Finite-volume Hamiltonian method for $\pi\pi$ scattering in lattice QCD

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Something New …

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Outline

- Introduction
- Hamiltonian for $\pi\pi$ scattering
- Finite-box Hamiltonian method
- Summary

Introduction

Resonance Region

QCD → ? → Experiment Data (cross section)

Nonperturbative

One way

Lattice QCD

Finite-volume & Euclidean time

Finite-Volume energy eigenstate's spectrum

Partial Wave Analysis

Partial Wave S matrix (phase shift and inelasticity)

?

?

Introduction

Finite-Volume energy eigenstate's spectrum

Luescher's method ⟷

Partial Wave S matrix (phase shift and inelasticity)

Introduction

Finite-Volume energy eigenstate's spectrum

Luescher's method ↔

Partial Wave S matrix (phase shift and inelasticity)

Three ($E \sim L_1, L_2, L_3$) ↔ Three ($E \sim \delta_1, \delta_2, \eta$)

↕ Difficult !

Lattice spectrum

One L → Several E

One E ↛ Several L

Introduction

Finite-Volume energy eigenstate's spectrum

Partial Wave S matrix (phase shift and inelasticity)

Hamiltonian Model

Introduction

Finite-Volume energy eigenstate's spectrum

Partial Wave S matrix (phase shift and inelasticity)

Hamiltonian Model

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Hamiltonian for ππ scattering

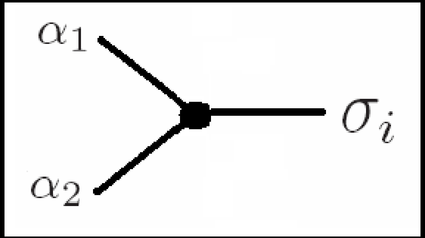
$$H = H_0 + H_I$$

$$H_0 = \sum_{i=1,n} |\sigma_i\rangle m_i \langle\sigma_i| + \sum_{\alpha} \int d\vec{k}_\alpha |\alpha(\vec{k}_\alpha)\rangle \left[2\sqrt{m_\alpha^2 + \vec{k}_\alpha^{\,2}} \right] \langle\alpha(\vec{k}_\alpha)|$$

$|\sigma_i>$ bare state with mass m_i

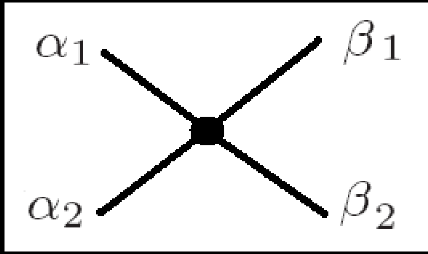
$|\alpha(k_\alpha)>$ the channels such as ππ, $\bar{K}$K, …

$$H_I = \hat{g} + \hat{v}$$



$$\hat{g} = \int d\vec{k}_\alpha \sum_{\alpha} \sum_{i=1,n} \left[|\alpha(\vec{k}_\alpha)\rangle g^{+}_{i,\alpha} \langle\sigma_i| + |\sigma_i\rangle g_{i,\alpha} \langle\alpha(\vec{k}_\alpha)| \right]$$

$$\hat{v} = \int d\vec{k}_\alpha d\vec{k}_\beta \sum_{\alpha,\beta} |\alpha(\vec{k}_\alpha)\rangle v_{\alpha,\beta} \langle\beta(\vec{k}_\beta)|$$

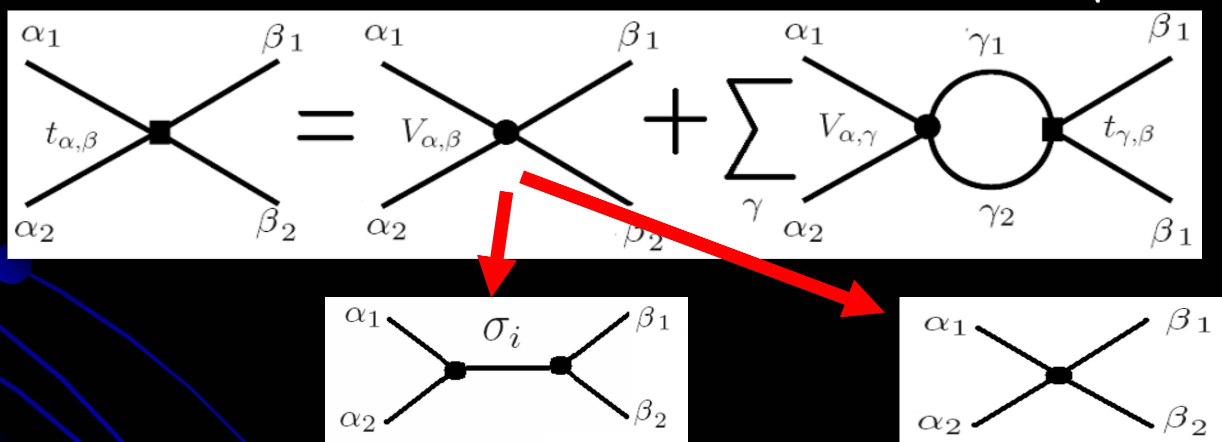


$$\langle\beta(\vec{k}_\beta)|\alpha(\vec{k}_\alpha)\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta) \qquad \langle\sigma_j|\sigma_i\rangle = \delta_{ij}$$

Hamiltonian for ππ scattering

Scattering Equation: (Partial Wave)

$$t^{L}_{\alpha,\beta}(k_\alpha, k_\beta, E) = V^{L}{}_{\alpha,\beta}(k_\alpha, k_\beta) + \sum_{\gamma} \int k_\gamma{}^2 dk_\gamma \frac{V^{L}_{\alpha,\gamma}(k_\alpha, k_\gamma) t^{L}_{\gamma,\beta}(k_\gamma, k_\beta, E)}{E - 2\sqrt{m^2_{\gamma 1} + k^2_\gamma} + i\varepsilon}$$



$$V(\vec{k}_\alpha, \vec{k}_\beta) = \sum_{\mathrm{i}} \frac{\langle \sigma_i | \hat{g} | \alpha(k_\alpha) \rangle \langle \alpha(k_\beta) | \hat{g} | \sigma_i \rangle}{E - m_i} + \langle \alpha(k_\beta) | \hat{v} | \alpha(k_\alpha) \rangle$$

$$\mathrm{V}^{\mathrm{L}}(k_\alpha, k_\beta) = \sum_{\mathrm{m}} \int d\,\Omega_{\vec{k}_\alpha} d\Omega_{\vec{k}_\beta} V(\vec{k}_\alpha, \vec{k}_\beta) Y_{Lm}(\Omega_{\vec{k}_\alpha}) Y^{*}_{Lm}(\Omega_{\vec{k}_\beta})$$

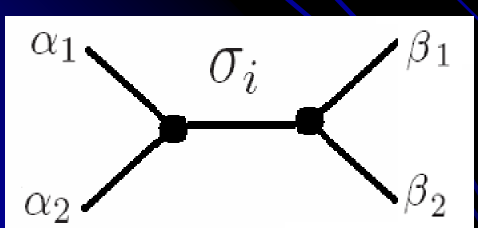
Hamiltonian for ππ scattering

Observations & t martix

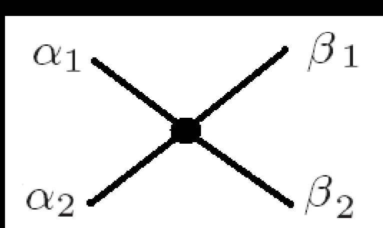
$$S_{\alpha,\beta} = 1 - i2\sqrt{\rho_\alpha}\, t_{\alpha,\beta}(k_{0\alpha}, k_{0\beta}, E)\sqrt{\rho_\beta}$$

$$\rho_\alpha = \frac{\pi k_{0\alpha}\left(\sqrt{m_{\alpha 1}^2 + k_{0\alpha}^2}\right)^2}{E} = \frac{\pi k_{0\alpha} E}{2}$$

$$\eta e^{2i\delta_\alpha} = S_{\alpha,\alpha}$$



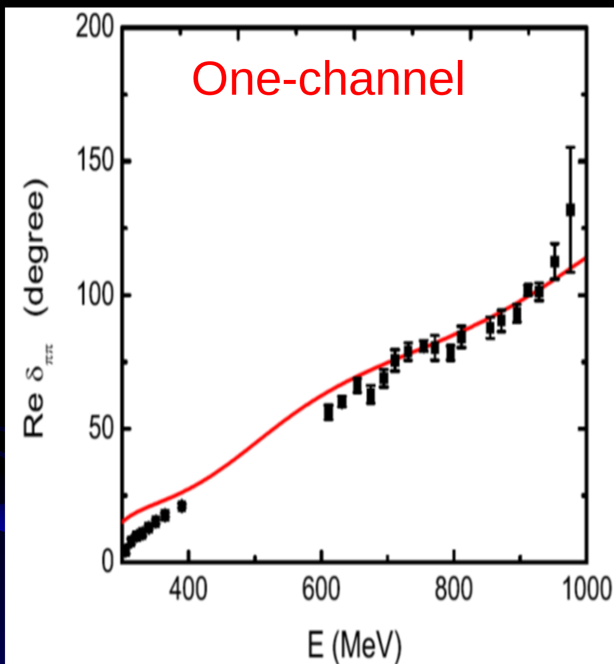
$$g_{i,\alpha}^{*}\frac{1}{E - m_i}g_{i,\beta} \qquad V_{\alpha,\beta}$$

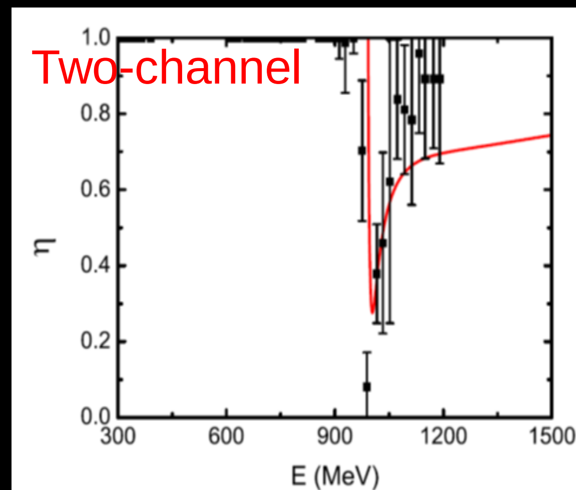


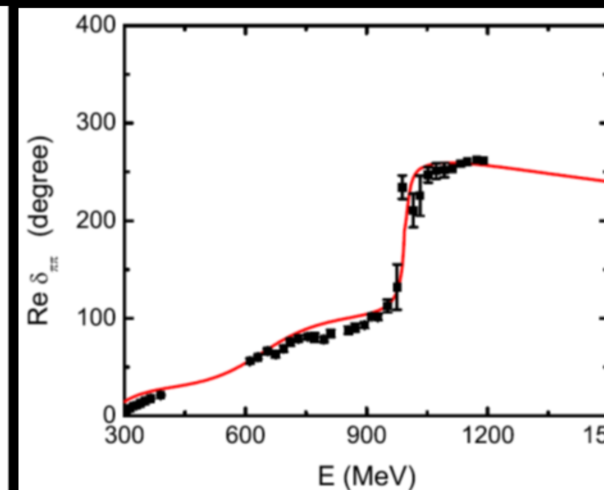
Hamiltonian for $\pi\pi$ scattering

S-wave

$$g_{i,\alpha}(k_\alpha) = \frac{\tilde{g}_{i,\alpha}}{\sqrt{m_\pi}} \frac{1}{(1+(c_\alpha k_\alpha)^2)} \qquad v_{\alpha,\beta}(k_\alpha,k_\beta) = \frac{G_{\alpha,\beta}}{m_\pi^2} \frac{1}{(1+(d_\alpha k_\alpha)^2)^2} \frac{1}{(1+(d_\beta k_\beta)^2)^2}$$

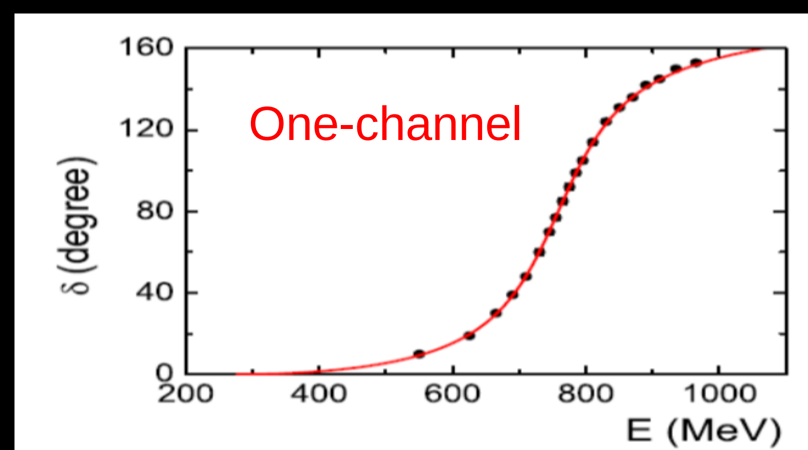






P-wave

$$g_{\rho,\pi\pi}(k_{\pi\pi}) = \frac{\tilde{g}_{\rho,\pi\pi}}{m_\pi^{3/2}} \frac{\varepsilon_\mu k_{\pi\pi}^\mu}{\left(1+(c_\alpha k_{\pi\pi})^2\right)^{3/2}}$$



Introduction

Finite-Volume energy eigenstate's spectrum

Partial Wave S matrix (phase shift and inelasticity)

Hamiltonian Model

Outline

- Introduction
- Hamiltonian for $\pi\pi$ scattering
- Finite-box Hamiltonian method
- Compare to the other methods
- Summary and Outlook

Finite-box Hamiltonian method

1. S-wave CM 1-channel 2-channel
2. P-wave CM 1-channel
3. S-wave Boost 1-channel
4. P-wave Boost 1-channel

Finite-box Hamiltonian method

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle$$

Eigenvalue
Energy

$$\mathrm{Det}[H_0 + H_I - E\mathrm{I}] = 0$$

$$\vec{k} = \vec{n}\frac{2\pi}{L} \qquad \vec{n} \in \mathrm{Z}^3$$

Lattice Size

Finite-box Hamiltonian method

$$H|\psi\rangle = E|\psi\rangle$$

Eigenvalue Energy

$$\mathrm{Det}[H_0 + H_I - E\mathrm{I}]=0$$

$$\vec{k} = \vec{n}\frac{2\pi}{L} \qquad \vec{n} \in Z^3$$

Lattice Size

$$\vec{\mathrm{P}}_{\mathrm{tot}} = 0$$ (CM)

Continue	$\int d\vec{k}$	and	$\left|\alpha(\vec{k}_\alpha)\right\rangle$	and	$\left\langle\beta(\vec{k}_\beta)\middle|\alpha(\vec{k}_\alpha)\right\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$
↓	↓		↓		↓
Discrete	$\sum_{\mathrm{i}}\left(2\pi/\mathrm{L}\right)^3$	and	$\left(2\pi/\mathrm{L}\right)^{-3/2}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha$	and	${}_\beta\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\middle|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha = \delta_{\alpha\beta}\delta_{\mathrm{ij}}$

$$H_0 = \sum_{i=1,n}|\sigma_i\rangle m_i\langle\sigma_i| + \sum_{\alpha,\mathrm{i}}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha\left[2\sqrt{m_\alpha^2 + k_\alpha^2}\right]_\alpha\left\langle\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right|$$

$$H_I = \sum_{\mathrm{j}}\left(2\pi/\mathrm{L}\right)^{3/2}\sum_\alpha\sum_{i=1,n}\left[\left|\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right\rangle_\alpha g_{i,\alpha}^{+}\langle\sigma_i| + |\sigma_i\rangle g_{i,\alpha}\ {}_\alpha\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right|\right]$$

$$+\sum_{\mathrm{i,j}}\left(2\pi/\mathrm{L}\right)^3\sum_{\alpha,\beta}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha v_{\alpha,\beta}\ {}_\beta\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right|$$

Finite-box Hamiltonian method

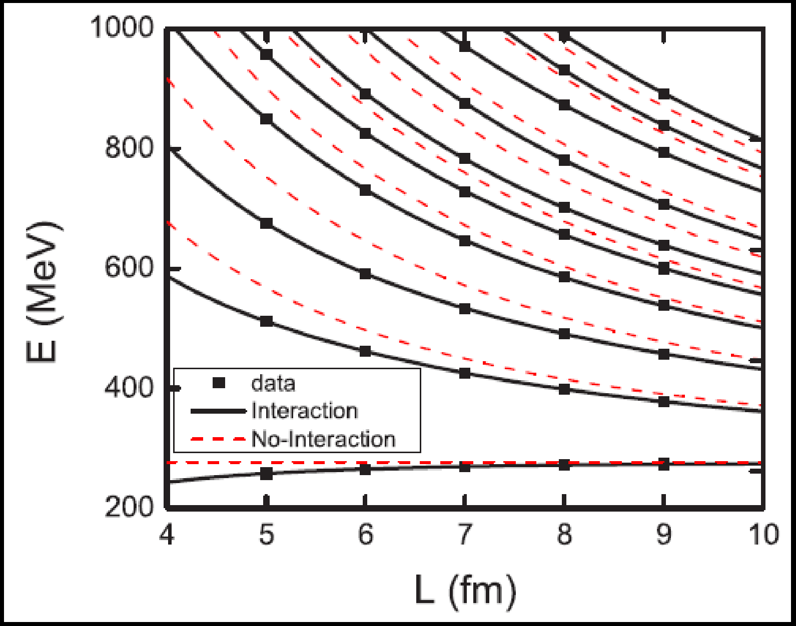
One channel case (**CM**):

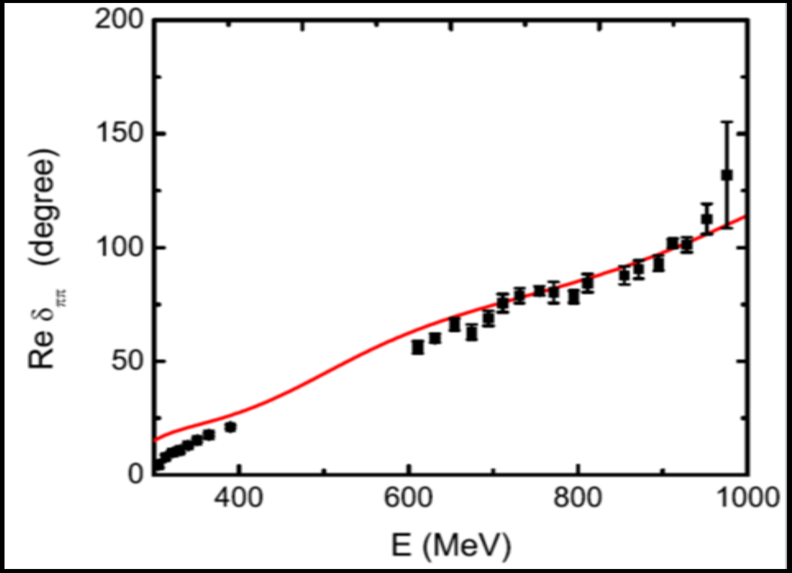
$$H_0=\begin{pmatrix} m_1 & 0 & 0 & \cdots \\ 0 & 2\sqrt{k_0^2+m_\pi^2} & 0 & \cdots \\ 0 & 0 & 2\sqrt{k_1^2+m_\pi^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$H_I=\begin{pmatrix} 0 & g_{\pi\pi}^{fin}(k_0) & g_{\pi\pi}^{fin}(k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_1) & v_{\pi\pi,\pi\pi}^{fin}(k_1,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$g_{\pi\pi}^{fin}(k_n)=\left(\frac{2\pi}{L}\right)^{\frac{3}{2}} g_{\pi\pi}(k_n) \qquad v_{\pi\pi,\pi\pi}^{fin}(k_n,k_m)=\left(\frac{2\pi}{L}\right)^{3} v_{\pi\pi,\pi\pi}(k_n,k_m)$$

$$\mathrm{Det}[H_0+H_I-E\mathrm{I}]=0$$





Finite-box Hamiltonian method

One channel case (**CM**):

$$H_0 = \begin{pmatrix} m_1 & 0 & 0 & \cdots \\ 0 & 2\sqrt{k_0^2 + m_\pi^2} & 0 & \cdots \\ 0 & 0 & 2\sqrt{k_1^2 + m_\pi^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

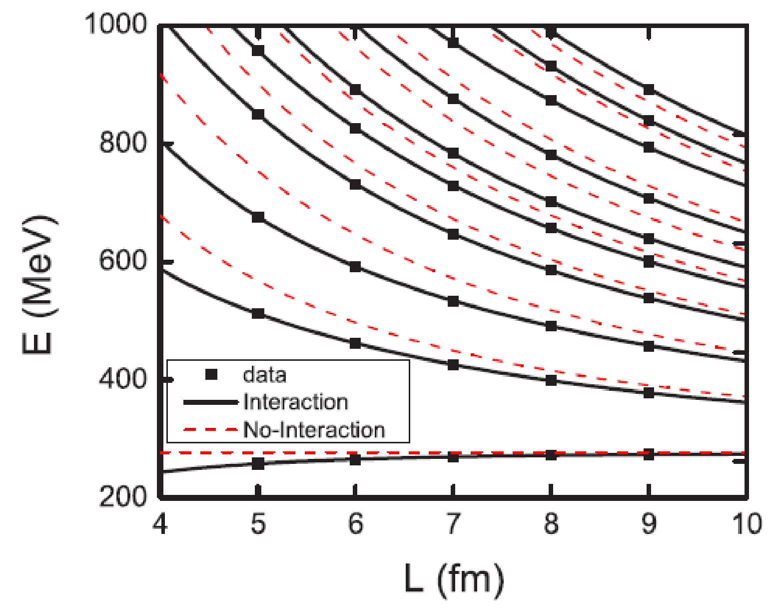
$$H_I = \begin{pmatrix} 0 & g_{\pi\pi}^{fin}(k_0) & g_{\pi\pi}^{fin}(k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_1) & v_{\pi\pi,\pi\pi}^{fin}(k_1,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

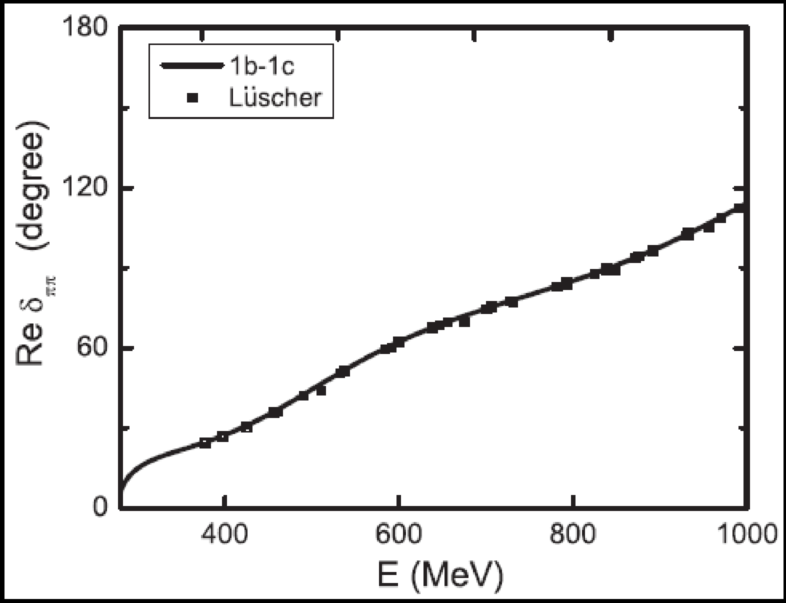
$$g_{\pi\pi}^{fin}(k_n) = \left(\frac{2\pi}{L}\right)^{\frac{3}{2}} g_{\pi\pi}(k_n) \qquad v_{\pi\pi,\pi\pi}^{fin}(k_n,k_m) = \left(\frac{2\pi}{L}\right)^{3} v_{\pi\pi,\pi\pi}(k_n,k_m)$$

Luescher Method

$$\delta(k) = -\phi(q) \bmod \pi \qquad q = \frac{kL}{2\pi} = \frac{2\sqrt{E^2/4 - m_\pi^2}\,L}{2\pi}$$

$$-\phi(q) = \tan^{-1}\left(\frac{q\pi^{3/2}}{Z_{00}(1;q^2)}\right) \qquad Z_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in Z^3} \frac{1}{\vec{n}^2 - q^2}$$





Finite-box Hamiltonian method

Two channels case (CM):

$$\mathrm{Det}[H_0 + H_I - E\mathrm{I}] = 0$$

$$H_0 = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\sqrt{k_0^2 + m_\pi^2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2\sqrt{k_0^2 + m_K^2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2\sqrt{k_1^2 + m_\pi^2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2\sqrt{k_1^2 + m_K^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

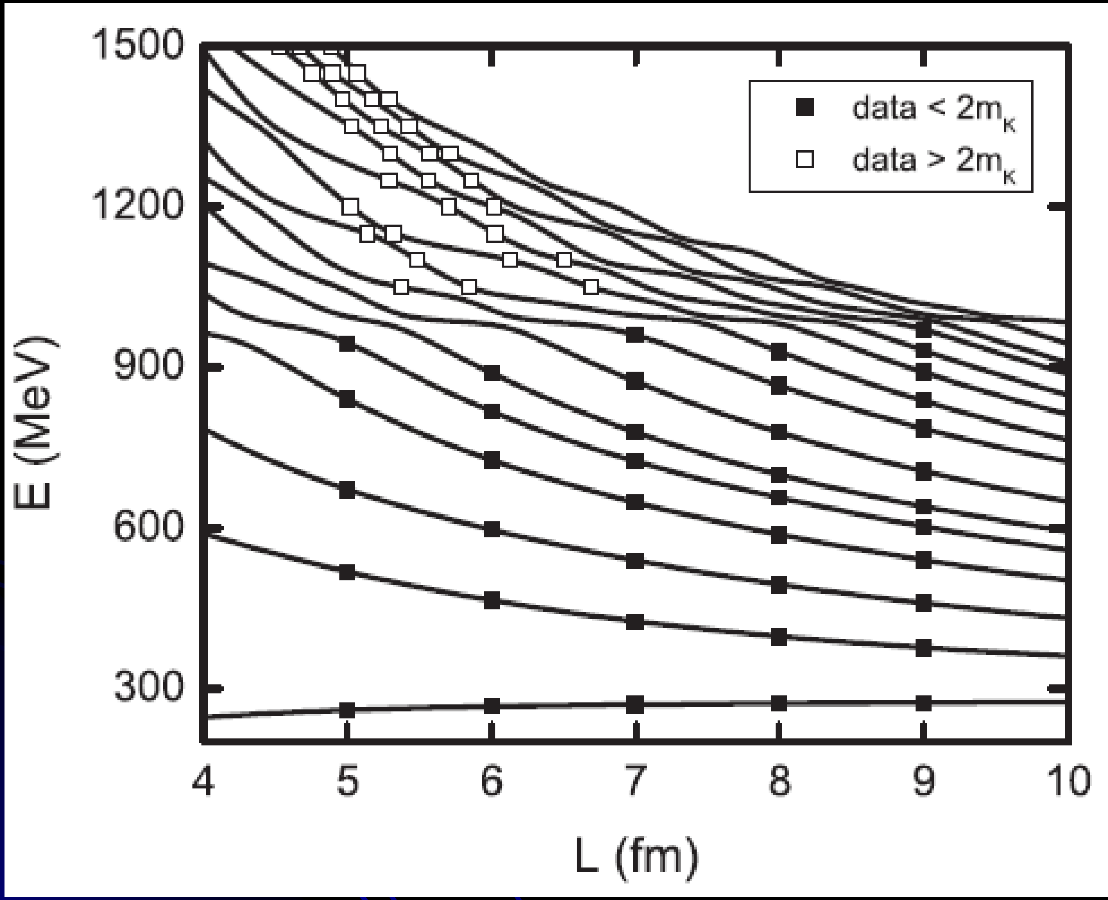
$$g_{\pi\pi}^{fin}(k_n) = \left(\frac{2\pi}{L}\right)^{\frac{3}{2}} g_{\pi\pi}(k_n)$$

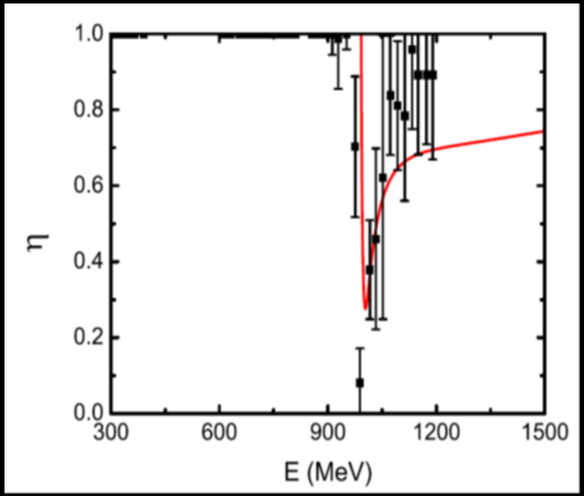
$$v_{\pi\pi,\pi\pi}^{fin}(k_n,k_m) = \left(\frac{2\pi}{L}\right)^{3} v_{\pi\pi,\pi\pi}(k_n,k_m)$$

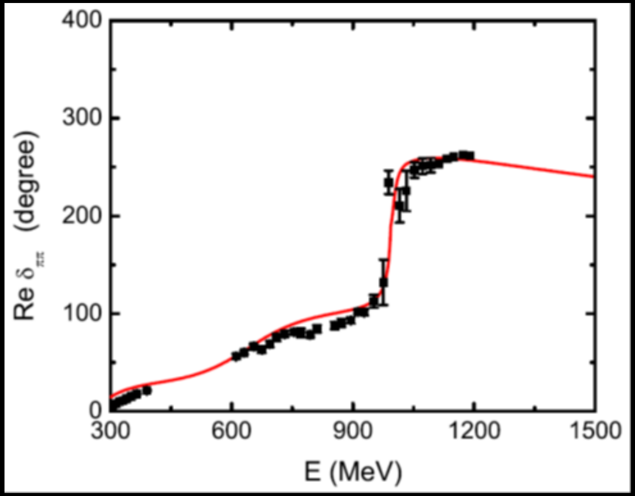
$$H_I = \begin{pmatrix} 0 & g_{\pi\pi}^{fin}(k_0) & g_{KK}^{fin}(k_0) & g_{\pi\pi}^{fin}(k_1) & g_{KK}^{fin}(k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_0) & v_{\pi\pi,KK}^{fin}(k_0,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_0,k_1) & v_{\pi\pi,KK}^{fin}(k_0,k_1) & \cdots \\ g_{KK}^{fin}(k_0) & v_{KK,\pi\pi}^{fin}(k_0,k_0) & v_{KK,KK}^{fin}(k_0,k_0) & v_{KK,\pi\pi}^{fin}(k_0,k_1) & v_{KK,KK}^{fin}(k_0,k_1) & \cdots \\ g_{\pi\pi}^{fin}(k_1) & v_{\pi\pi,\pi\pi}^{fin}(k_1,k_0) & v_{\pi\pi,KK}^{fin}(k_1,k_0) & v_{\pi\pi,\pi\pi}^{fin}(k_1,k_1) & v_{\pi\pi,KK}^{fin}(k_1,k_1) & \cdots \\ g_{KK}^{fin}(k_1) & v_{KK,\pi\pi}^{fin}(k_1,k_0) & v_{KK,KK}^{fin}(k_1,k_0) & v_{KK,\pi\pi}^{fin}(k_0,k_1) & v_{KK,KK}^{fin}(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Finite-box Hamiltonian method

Two channels:

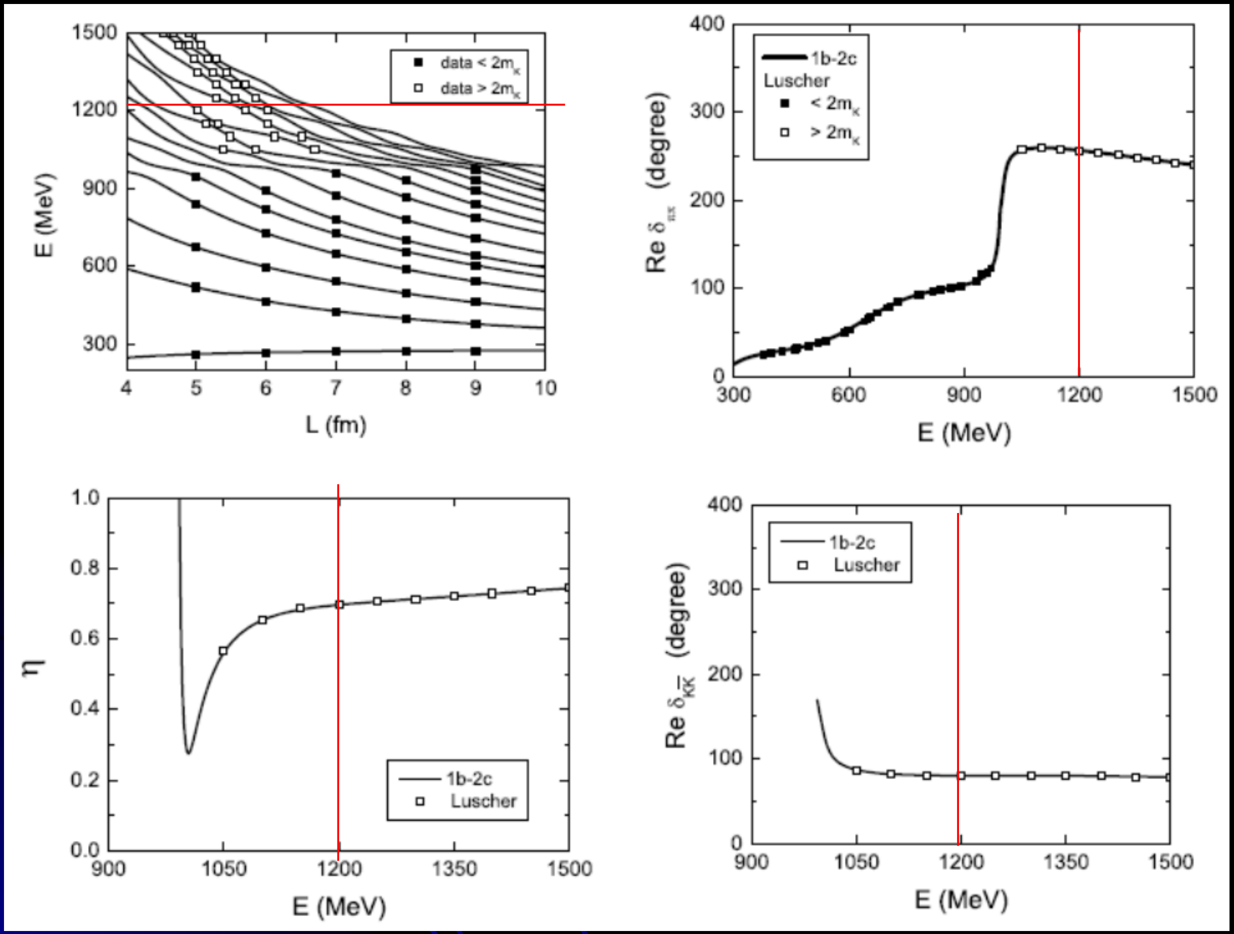






Finite-box Hamiltonian method

Two channels:



L_1, L_2, L_3 — E

5.022, 5.708, 6.014 —— 1200

fm fm fm MeV

T: 256.5° 80.18° 0.697

$\delta_{\pi\pi}(E)$, $\delta_{K\bar{K}}(E)$, $\eta(E)$

L: 256.6° 79.84° 0.698

$$\Delta_\alpha(L) = \tan^{-1}\left(\frac{q_\alpha \pi^{3/2}}{Z_{00}(1, q_\alpha^2)}\right)$$

$$0 = \cos\left(\Delta_{\pi\pi}(L) + \Delta_{K\bar{K}}(L) - \delta_{\pi\pi}(E) - \delta_{K\bar{K}}(E)\right)$$
$$-\eta\cos\left(\Delta_{\pi\pi}(L) - \Delta_{K\bar{K}}(L) - \delta_{\pi\pi}(E) + \delta_{K\bar{K}}(E)\right)$$

S. He, X. Feng, and C. Liu, J. High Energy Phys. 07 (2005) 011

Different Hamiltonian Models

A $$g_{i,\alpha}(k_\alpha) = \frac{\tilde{g}_{i,\alpha}}{\sqrt{\pi}} \frac{1}{(1+(c_\alpha k_\alpha)^2)}$$

$$v_{\alpha,\beta}(k_\alpha,k_\beta) = \frac{G_{\alpha,\beta}}{m_\pi^2} \frac{1}{(1+(d_\alpha k_\alpha)^2)^2} \frac{1}{(1+(d_\beta k_\beta)^2)^2}$$

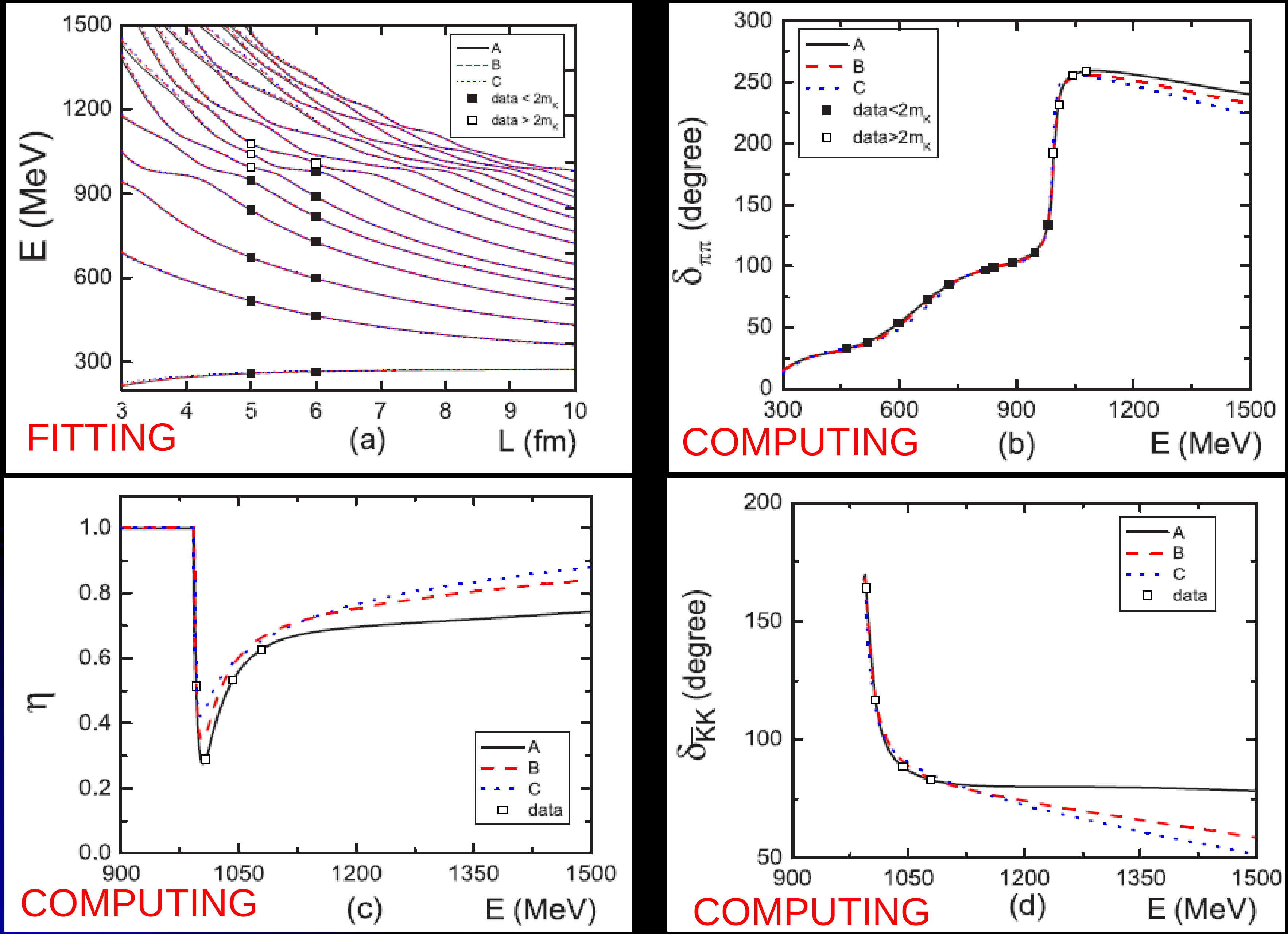
B $$g_{i,\alpha}(k_\alpha) = \frac{\tilde{g}_{i,\alpha}}{\sqrt{\pi}} \frac{1}{(1+(c_\alpha k_\alpha)^2)^2}$$

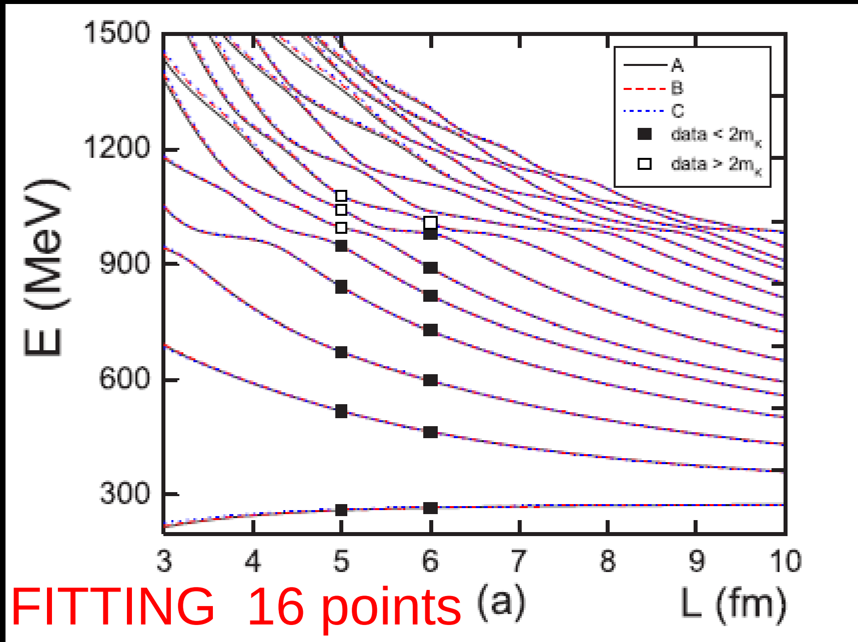
$$v_{\alpha,\beta}(k_\alpha,k_\beta) = \frac{G_{\alpha,\beta}}{m_\pi^2} \frac{1}{(1+(d_\alpha k_\alpha)^2)^4} \frac{1}{(1+(d_\beta k_\beta)^2)^4}$$

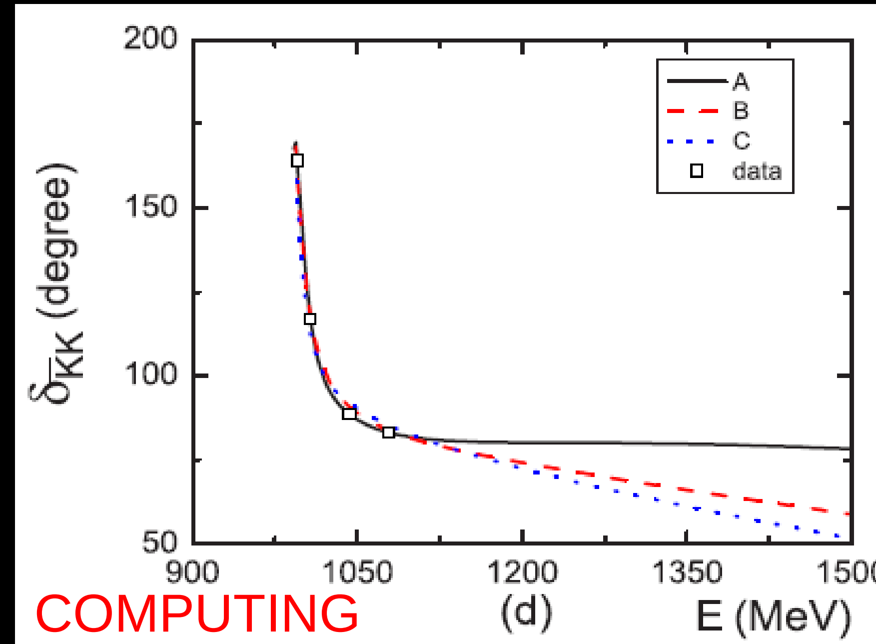
C $$g_{i,\alpha}(k_\alpha) = \frac{\tilde{g}_{i,\alpha}}{\sqrt{\pi}} e^{-(c_\alpha k_\alpha)^2}$$

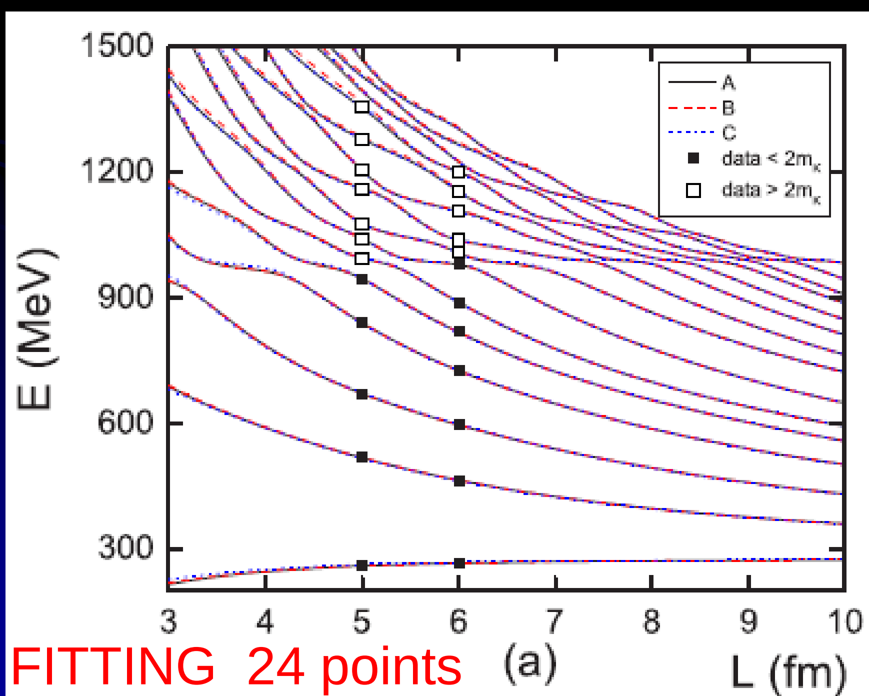
$$v_{\alpha,\beta}(k_\alpha,k_\beta) = \frac{G_{\alpha,\beta}}{m_\pi^2} e^{-(d_\alpha k_\alpha)^2} e^{-(d_\beta k_\beta)^2}$$

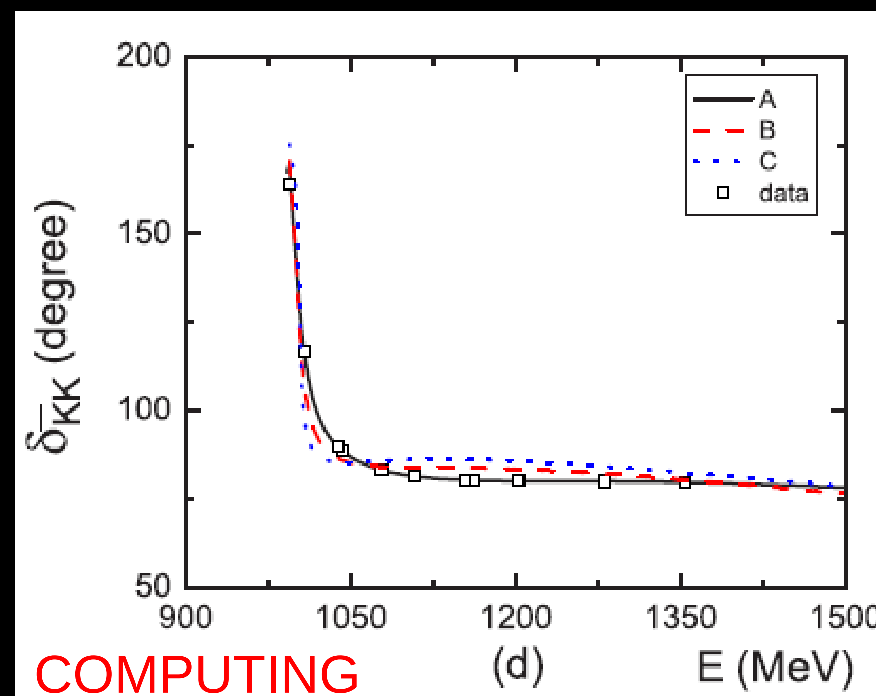
Two channels case:

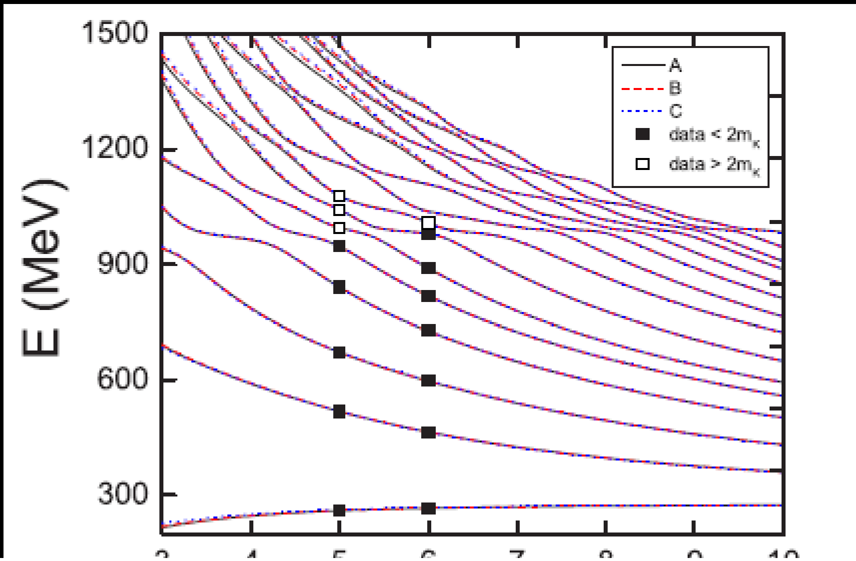


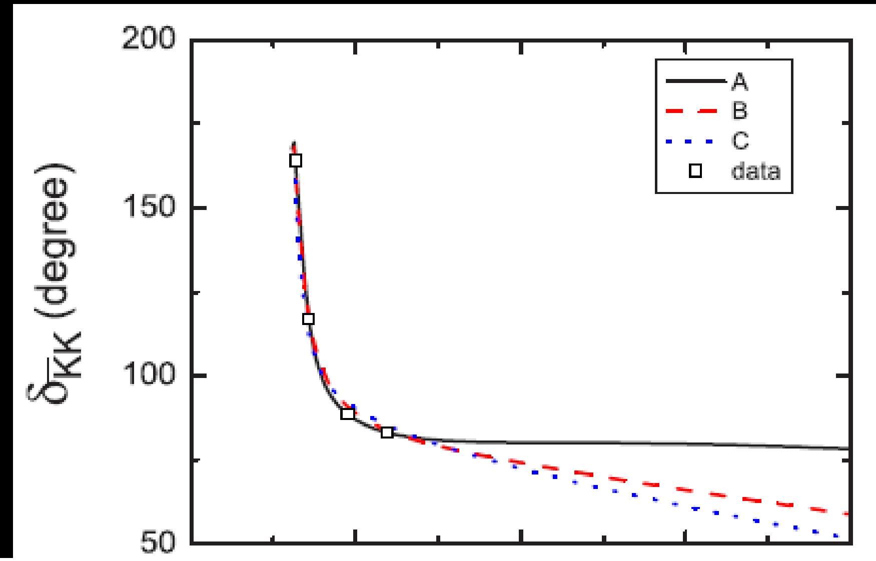

1500
1200
900
600
300
E (MeV)
3 4 5 6 7 8 9 10
L (fm)
A
B
C
data < 2m_K
data > 2m_K
FITTING 16 points (a)


200
150
100
50
δ_KK (degree)
900 1050 1200 1350 1500
E (MeV)
A
B
C
data
COMPUTING (d)

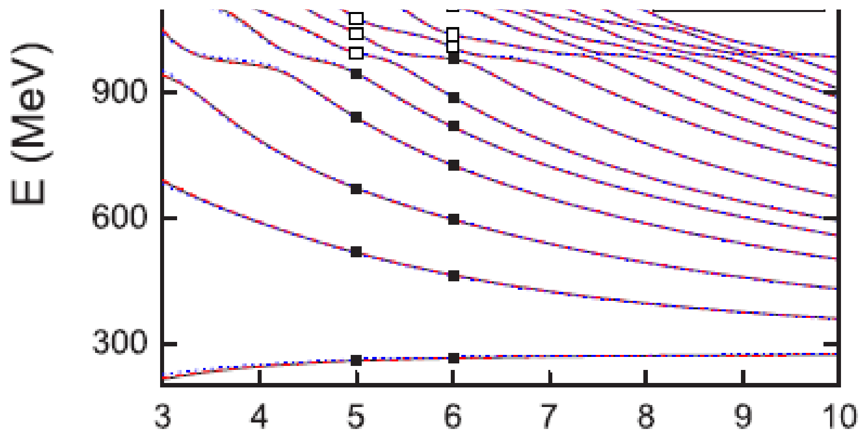

1500
1200
900
600
300
E (MeV)
3 4 5 6 7 8 9 10
L (fm)
A
B
C
data < 2m_K
data > 2m_K
FITTING 24 points (a)

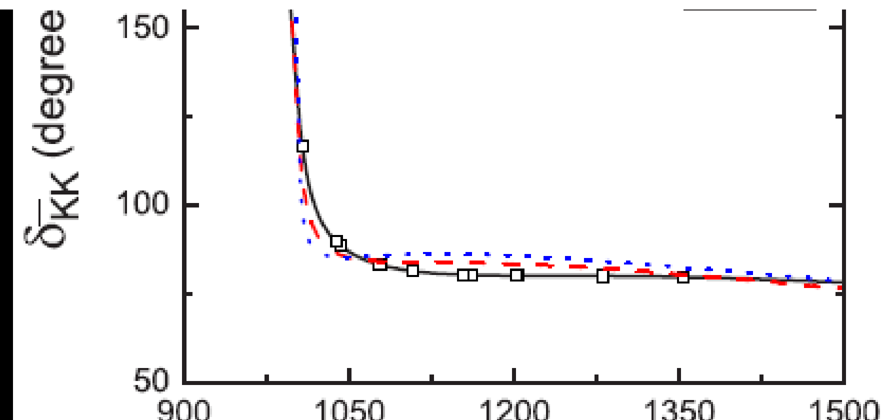

200
150
100
50
δ_KK (degree)
900 1050 1200 1350 1500
E (MeV)
A
B
C
data
COMPUTING (d)





By 16 or 24 points on the two different Lattice sizes (L=5, 6 fm), Luescher method can tell us NOTHING, but our approach can give a good description of observations. And it is also independent on the Hamiltonian model.





FITTING 24 points (a) L (fm)

COMPUTING (d) E (MeV)

Finite-box Hamiltonian method

1. S-wave CM 1-channel 2-channel
2. P-wave CM 1-channel
3. S-wave Boost 1-channel
4. P-wave Boost 1-channel

Finite-box Hamiltonian method

Continue

↓

Discrete

$\vec{P}_{tot} = 0$ (CM)

$\vec{P}_{tot} \neq 0$ (Boost)

$$\int d\vec{k} \quad \text{and} \quad \left|\alpha(\vec{k}_\alpha)\right\rangle \quad \text{and} \quad \left\langle \beta(\vec{k}_\beta) \middle| \alpha(\vec{k}_\alpha) \right\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$$

↓

$$\sum_i \left(2\pi/L\right)^3 \quad \text{and} \quad \left(2\pi/L\right)^{-3/2} \left|\vec{k}_i, -\vec{k}_i\right\rangle_\alpha \quad \text{and} \quad {}_\beta\left\langle \vec{k}_j, -\vec{k}_j \middle| \vec{k}_i, -\vec{k}_i \right\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

↓

$$\left(2\pi/L\right)^{-3/2} \left|\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\right\rangle_\alpha \quad \text{and} \quad {}_\beta\left\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j \middle| \vec{k}_i, \vec{P}_{tot} - \vec{k}_i \right\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

Finite-box Hamiltonian method

$\int d\vec{k}$ and $|\alpha(\vec{k}_\alpha)\rangle$ and $\langle\beta(\vec{k}_\beta)|\alpha(\vec{k}_\alpha)\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$

$\downarrow$ $\downarrow$ $\downarrow$

$\vec{P}_{tot} \neq 0$ (Boost)

$\sum_i \left(2\pi/L\right)^3$ and $\left(2\pi/L\right)^{-3/2} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha$ and ${}_\beta\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j | \vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$

$$H_0 = \sum_{i=1,n} |\sigma_i\rangle m_i \langle\sigma_i| + \sum_{\alpha,i} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \, E^*(\vec{P}_{tot}, \vec{k}_i) \; {}_\alpha\langle \vec{k}_i, \vec{P}_{tot} - \vec{k}_i|$$

$$E^*(\vec{P}_{tot}, \vec{k}) = \sqrt{\left(\sqrt{m_\alpha^2 + \vec{k}_i^{\,2}} + \sqrt{m_\alpha^2 + \left(\vec{P}_{tot} - \vec{k}_i\right)^2}\right)^2 - \vec{P}_{tot}^{\,2}}$$

Finite-box Hamiltonian method

$$\int d\vec{k} \quad \text{and} \quad \left|\alpha(\vec{k}_\alpha)\right\rangle \quad \text{and} \quad \left\langle \beta(\vec{k}_\beta) \middle| \alpha(\vec{k}_\alpha)\right\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$$

$\downarrow$

$\vec{P}_{tot} \neq 0$ (Boost)

$$\sum_i \left(2\pi/L\right)^3 \quad \text{and} \quad \left(2\pi/L\right)^{-3/2} \left|\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\right\rangle_\alpha \quad \text{and} \quad {}_\beta\left\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j \middle| \vec{k}_i, \vec{P}_{tot} - \vec{k}_i\right\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

$$H_I = \left(2\pi/L\right)^{3/2} \sum_j \sum_\alpha \sum_{i=1,n} \left[\left|\vec{k}_j, \vec{P}_{tot} - \vec{k}_j\right\rangle_\alpha g^+_{i,\alpha} \left\langle \sigma_i \right| + \left|\sigma_i\right\rangle g_{i,\alpha}\, {}_\alpha\left\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j \right| \right]$$

$$+ \left(2\pi/L\right)^3 \sum_{i,j} \sum_{\alpha,\beta} \left|\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\right\rangle_\alpha v_{\alpha,\beta}\, {}_\beta\left\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j \right|$$

$\downarrow$

$$\left|\vec{P}_{tot}, \vec{k}^*_i\right\rangle_\alpha$$

Finite-box Hamiltonian method

$$\int d\vec{k} \quad \text{and} \quad |\alpha(\vec{k}_\alpha)\rangle \quad \text{and} \quad \langle \beta(\vec{k}_\beta)|\alpha(\vec{k}_\alpha)\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$$

$\vec{P}_{tot} \neq 0$ (Boost)

$$\sum_i \left(2\pi/L\right)^3 \quad \text{and} \quad \left(2\pi/L\right)^{-3/2} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \quad \text{and} \quad {}_\beta\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j | \vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

$$H_I = \left(2\pi/L\right)^{3/2} \sum_j \sum_\alpha \sum_{i=1,n} \left[|\vec{k}_j, \vec{P}_{tot} - \vec{k}_j\rangle_\alpha \, g^+_{i,\alpha} \langle \sigma_i | + |\sigma_i\rangle g_{i,\alpha} \, {}_\alpha\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j | \right]$$

$$+ \left(2\pi/L\right)^3 \sum_{i,j} \sum_{\alpha,\beta} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \, v_{\alpha,\beta} \, {}_\beta\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j |$$

$$|\vec{P}_{tot}, \vec{k}_i^*\rangle_\alpha$$

$$\langle \vec{P}_{tot}, \vec{k}_i^* | \vec{P}_{tot}, \vec{k}_i^*\rangle d\vec{P}_{tot} d\vec{k}_i^*$$

$$= \langle \vec{k}_i, \vec{P}_{tot} - \vec{k}_i | \vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle d\vec{k}_i d(\vec{P}_{tot} - \vec{k}_i)$$

$$\sqrt{\frac{d\vec{P}_{tot} d\vec{k}_i^*}{d\vec{k}_i d(\vec{P}_{tot} - \vec{k}_i)}}$$

$${}_\beta\langle \vec{k}_j, \vec{P}_{tot} - \vec{k}_j | \vec{P}_{tot}, \vec{k}_i^*\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij} \sqrt{\frac{\varpi_\alpha(\vec{k}_j) + \varpi_\alpha(\vec{P}_{tot} - \vec{k}_j)}{\varpi_\alpha(\vec{k}_j)\varpi_\alpha(\vec{P}_{tot} - \vec{k}_j)} \frac{\varpi_\alpha(\vec{k}_i^*)}{2}}$$

$$\varpi_\alpha(\vec{k}_j) = \sqrt{m_\alpha^2 + \vec{k}_j^2}$$

Finite-box Hamiltonian method

$\vec{P}_{tot} \neq 0$ (Boost)

$$\int d\vec{k} \quad \text{and} \quad |\alpha(\vec{k}_\alpha)\rangle \quad \text{and} \quad \langle\beta(\vec{k}_\beta)|\alpha(\vec{k}_\alpha)\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$$

$$\downarrow$$

$$\sum_i \left(\tfrac{2\pi}{L}\right)^3 \quad \text{and} \quad \left(\tfrac{2\pi}{L}\right)^{-3/2} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \quad \text{and} \quad {}_\beta\langle\vec{k}_j, \vec{P}_{tot} - \vec{k}_j|\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

$$H_0 = \sum_{i=1,n} |\sigma_i\rangle m_i \langle\sigma_i| + \sum_{\alpha,i} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \, E^*(\vec{P}_{tot}, \vec{k}_i) \; {}_\alpha\langle\vec{k}_i, \vec{P}_{tot} - \vec{k}_i|$$

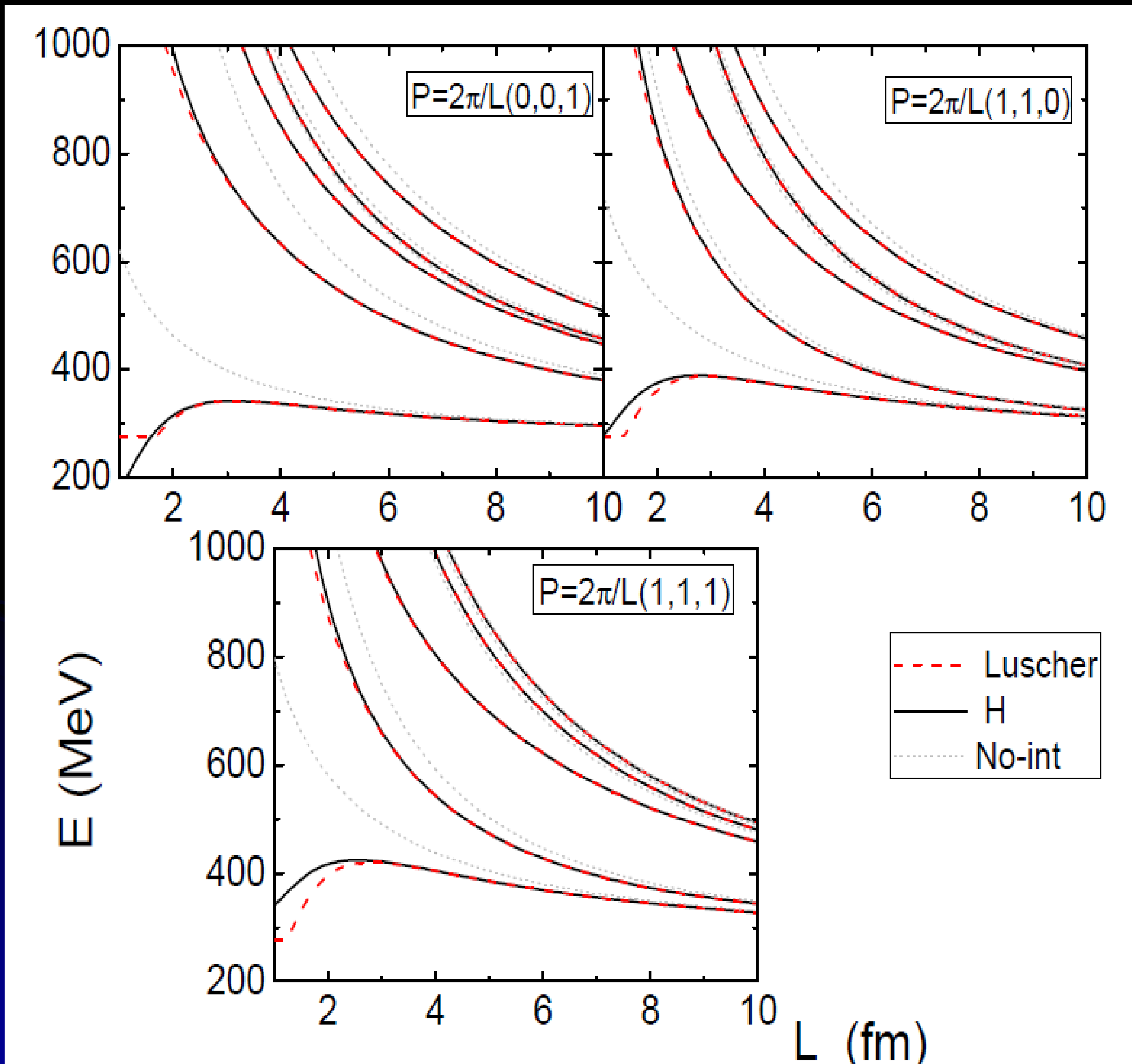
$$E^*(\vec{P}_{tot}, \vec{k}) = \sqrt{\left(\sqrt{m_\alpha^2 + \vec{k}_i^2} + \sqrt{m_\alpha^2 + (\vec{P}_{tot} - \vec{k}_i)^2}\right)^2 - \vec{P}_{tot}^{\,2}}$$

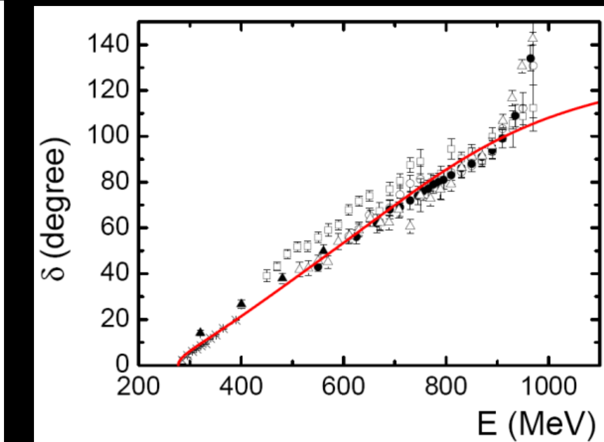
$$H_I = \left(\tfrac{2\pi}{L}\right)^{3/2} \sum_j \sum_\alpha \sum_{i=1,n} \left[C_\alpha(\vec{k}_j, \vec{P}_{tot}) |\vec{k}_j, \vec{P}_{tot} - \vec{k}_j\rangle_\alpha \, g^+_{i,\alpha} \langle\sigma_i| + |\sigma_i\rangle g_{i,\alpha} \; {}_\alpha\langle\vec{k}_j, \vec{P}_{tot} - \vec{k}_j| C_\alpha(\vec{k}_j, \vec{P}_{tot}) \right]$$

$$+ \left(\tfrac{2\pi}{L}\right)^3 \sum_{i,j} C_\alpha(\vec{k}_i, \vec{P}_{tot}) C_\beta(\vec{k}_j, \vec{P}_{tot}) \sum_{\alpha,\beta} |\vec{k}_i, \vec{P}_{tot} - \vec{k}_i\rangle_\alpha \, v_{\alpha,\beta} \; {}_\beta\langle\vec{k}_j, \vec{P}_{tot} - \vec{k}_j|$$

$$C_\alpha(\vec{k}_i, \vec{P}_{tot}) = \sqrt{\frac{\varpi_\alpha(\vec{k}_j) + \varpi_\alpha(\vec{P}_{tot} - \vec{k}_j)}{\varpi_\alpha(\vec{k}_j)\varpi_\alpha(\vec{P}_{tot} - \vec{k}_j)} \frac{\varpi_\alpha(\vec{k}_i^*)}{2}}$$

Finite-box Hamiltonian method





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$$\frac{\tilde{q}}{2}\cot(\delta_0) = \frac{1}{\sqrt{\pi} L\gamma} Z_{00}^{\vec{d}}\left(1; \left(\frac{L\tilde{q}}{2\pi}\right)^2\right)$$

Finite-box Hamiltonian method

1. S-wave CM 1-channel 2-channel
2. P-wave CM 1-channel
3. S-wave Boost 1-channel
4. P-wave Boost 1-channel

Finite-box Hamiltonian method

P wave

$\vec{\mathrm{P}}_{\mathrm{tot}} \neq 0$ (Boost)

$$\sum_{\mathrm{i}} \left(2\pi/\mathrm{L}\right)^3 \quad \text{and} \quad \left(2\pi/\mathrm{L}\right)^{-3/2} \left|\vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_{\alpha} \quad \text{and} \quad {}_{\beta}\left\langle \vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}} \middle| \vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_{\alpha} = \delta_{\alpha\beta}\delta_{\mathrm{ij}}$$

$$H_0 = |\rho\rangle m_\rho \langle\rho| + \sum_{\mathrm{i}} \left|\vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_{\pi\pi} \mathrm{E}^*(\vec{\mathrm{P}}_{\mathrm{tot}}, \vec{k}_{\mathrm{i}})\ {}_{\pi\pi}\left\langle \vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right|$$

$$H_I = \left(2\pi/\mathrm{L}\right)^{3/2} \sum_{\mathrm{j}} \left[C_{\pi\pi}(\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}}) \left|\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}}\right\rangle_{\pi\pi} g^+_{\rho\pi\pi} \langle\rho| + |\rho\rangle g_{\rho\pi\pi}\ {}_{\pi\pi}\left\langle \vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}}\right| C_{\pi\pi}(\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}}) \right]$$

J_ρ, $\vec{\mathrm{P}}_{\mathrm{tot}}$ $\quad |\rho, A_1\rangle$

$\delta\vec{\mathrm{P}}_{//} \neq 0 \qquad \vec{P}_{\pi_1} = (0,0,0) \qquad \vec{P}_{\pi_2} = (0,0,1)$

J_ρ, $\vec{\mathrm{P}}_{\mathrm{tot}}$ $\quad |\rho, E_2\rangle$

$\delta\vec{\mathrm{P}}_{\perp} \neq 0 \qquad \vec{P}_{\pi_1} = (0,0,0) \qquad \vec{P}_{\pi_2} = (0,0,1)$

(0,0,1)	A_1	1	$\|1,0>$
	E_2	1	$\frac{1-i}{2}\|1,-1> + \frac{-1-i}{2}\|1,+1>$
		2	$\frac{-i}{\sqrt{2}}\|1,-1> + \frac{1}{\sqrt{2}}\|1,+1>$

PRD86 094513, M. Gockeler, R. Horsley, M. Lage, U. -G. Meissner, P. E. L. Rakow, A. Rusetsky, G. Schierholz and J. M. Zanotti

Finite-box Hamiltonian method

P wave

$\vec{\mathrm{P}}_{\mathrm{tot}} \neq 0$ (Boost)

$$\sum_{\mathrm{i}} \left(2\pi/\mathrm{L}\right)^3 \quad \text{and} \quad \left(2\pi/\mathrm{L}\right)^{-3/2} \left|\vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_\alpha \quad \text{and} \quad {}_\beta\left\langle \vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}} \middle| \vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_\alpha = \delta_{\alpha\beta}\delta_{\mathrm{ij}}$$

$$H_0 = |\rho\rangle m_\rho \langle\rho| + \sum_{\mathrm{i}} \left|\vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right\rangle_{\pi\pi} \mathrm{E}^*(\vec{\mathrm{P}}_{\mathrm{tot}}, \vec{k}_{\mathrm{i}})\; {}_{\pi\pi}\left\langle \vec{k}_{\mathrm{i}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{i}}\right|$$

$$H_I = \left(2\pi/\mathrm{L}\right)^{3/2} \sum_{\mathrm{j}} \left[C_{\pi\pi}(\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}}) \left|\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}}\right\rangle_{\pi\pi} g^+_{\rho\pi\pi} \langle\rho| + |\rho\rangle g_{\rho\pi\pi}\; {}_{\pi\pi}\left\langle \vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}} - \vec{k}_{\mathrm{j}}\right| C_{\pi\pi}(\vec{k}_{\mathrm{j}}, \vec{\mathrm{P}}_{\mathrm{tot}}) \right]$$

J_ρ, $\vec{\mathrm{P}}_{\mathrm{tot}}$, $|\rho, A_1\rangle$

$\delta\vec{\mathrm{P}}_{//} \neq 0 \qquad \vec{P}_{\pi_1} = (0,0,0) \qquad \vec{P}_{\pi_2} = (0,0,1)$

J_ρ, $\vec{\mathrm{P}}_{\mathrm{tot}}$, $|\rho, E_2\rangle$

$\delta\vec{\mathrm{P}}_{\perp} \neq 0 \qquad \vec{P}_{\pi_1} = (0,0,0) \qquad \vec{P}_{\pi_2} = (0,0,1)$ (crossed out)

$|\rho, \Gamma, \Gamma_\alpha\rangle$

$$g_{\rho,\pi\pi}(k_{\pi\pi}) = \frac{\tilde{g}_{\rho,\pi\pi}}{m_\pi^{3/2}} \frac{\varepsilon_\mu k^\mu_{\pi\pi}}{\left(1 + (c_\alpha \vec{k}^*_{\pi\pi})^2\right)^{3/2}}$$

$$\varepsilon_\mu(\vec{\mathrm{P}}_{\mathrm{tot}}, |\Gamma, \Gamma_\alpha\rangle) k^\mu_{\pi\pi}$$

(0,0,1)	A_1	1	$\|1,0>$
	E_2	1	$\frac{1-i}{2}\|1,-1> + \frac{-1-i}{2}\|1,+1>$
		2	$\frac{-i}{\sqrt{2}}\|1,-1> + \frac{1}{\sqrt{2}}\|1,+1>$

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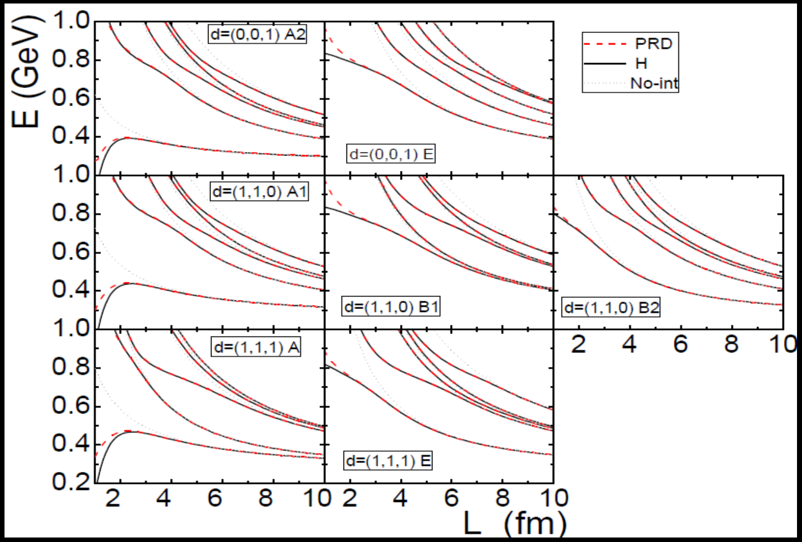
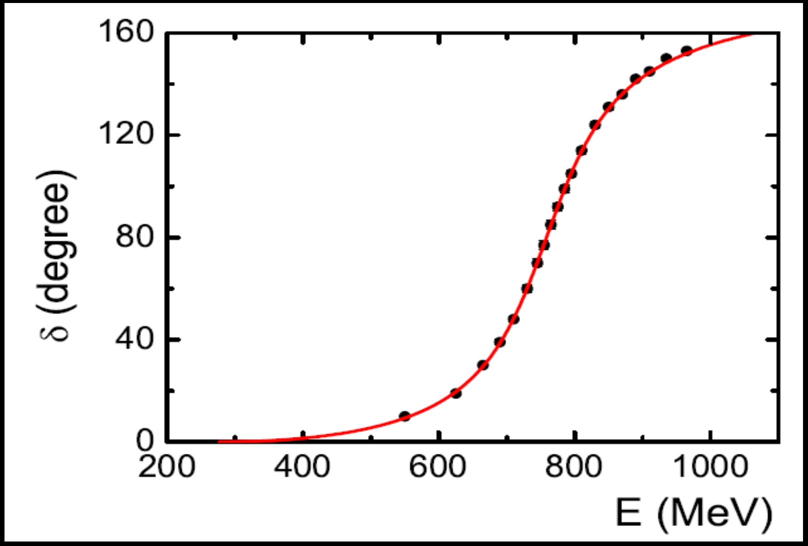
Finite-box Hamiltonian method





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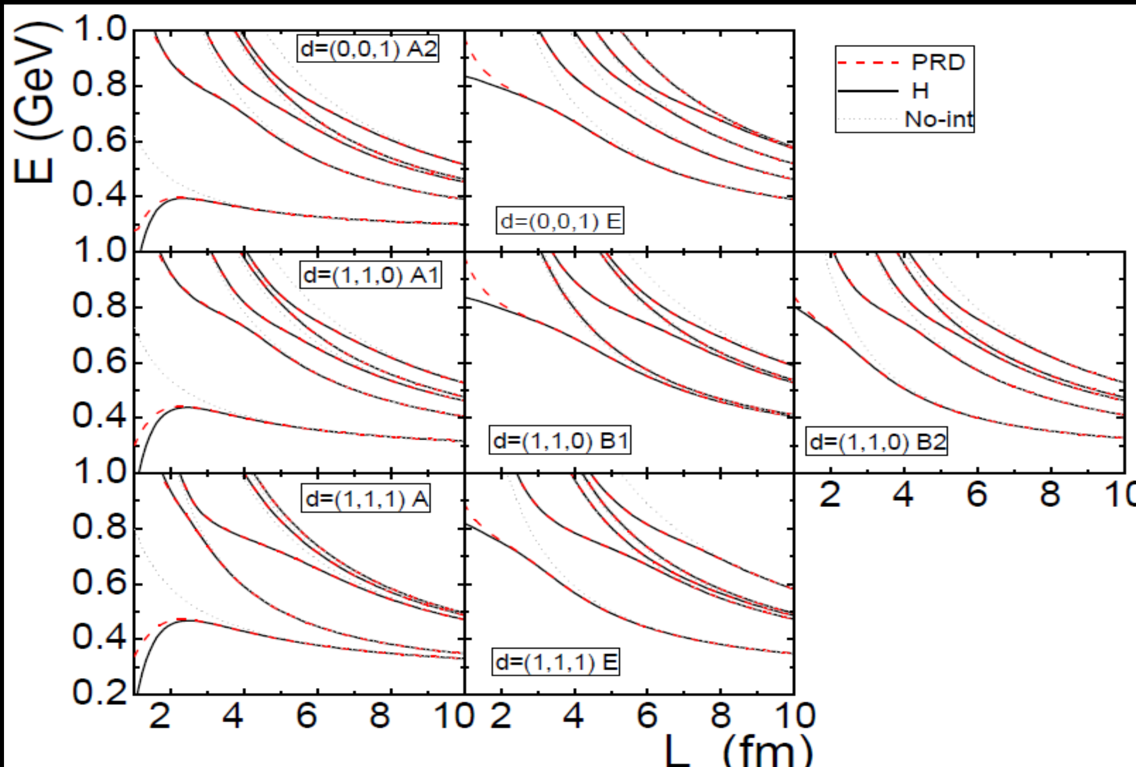
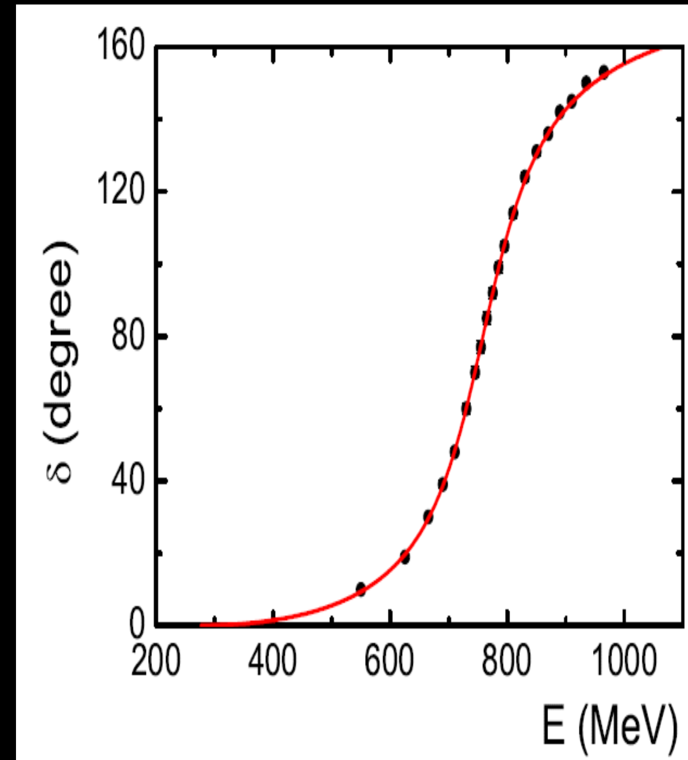




$$[00n]\,A_1:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[00n]}(q^2) + \frac{2}{\sqrt{5}}\frac{1}{q^2}Z_{2,0}^{[00n]}(q^2)\right],$$

$$[00n]\,E_2:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[00n]}(q^2) - \frac{1}{\sqrt{5}}\frac{1}{q^2}Z_{2,0}^{[00n]}(q^2)\right],$$

$$[0nn]\,A_1:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[0nn]}(q^2) + \frac{1}{2\sqrt{5}}\frac{1}{q^2}Z_{2,0}^{[0nn]}(q^2) + i\sqrt{\frac{6}{5}}\frac{1}{q^2}Z_{2,1}^{[0nn]}(q^2) - \sqrt{\frac{3}{10}}\frac{1}{q^2}Z_{2,2}^{[0nn]}(q^2)\right],$$

$$[0nn]\,B_1:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[0nn]}(q^2) + \frac{1}{2\sqrt{5}}\frac{1}{q^2}Z_{2,0}^{[0nn]}(q^2) - i\sqrt{\frac{6}{5}}\frac{1}{q^2}Z_{2,1}^{[0nn]}(q^2) - \sqrt{\frac{3}{10}}\frac{1}{q^2}Z_{2,2}^{[0nn]}(q^2)\right],$$

$$[0nn]\,B_2:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[0nn]}(q^2) - \frac{1}{\sqrt{5}}\frac{1}{q^2}Z_{2,0}^{[0nn]}(q^2) + \sqrt{\frac{6}{5}}\frac{1}{q^2}Z_{2,2}^{[0nn]}(q^2)\right],$$

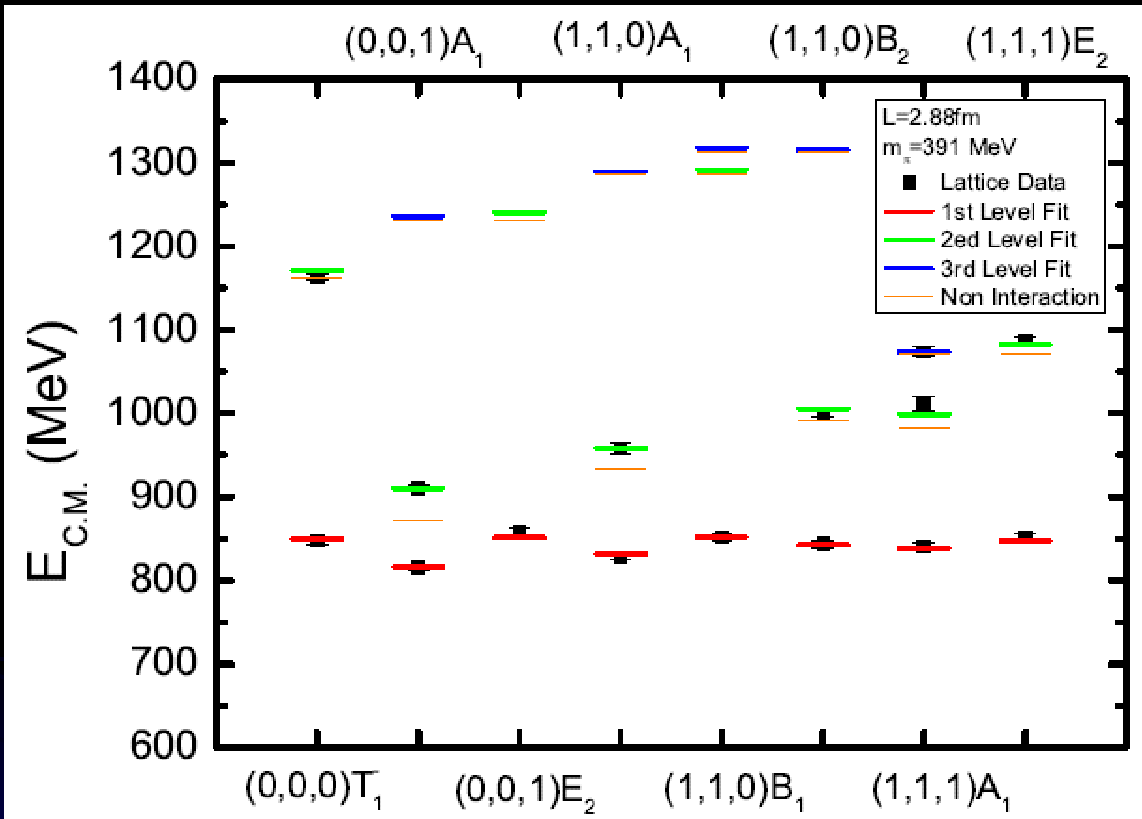
$$[nnn]\,A_1:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[nnn]}(q^2) - i\sqrt{\frac{8}{15}}\frac{1}{q^2}Z_{2,2}^{[nnn]}(q^2) - \sqrt{\frac{8}{15}}\frac{1}{q^2}\,{\rm Re}[Z_{2,1}^{[nnn]}(q^2)] - \sqrt{\frac{8}{15}}\frac{1}{q^2}\,{\rm Im}[Z_{2,1}^{[nnn]}(q^2)]\right],$$

$$[nnn]\,E_2:\ \cot\delta_1(E_{\rm cm}) = \frac{1}{\gamma\pi^{3/2}q}\left[Z_{0,0}^{[nnn]}(q^2) + i\sqrt{\frac{6}{5}}\frac{1}{q^2}Z_{2,2}^{[nnn]}(q^2)\right].$$

	experimental data
m_π (MeV)	138.5 (fixed)
m_ρ (MeV)	852.50±0.04
$g_{\rho\pi\pi}$	0.09563±0.00002
$c_{\rho\pi\pi}$ (fm)	0.48477±0.00006
Pole position (Z)	
$Re[Z] = m_{Pole}$ (MeV)	758.80
$-Im[Z] = \Gamma/2$	79.87

J. J. Dudek *et al.* [Hadron Spectrum Collaboration], **PRD 87, 034505 (2013)**

Finite-box Hamiltonian method



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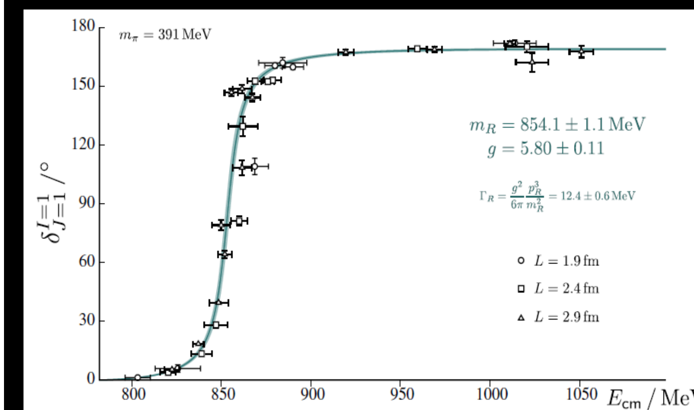
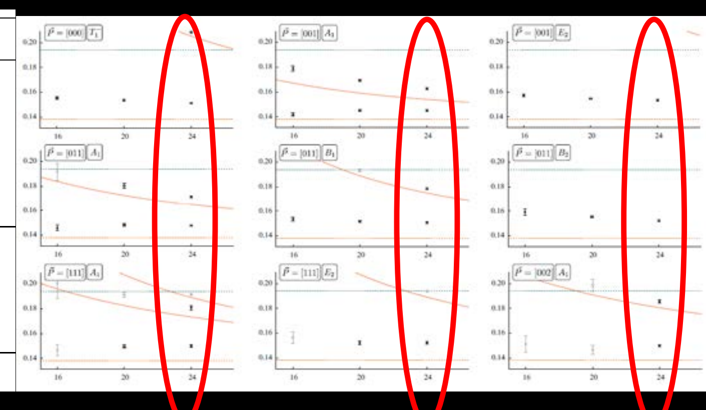
Breit-Wigner
M = 854.1 MeV
Γ = 12.1 MeV



	experimental data	Lattice data(PRD)
m_π (MeV)	138.5 (fixed)	391.0 (fixed)
m_ρ (MeV)	852.50±0.04	869.17± 1.54
$g_{\rho\pi\pi}$	0.09563±0.00002	0.04556± 0.00112
$c_{\rho\pi\pi}$ (fm)	0.48477±0.00006	0.48477(fixed)
Pole position (Z)		
$Re[Z] = m_{Pole}$ (MeV)	758.80	840.54
$-Im[Z] = \Gamma/2$	79.87	5.01



Outline

- Introduction
- Hamiltonian for $\pi\pi$ scattering
- Finite-box Hamiltonian method
- Summary

Summary

- This method has been developed to

1. S-wave CM 1-channel 2-channel
2. P-wave CM 1-channel
3. S-wave Boost 1-channel
4. P-wave Boost 1-channel

- Fitting approach can extract resonance information directly from the Lattice data. The predictions of scattering observables satisfy:

  1) It is independent of the form of the Hamiltonian.

  2) it is valid for multi-channel case.

  3) It is valid in the energy region where the spectrum data are fitted.

Thank you very much

$$\Gamma = \frac{g_{dressed}^{2}\, p_R^3}{6\pi m_R^2}$$

$$\Gamma = -2\,\mathrm{Im}[\Sigma] = \frac{4\pi^2}{3}\frac{m_R}{m_\pi^3}\, g_{bare}^{2}\, p_R^3 \frac{1}{\left(1+\left(c_{\pi\pi}\, p_R\right)^2\right)^3}$$

Outline

- Introduction
- Hamiltonian for $\pi\pi$ scattering
- Finite-box Hamiltonian method
- Compare to the other methods
- Summary and Outlook

Compare to the other methods

Beth-Salpeter Equation

$$T(q,q',P)=V(q,q',P)+i\int\frac{d^4k}{(2\pi)^4}V(q,k,P)G(k,P)T(k,q',P)$$

$$G^{-1}(k,P)=(k^2-m^2+i\varepsilon)((P-k)^2-m^2+i\varepsilon)$$

$$V(q,q',P)=4\pi\sum_{l,m}Y_{lm}(\Omega_{\vec{q}^*})Y^*_{lm}(\Omega_{\vec{q}'^*})V_l(q_0^*,|\vec{q}\,|,q_0'^*,|\vec{q}'\,|,P^*)$$

Compare to the other methods

Beth-Salpeter Equation

$$T(q,q',P) = V(q,q',P) + i\int \frac{d^4k}{(2\pi)^4} V(q,k,P)G(k,P)T(k,q',P)$$

$$G^{-1}(k,P) = (k^2 - m^2 + i\varepsilon)((P-k)^2 - m^2 + i\varepsilon)$$

$$V(q,q',P) = 4\pi \sum_{l,m} Y_{lm}(\Omega_{\vec{q}^*}) Y^*_{lm}(\Omega_{\vec{q}'^*}) f_l(q_0^*, |\vec{q}|, P^*) f_l(q_0'^*, |\vec{q}'|, P^*)$$

Infinite Volume

$$T(q,q',P) = 4\pi \sum_{l,m} Y_{lm}(\Omega_{\vec{q}^*}) Y^*_{lm}(\Omega_{\vec{q}'^*}) T_l(q_0^*, |\vec{q}|, q_0'^*, |\vec{q}'|, P^*)$$

Finite Volume

$$T^L(q,q',P) = 4\pi \sum_{l_1,m_1} \sum_{l_2,m_2} Y_{l_1 m_1}(\Omega_{\vec{q}^*}) Y^*_{l_2 m_2}(\Omega_{\vec{q}'^*}) T^L_{l_1 m_1, l_2 m_2}(q_0^*, |\vec{q}|, q_0'^*, |\vec{q}'|, P^*)$$

Compare to the other methods

Beth-Salpeter Equation

$$T(q,q',P) = V(q,q',P) + i\int \frac{d^4k}{(2\pi)^4} V(q,k,P)G(k,P)T(k,q',P)$$

$$G^{-1}(k,P) = (k^2 - m^2 + i\varepsilon)((P-k)^2 - m^2 + i\varepsilon)$$

$$V(q,q',P) = 4\pi \sum_{l,m} Y_{lm}(\Omega_{\vec{q}^*}) Y^*_{lm}(\Omega_{\vec{q}'^*})\, f_l(q_0^*, |\vec{q}^*|, P^*)\, f_l(q_0'^*, |\vec{q}'^*|, P^*)$$

Infinite Volume $$T(q,q',P) = 4\pi \sum_{l,m} Y_{lm}(\Omega_{\vec{q}^*}) Y^*_{lm}(\Omega_{\vec{q}'^*}) T_l(q_0^*, |\vec{q}|, q_0'^*, |\vec{q}'|, P^*)$$

Finite Volume $$T^L(q,q',P) = 4\pi \sum_{l_1,m_1} \sum_{l_2,m_2} Y_{l_1 m_1}(\Omega_{\vec{q}^*}) Y^*_{l_2 m_2}(\Omega_{\vec{q}'^*}) T^L_{l_1 m_1, l_2 m_2}(q_0^*, |\vec{q}|, q_0'^*, |\vec{q}'|, P^*)$$

Infinite Volume $$\overline{t} = \overline{v} + i\overline{v}\overline{M}\overline{t}$$

Finite Volume $$\overline{t}^L = \overline{v} + i\overline{v}\overline{M}^L\overline{t}^L$$

$$\overline{t}_{l_1 m_1, l_2 m_2} = \delta_{l_1 l_2}\delta_{m_1 m_2} \frac{8\pi P_0^*}{\tilde{q}\tilde{f}_l^{\,2}(i - \cot\delta_l)}$$

Compare to the other methods

Beth-Salpeter Equation

$$T(q,q',P)=V(q,q',P)+i\int\frac{d^4k}{(2\pi)^4}V(q,k,P)G(k,P)T(k,q',P)$$

$$G^{-1}(k,P)=(k^2-m^2+i\varepsilon)((P-k)^2-m^2+i\varepsilon)$$

$$V(q,q',P)=4\pi\sum_{l,m}Y_{lm}(\Omega_{\vec{q}^*})Y^*_{lm}(\Omega_{\vec{q}'^*})\,f_l(q_0^*,|\vec{q}^*|,P^*)\,f_l(q_0'^*,|\vec{q}'^*|,P^*)$$

Infinite Volume $\quad \bar{t}=\bar{v}+i\bar{v}\bar{M}\bar{t}$

$$\bar{t}_{l_1m_1,l_2m_2}=\delta_{l_1l_2}\delta_{m_1m_2}\frac{8\pi P_0^*}{\tilde{q}\tilde{f}_l^{\,2}(i-\cot\delta_l)}$$

Finite Volume $\quad \bar{t}^L=\bar{v}+i\bar{v}\bar{M}^L\bar{t}^L$

$$\bar{t}^L=\bar{t}(1-i\bar{\Delta}\bar{t})^{-1}$$

$$\bar{\Delta}_{l_1m_1,l_2m_2}=\left(\bar{M}^L-\bar{M}\right)_{l_1m_1,l_2m_2}=\frac{1}{L^3}\sum_{\vec{k}=2\pi\vec{n}/L}\int\frac{dk_0}{8\pi^2}G(k,P)F_{l_1m_1,l_2m_2}-\int\frac{d\vec{k}}{(2\pi)^3}\int\frac{dk_0}{8\pi^2}G(k,P)F_{l_1m_1,l_2m_2}$$

$$=\left[-i\frac{1}{L^3}\sum_{\vec{k}=2\pi\vec{n}/L}+ip\int\frac{d\vec{k}}{(2\pi)^3}\right]\left[\frac{F_{l_1m_1,l_2m_2}(\varpi(\vec{k}),\vec{k},P)}{2\varpi(\vec{k})\left(\left(P_0-\varpi(\vec{k})\right)^2-\varpi^2(\vec{P}-\vec{k})\right)}+\frac{F_{l_1m_1,l_2m_2}(P_0+\varpi(\vec{P}-\vec{k}),\vec{k},P)}{2\varpi(\vec{P}-\vec{k})\left(\left(P_0+\varpi(\vec{P}-\vec{k})\right)^2-\varpi^2(\vec{k})\right)}\right]$$

$$+\delta_{l_1l_2}\delta_{m_1m_2}\frac{\tilde{q}\tilde{f}_{l_1}^{\,2}}{8\pi P_0^*}$$

$$F_{l_1m_1,l_2m_2}(k_0,\vec{k},P)=4\pi Y_{l_1m_1}(\Omega_{\vec{k}^*})Y^*_{l_2m_2}(\Omega_{\vec{k}^*})\,f_{l_1}(k_0^*,|\vec{k}^*|,P^*)\,f_{l_2}(k_0^*,|\vec{k}^*|,P^*)$$

Compare to the other methods

S wave

$$t^L_{00,00} = t_0(1 - i\Delta_{00,00}t_0)^{-1} = 0$$

$$\frac{\tilde{q}\tilde{f}_0^{\,2}}{8\pi P_0^*}\cot(-\delta_0) = \left[\frac{1}{L^3}\sum_{\vec{k}=2\pi\vec{n}/L} - p\int\frac{d\vec{k}}{(2\pi)^3}\right]\left[\frac{F_{00,00}(\varpi(\vec{k}),\vec{k},P)}{2\varpi(\vec{k})\left(\left(P_0-\varpi(\vec{k})\right)^2-\varpi^2(\vec{P}-\vec{k})\right)} + \frac{F_{00,00}(P_0+\varpi(\vec{P}-\vec{k}),\vec{k},P)}{2\varpi(\vec{P}-\vec{k})\left(\left(P_0+\varpi(\vec{P}-\vec{k})\right)^2-\varpi^2(\vec{k})\right)}\right]$$

Boost Method

$$\vec{k}^* = \hat{A}\left(\vec{k} - B\vec{P}\right) = A\left(\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2} - B\right)\vec{P} + \vec{k} - \frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P} \qquad d\vec{k}^* = Jd\vec{k}$$

Model Independent

$$\left[J\frac{\tilde{f}_0^{\,2}/(2P_0^*)}{\tilde{q}^2-\vec{k}^{*}} + \left(\frac{F_{00,00}(\varpi(\vec{k}),\vec{k},P)}{2\varpi(\vec{k})\left(\left(P_0-\varpi(\vec{k})\right)^2-\varpi^2(\vec{P}-\vec{k})\right)} - J\frac{\tilde{f}_0^{\,2}/(2P_0^*)}{\tilde{q}^2-\vec{k}^{*}} + \frac{F_{00,00}(P_0+\varpi(\vec{P}-\vec{k}),\vec{k},P)}{2\varpi(\vec{P}-\vec{k})\left(\left(P_0+\varpi(\vec{P}-\vec{k})\right)^2-\varpi^2(\vec{k})\right)}\right)\right]$$

Non-Singularity

$$\cot(-\delta_0) = \frac{4\pi}{\tilde{q}}\Delta = \frac{4\pi}{\tilde{q}L}\left[\sum_{\vec{k}=2\pi\vec{n}/L} J\frac{1}{\tilde{q}^2-\vec{k}^{*\,2}} - p\int\frac{d\vec{k}^*}{(2\pi)^3}\frac{1}{\tilde{q}^2-\vec{k}^{*\,2}}\right]$$

Compare to the other methods

S wave $$t^{L}_{00,00}=t_0(1-i\Delta_{00,00}t_0)^{-1}=0$$

$$\frac{\tilde{q}\tilde{f}_0^{\,2}}{8\pi P_0^*}\cot(-\delta_0)=\left[\frac{1}{L^3}\sum_{\vec{k}=2\pi\vec{n}/L}-\mathcal{P}\int\frac{d\vec{k}}{(2\pi)^3}\right]\left[\frac{F_{00,00}(\varpi(\vec{k}),\vec{k},P)}{2\varpi(\vec{k})\left(\left(P_0-\varpi(\vec{k})\right)^2-\varpi^2(\vec{P}-\vec{k})\right)}+\frac{F_{00,00}(P_0+\varpi(\vec{P}-\vec{k}),\vec{k},P)}{2\varpi(\vec{P}-\vec{k})\left(\left(P_0+\varpi(\vec{P}-\vec{k})\right)^2-\varpi^2(\vec{k})\right)}\right]$$

Boost Method

$$\vec{k}^*=\hat{A}\left(\vec{k}-B\vec{P}\right)=A\left(\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}-B\right)\vec{P}+\vec{k}-\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P}\qquad d\vec{k}^*=Jd\vec{k}$$

Model Independent

$$\left[J\frac{\tilde{f}_0^{\,2}/(2P_0^*)}{\tilde{q}^2-\vec{k}^*}+\left(\frac{F_{00,00}(\varpi(\vec{k}),\vec{k},P)}{2\varpi(\vec{k})\left(\left(P_0-\varpi(\vec{k})\right)^2-\varpi^2(\vec{P}-\vec{k})\right)}-J\frac{\tilde{f}_0^{\,2}/(2P_0^*)}{\tilde{q}^2-\vec{k}^*}+\frac{F_{00,00}(P_0+\varpi(\vec{P}-\vec{k}),\vec{k},P)}{2\varpi(\vec{P}-\vec{k})\left(\left(P_0+\varpi(\vec{P}-\vec{k})\right)^2-\varpi^2(\vec{k})\right)}\right)\right]$$

Non-Singularity

$$\cot(-\delta_0)=\frac{4\pi}{\tilde{q}}\Delta=\frac{4\pi}{\tilde{q}L}\left[\sum_{\vec{k}=2\pi\vec{n}/L}J\frac{1}{\tilde{q}^2-\vec{k}^{*\,2}}-\mathcal{P}\int\frac{d\vec{k}^*}{(2\pi)^3}\frac{1}{\tilde{q}^2-\vec{k}^{*\,2}}\right]$$

Model Dependent

$$\left[\frac{1}{P_0^{*2}}\frac{\varpi(\vec{k}^*)}{2}\frac{\varpi(\vec{k})+\varpi(\vec{P}-\vec{k})}{\varpi(\vec{k})\varpi(\vec{P}-\vec{k})}\frac{f_0^{\,2}(\varpi(\vec{k}_C^*),\vec{k}_C^*,P)}{P_0^*-2\varpi(\vec{k}_C^*)}+(\ldots\ldots)\right]$$

Compare to the other methods

$$\cot(-\delta_0) = \frac{4\pi}{\tilde{q}}\Delta = \frac{4\pi}{\tilde{q}L}\left[\sum_{\vec{k}=2\pi\vec{n}/L} J\,\frac{1}{\tilde{q}^2 - \vec{k}^{*\,2}} - \mathcal{P}\int \frac{d\vec{k}^{\,*}}{(2\pi)^3}\frac{1}{\tilde{q}^2 - \vec{k}^{*\,2}}\right]$$

$$\vec{k}^{\,*} = A\left(\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2} - B\right)\vec{P} + \vec{k} - \frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P}$$

$$d\vec{k}^{\,*} = J d\vec{k}$$

A boost method

$$A = \gamma_A = P_0^*/P_0 \qquad B = \varpi(\vec{k}_A^*)/P_0^* \qquad J = \varpi(\vec{k}_A^*)/\varpi(\vec{k})$$

$$\cot(-\delta_0) = \frac{4\pi}{\tilde{q}L^3}\left[\sum_{\vec{k}=2\pi\vec{n}/L} \frac{\varpi(\vec{k}_A^*)}{\varpi(\vec{k})}\frac{e^{\alpha(\tilde{q}^2-\vec{k}^{*\,2})}}{\tilde{q}^2 - \vec{k}^{*\,2}} - \mathcal{P}\int \frac{d\vec{k}^{\,*}}{(2\pi)^3}\frac{e^{\alpha(\tilde{q}^2-\vec{k}^{*\,2})}}{\tilde{q}^2 - \vec{k}^{*\,2}}\right] - \sum_{\vec{n}\neq 0}\int_0^{\alpha(2\pi/L)^2} dt\,\frac{F_A(t)}{\pi\tilde{q}L}$$

$$F_A(t) = e^{t\left(L\tilde{q}/2\pi\right)^2}\int dq\; 2qe^{-tq^2}\cos\left[\frac{2\pi\sqrt{(mL/2\pi)^2+q^2}}{P_0^*}\vec{n}\cdot\vec{P}\right]\frac{\sin\left[2\pi q\sqrt{\vec{n}^2+\left(\frac{\vec{n}\cdot\vec{P}}{P_0^*}\right)^2}\right]}{\sqrt{\vec{n}^2+\left(\frac{\vec{n}\cdot\vec{P}}{P_0^*}\right)^2}}$$

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. Sharpe, NPB **727**, 218 (2005)

$$\tan[\delta(q^*)] = -\tan[\phi^P(q^*)], \qquad (1)$$

where $\delta(q^*)$ is the physical s-wave phase-shift and the function $\phi^P(q^*)$ is defined by

$$\tan[\phi^P(q^*)] = \frac{q^*}{4\pi}\left[c^P(q^{*2})\right]^{-1}, \qquad (2)$$

with the box size entering through the following regularized sum

$$c^P(q^{*2}) \equiv \frac{1}{L^3}\sum_{\vec{k}}\frac{\omega_k^*}{\omega_k}\frac{e^{\alpha(q^{*2}-k^{*2})}}{q^{*2}-k^{*2}} - \mathcal{P}\int\frac{d^3k^*}{(2\pi)^3}\frac{e^{\alpha(q^{*2}-\vec{k}^{*2})}}{q^{*2}-\vec{k}^{*2}}. \qquad (3)$$

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ials multiplied by powers of L. This
. in physical units, the result for $\mathcal{Z}_{lm}$
ıe should choose α sufficiently small
that the α-dependent terms are negligible. The simplest choice is to send $\alpha \to 0^+$.

Compare to the other methods

$$\cot(-\delta_0) = \frac{4\pi}{\tilde{q}}\Delta = \frac{4\pi}{\tilde{q}L}\left[\sum_{\vec{k}=2\pi\vec{n}/L} J\frac{1}{\tilde{q}^2 - \vec{k}^{*\,2}} - \mathcal{P}\int \frac{d\vec{k}^*}{(2\pi)^3}\frac{1}{\tilde{q}^2 - \vec{k}^{*\,2}}\right]$$

$$\vec{k}^* = A\left(\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2} - B\right)\vec{P} + \vec{k} - \frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P}$$

$$d\vec{k}^* = Jd\vec{k}$$

B boost method

$$A = 1/\gamma_B = P_0^*/P_0 \qquad B = 1/2 \qquad J = P_0^*/P_0$$

$$\cot(-\delta_0) = -\frac{4\pi}{\tilde{q}}\frac{1}{2\sqrt{\pi}^3 L\gamma_B} Z_{00}^{\vec{d}}\left(1;\left(\frac{L\tilde{q}}{2\pi}\right)^2\right) \Longrightarrow \tan(\delta_0) = -\frac{L\tilde{q}}{2\pi}\frac{2\sqrt{\pi}^3\gamma_B}{Z_{00}^{\vec{d}}\left(1;\left(\frac{L\tilde{q}}{2\pi}\right)^2\right)}$$

$$\delta_0(p^*) = -\phi^d(q) \bmod \pi, \qquad q = \frac{p^* L}{2\pi}, \tag{17}$$

where ϕ^d is a continuous function defined by the equation

$$\tan(-\phi^d(q)) = \frac{\gamma q \pi^{3/2}}{Z_{00}^d(1;q^2)} \qquad \phi^d(0) = 0. \tag{18}$$

Function Z_{00}^d is generalized zeta function, and is formally given by

$$Z_{00}^d(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{r\in P_d}(r^2 - q^2)^{-s}, \tag{19}$$

where the set P_d is

$$P_d = \{\boldsymbol{r}\in\mathbb{R}^3 | \boldsymbol{r} = \gamma^{-1}(\boldsymbol{n} + \boldsymbol{d}/2),\ \boldsymbol{n}\in\mathbb{Z}^3\}. \tag{20}$$

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Compare to the other methods

$$\cot(-\delta_0)=\frac{4\pi}{\tilde q}\Delta=\frac{4\pi}{\tilde q L}\left[\sum_{\vec k=2\pi\vec n/L} J\frac{1}{\tilde q^2-\vec k^{*\,2}} - P\int\frac{d\vec k^{*}}{(2\pi)^3}\frac{1}{\tilde q^2-\vec k^{*\,2}}\right]$$

$$\vec k^{*}=A\left(\frac{\vec k\cdot\vec P}{|\vec P|^2}-B\right)\vec P+\vec k-\frac{\vec k\cdot\vec P}{|\vec P|^2}\vec P$$

$$d\vec k^{*}=Jd\vec k$$

C boost method

$$A=\frac{\sqrt{(\varpi(\vec k)+\varpi(\vec P-\vec k))^2-\vec P^2}}{\varpi(\vec k)+\varpi(\vec P-\vec k)}\qquad B=1/2\qquad J=\frac{\varpi(\vec k^{*})}{2}\frac{\varpi(\vec k)+\varpi(\vec P-\vec k)}{\varpi(\vec k)\varpi(\vec P-\vec k)}$$

$$\cot(-\delta_0)=\frac{4\pi}{\tilde q L^3}\left[\sum_{\vec k=2\pi\vec n/L} J\frac{e^{\alpha(\tilde q^2-\vec k^{*\,2})}}{\tilde q^2-\vec k^{*\,2}} - P\int\frac{d\vec k^{*}}{(2\pi)^3}\frac{e^{\alpha(\tilde q^2-\vec k^{*\,2})}}{\tilde q^2-\vec k^{*\,2}}\right]-\sum_{\vec n\neq 0}\int_0^{\alpha(2\pi/L)^2}dt\ \frac{F_C(t)}{\pi\tilde q L}$$

$$F_C(t)=e^{t\left(L\tilde q/2\pi\right)^2}\cos\left[\frac{L}{2}\vec n\cdot\vec P\right]\int dq\ 2qe^{-tq^2}\frac{\sin[2\pi q\sqrt{\vec n^2+\left(\frac{\vec n\cdot\vec P}{2\varpi()}\right)^2}]}{\sqrt{\vec n^2+\left(\frac{\vec n\cdot\vec P}{2\varpi()}\right)^2}}$$

Compare to the other methods

Model Dependent

$$\left[\frac{1}{P_0^{*2}}\frac{\varpi(\vec{k}^*)}{2}\frac{\varpi(\vec{k})+\varpi(\vec{P}-\vec{k})}{\varpi(\vec{k})\varpi(\vec{P}-\vec{k})}\frac{f_0^{\,2}(\varpi(\vec{k}_C^*),\vec{k}_C^*,P)}{P_0^*-2\varpi(\vec{k}_C^*)}+(\ldots\ldots)\right]$$

$$cot(-\delta_0) = \frac{4\pi}{\tilde{k}}\frac{2}{P_0^*\tilde{f}_0^2}\left\{\frac{1}{L^3}\sum_{\vec{k}=\frac{2\pi}{L}\vec{n},\,\vec{n}\in\mathbb{Z}}\frac{\omega_{q_C}}{2}\frac{\omega_k+\omega_{Pk}}{\omega_k\omega_{Pk}}\frac{f_0^2(\omega_{q_C},|\vec{q}_C|,P^*)}{(P_0^*-2\omega_{q_C})}\right.$$
$$\left.-\mathscr{P}\int\frac{d^3\vec{q}_C}{(2\pi)^3}\frac{f_0^2(\omega_{q_C},|\vec{q}_C|,P^*)}{(P_0^*-2\omega_{q_C})}\right\}$$

$$cot(-\delta) = \frac{4}{q_{on}P_0^*\pi g^2(q_{on})}(E-m_\sigma-\mathscr{P}\int q^2dq\frac{g^2(q)}{P_0^*-2\sqrt{m_\pi^2+q^2}}).$$

$$P_0^*-m_\sigma = \frac{(2\pi)^3}{L^3}\frac{1}{4\pi}\sum_{\vec{k}=\frac{2\pi}{L}\vec{n},\,\vec{n}\in\mathbb{Z}}\frac{\omega_{q_C}}{2}\frac{\omega_k+\omega_{Pk}}{\omega_k\omega_{Pk}}\frac{g^2(|\vec{q}_C|)}{(P_0^*-2\omega_{q_C})}$$

Compare to the other methods