meson-ps-meson system,
  \[
  U(\vec{r},\vec{r}') = \left[ V_C(r) \right] + O(\nabla^2)
  \]
  Leading order part
Comparison between the potential method and Lüscher's method

\( \pi - \pi \) scattering with quench QCD

Resulting scattering phase shifts are consistent from both methods
We define potentials which satisfy a coupled channel Schrödinger equation

\[ \Psi^\alpha(E_i, \vec{r}) = \langle 0 | (B_1 B_2)^\alpha(\vec{r}) | E_i \rangle \]

\[ \Psi^\beta(E_i, \vec{r}) = \langle 0 | (B_1 B_2)^\beta(\vec{r}) | E_i \rangle \]

Two-channel coupling case

\[ \int dr \Psi^\beta(E', \vec{r}) \Psi^\gamma(E, \vec{r}) = \delta(E' - E) \delta^\gamma_\beta \]

Leading order of velocity expansion and time-derivative method

\[ \left[ \left( \frac{\partial}{\partial t} + \frac{\nabla^2}{2 \mu_\alpha} \right) R_{E_0}^\alpha(t, \vec{r}) - \left( \frac{\partial}{\partial t} + \frac{\nabla^2}{2 \mu_\beta} \right) R_{E_0}^\beta(t, \vec{r}) \right] = \left( \begin{array}{cc} V^\alpha_\alpha(\vec{r}) & V^\alpha_\beta(\vec{r}) \Delta^\alpha_\beta(t) \\ V^\beta_\alpha(\vec{r}) \Delta^\beta_\alpha(t) & V^\beta_\beta(\vec{r}) \end{array} \right) \left( \begin{array}{c} R_{E_0}^\alpha(t, \vec{r}) \\ R_{E_0}^\beta(t, \vec{r}) \end{array} \right) \]

Considering two different energy eigen states

\[ \Delta^\alpha_\beta = \frac{\exp(-m_\alpha + m_\alpha t)}{\exp(-m_\beta + m_\beta t)} \]

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Numerical results
2+1 flavor gauge configurations by PACS-CS collaboration.

- RG improved gauge action & O(a) improved Wilson quark action
- $\beta = 1.90$, $a^{-1} = 2.176$ [GeV], $32^3 \times 64$ lattice, $L = 2.902$ [fm].
- $\kappa_s = 0.13640$ is fixed, $\kappa_{ud} = 0.13700, 0.13727$ and $0.13754$ are chosen.

Flat wall source is considered to produce S-wave B-B state.

The KEK computer system A resources are used.

<table>
<thead>
<tr>
<th>In unit of MeV</th>
<th>$Esb 1$</th>
<th>$Esb 2$</th>
<th>$Esb 3$</th>
</tr>
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<tbody>
<tr>
<td>$\pi$</td>
<td>701±1</td>
<td>570±2</td>
<td>411±2</td>
</tr>
<tr>
<td>$K$</td>
<td>789±1</td>
<td>713±2</td>
<td>635±2</td>
</tr>
<tr>
<td>$m_\pi / m_K$</td>
<td>0.89</td>
<td>0.80</td>
<td>0.65</td>
</tr>
<tr>
<td>$N$</td>
<td>1585±5</td>
<td>1411±12</td>
<td>1215±12</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>1644±5</td>
<td>1504±10</td>
<td>1351±8</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>1660±4</td>
<td>1531±11</td>
<td>1400±10</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>1710±5</td>
<td>1610±9</td>
<td>1503±7</td>
</tr>
</tbody>
</table>

$\Lambda \Lambda : 3288$ MeV
$N \Xi : 3295$ MeV
$\Sigma \Sigma : 3320$ MeV

SU(3) breaking effects becomes larger
In this channel, our group found the “H-dibaryon” in the SU(3) limit.

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$\Lambda\Lambda$ and $N\Xi$ phase shifts

$N_f = 2+1$ full QCD with $L = 2.9$fm

- $m\pi = 700$ MeV: bound state
- $m\pi = 570$ MeV: resonance near $\Lambda\Lambda$ threshold
- $m\pi = 410$ MeV: resonance near $N\Xi$ threshold

H-dibaryon is unlikely bound state

Preliminary!
**Comparison of potential matrices**

Transformation of potentials from the particle basis to the SU(3) irreducible representation (irrep) basis. 

SU(3) Clebsh-Gordan coefficients

\[
\begin{pmatrix}
1 \\
8 \\
27 \\
\end{pmatrix} = U \begin{pmatrix}
\Lambda \Lambda \\
N \Xi \\
\Sigma \Sigma \\
\end{pmatrix}, \quad U \begin{pmatrix}
V_{\Lambda \Lambda}^{\Lambda \Lambda} & V_{N \Xi}^{\Lambda \Lambda} & V^{\Lambda \Lambda} \\
V_{\Lambda \Lambda}^{N \Xi} & V_{N \Xi}^{N \Xi} & V^{N \Xi} \\
V_{\Sigma \Sigma}^{\Lambda \Lambda} & V_{N \Xi}^{\Sigma \Sigma} & V^{\Sigma \Sigma} \\
\end{pmatrix} \quad U^t \rightarrow \begin{pmatrix}
V_1 \\
V_8 \\
V_{27} \\
\end{pmatrix}
\]

In the SU(3) irreducible representation basis, the potential matrix should be diagonal in the SU(3) symmetric configuration.

Off-diagonal part of the potential matrix in the SU(3) irrep basis would be an effectual measure of the SU(3) breaking effect.

We will see how the SU(3) symmetry of potential will be broken by changing the u,d quark masses lighter.

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$1, 8_s, 27 \ (l=0)^1S_0 \ channel$

**Strongly attractive**

H-dibaryon channel

**Pauli blocking effect**

**Mixture of singlet and octet**

Is relatively larger than the others

27 plet does not mix so much to the other representations

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- $E_{sb1} : m_{\pi} = 701 \ MeV$
- $E_{sb2} : m_{\pi} = 570 \ MeV$
- $E_{sb3} : m_{\pi} = 411 \ MeV$
We introduced the coupled channel HAL QCD method.

It allows us to tackle several exotic systems such as H-dibaryon.

Potentials are derived from NBS wave functions calculated with PACS-CS configurations.

H-dibaryon energy is going higher as a decreasing quark masses.

Small mixture between different SU(3) irreps can be seen as the flavor SU(3) breaking effect.

The preliminary results at physical situation \( \frac{m_\pi}{m_K} = 0.28 \) coming soon.