Hadron interaction from phase shift analysis in lattice QCD

NSTAR2015, 2015/05/27

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1. Calculation method of the scattering phase shift by Lattice QCD.

2. Some applications to resonance and bound state systems.

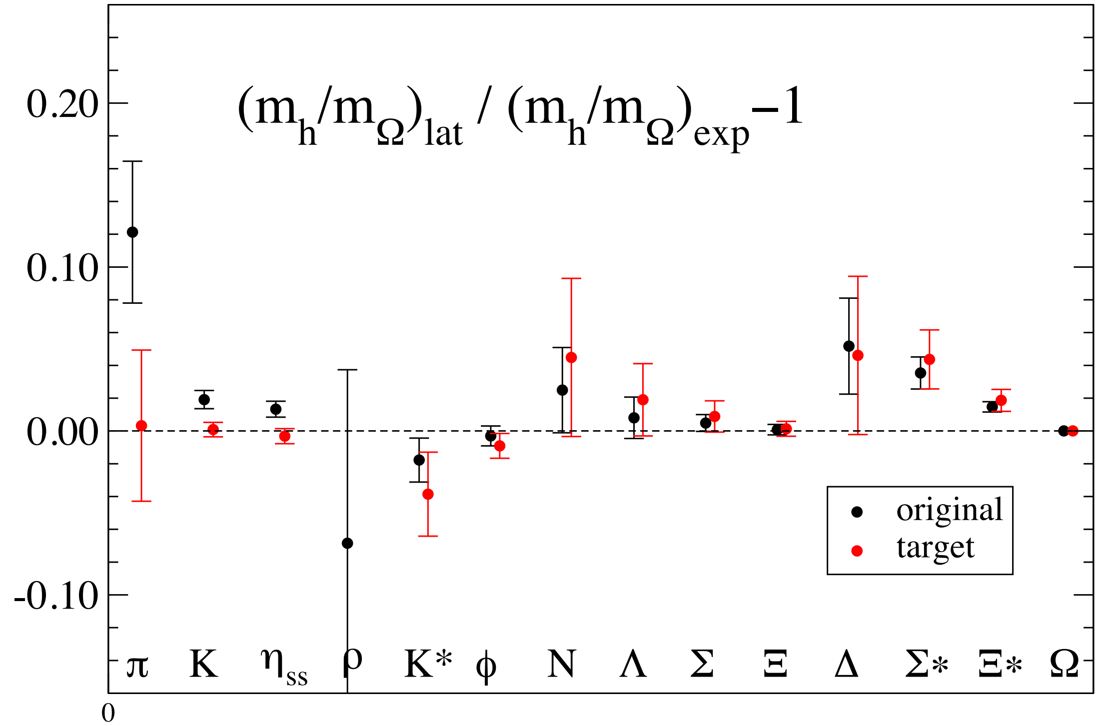
1. Introduction

Recent progress of simulation algorithm,
supported by the development of computer power,
makes it possible to calculate the hadron masses at physical quark mass by lattice QCD.

But, it is only for stable particles ......

Hadron masses at physical quark mass

PACS-CS Phys.Rev.D81:074503,2010.



for unstable particle (*ρ* , *K** , *Δ*)

energies of ground states
on finite volume are plotted.

(: not resonance masses)

Calculations of resonance mass
and decay width of unstable particles
at physical quark mass still remain.

These can be carried out
by analysis of the scattering phase shift.

Scattering phase shift
=> Information of Hadron interaction

: important for understanding properties of hadrons

Correlation function of resonance on the lattice

for ρ meson

$$\rho = \bar{u}\gamma_j u - \bar{d}\gamma_j d$$

$$\langle 0|\ \rho^\dagger(t)\ \rho(0)\ |0\rangle = \sum_\alpha |\langle 0|\rho|\alpha\rangle|^2 \cdot \mathrm{e}^{-E_\alpha t}$$

: multi exp. form with $E_\alpha \in \mathbb{R}$

(: same form as for stable particle)

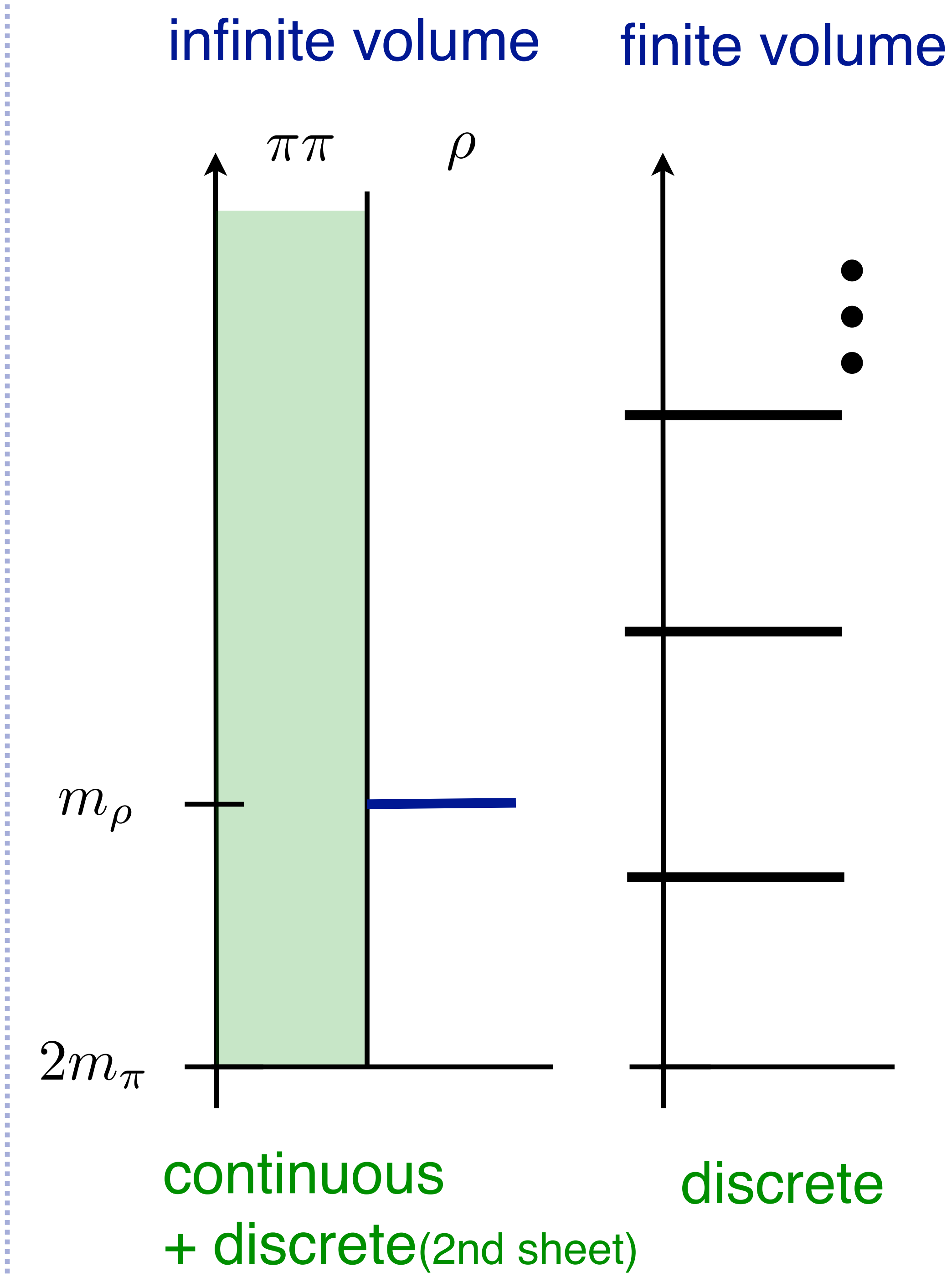
naive expectation :

for unstable particle

$$E = M + i\Gamma \qquad (\ M, \Gamma \in \mathbb{R}\ )$$

time correlator $\sim\ \mathrm{e}^{-Mt} \cdot \mathrm{e}^{-i\Gamma t}$

: ***This is only true in infinite volume !***



Decay width can not be directly obtained from correlation function on the lattice.

Calculation of scattering phase shift [Finite size method]

M. Lüscher , CMP105(86)153, NPB354(91)531.

Ex) for $\pi\pi$ S-wave (extension to other system is straight forward)

In $L \times L \times L$ periodic box (: lattice)

Energy of π :

$$E = \sqrt{m_\pi^2 + p^2} \qquad p^2 = (2\pi/L)^2 \cdot n \, , \quad \underline{n \in \mathbb{Z}}$$

Energy of $\pi\pi$:

$$E = 2\sqrt{m_\pi^2 + p^2} \qquad p^2 = (2\pi/L)^2 \cdot n \, , \quad \underline{n \notin \mathbb{Z}}$$

(: discrete)

$$p \cdot \cot \delta(p) = \frac{2}{\sqrt{\pi} L} \cdot \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{n^2 - q^2} \qquad (\, q = 2\pi/L \cdot p \,)$$

$\delta(p)$: SC. phase shift in infinite volume

: Lüscher's finite size formula

Energy of system on the lattice $\Longrightarrow$ SC. phase shift in infinite volume $\Longrightarrow$ Informations of hadron interaction

Other methods

1. From time correlation function

Extraction the resonance parameter
by fitting the time correlation function on the lattice
to effective theories.

: strongly relies on effective theory.

recently : model independent method at *L = huge*

U.-G. Meissner et al., NPB846(2011)1.

: very difficult in QCD !!

2. From spectrum density

V. Bernard et al., JHEP 08(2008) 024.

Calculations on many lattice volumes.
Extracting energies of very higher states.

: very difficult in QCD !!

3. From “potential” extracted from BS-function

HAL coll., PRL99(2007)02201, PTP123(2010)89.

Wave function calculated on the lattice.

=> extracting “potential”

=> solving Schrödinger eq. => SC. phase shift.

Contents

2. Finite size formula (S-wave $\pi\pi$ for $E < 4m_\pi$)

(extension to other system is straight forward)

Wave function on the lattice :

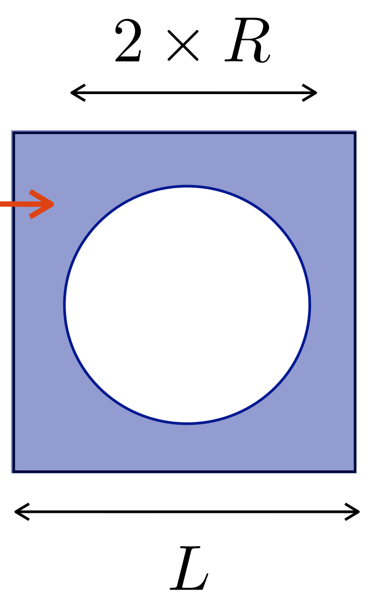
$$\Phi(\mathbf{x}) = \langle 0|\ \pi(\mathbf{x}/2)\ \pi(-\mathbf{x}/2)\ |E\rangle$$

energy eigenstate with $E = 2\sqrt{m_\pi^2 + p^2}$

assumption : interaction range : $R\ <\ L/2$

for $R < |\mathbf{x}| < L/2$

- $(\ \nabla^2\ +\ p^2\ )\,\Phi\ =\ 0$
- p.BC. : $\Phi(\mathbf{x}+\mathbf{n}L)\ =\ \Phi(\mathbf{x})\quad \text{for } \mathbf{n} \in \mathbb{Z}^3$



solution :

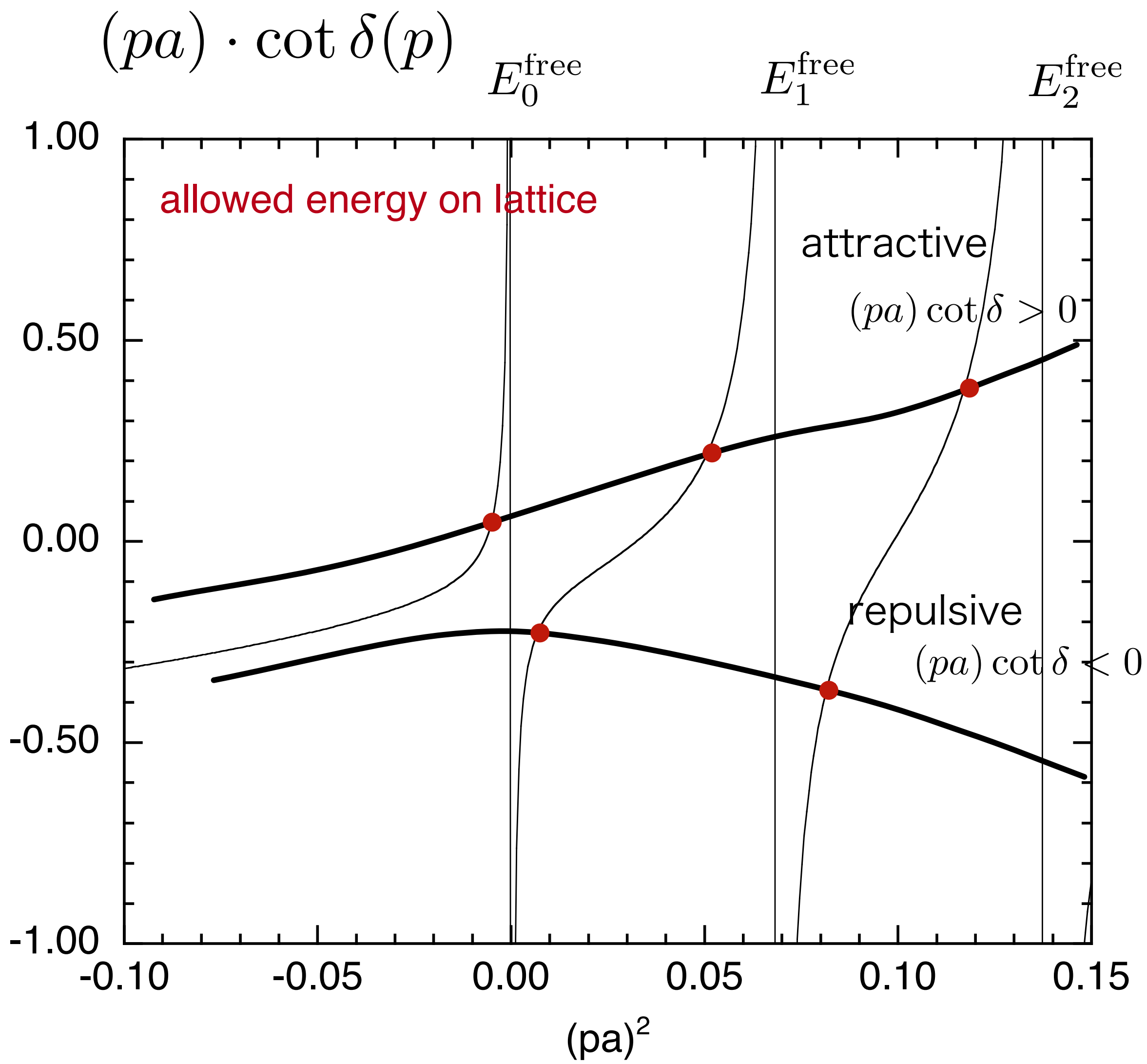
$$\Phi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \frac{\mathrm{e}^{i\mathbf{p}_n\cdot\mathbf{x}}}{p_n^2 - p^2} \qquad (\ \mathbf{p}_n = \mathbf{n}\cdot(2\pi)/L\ ) \qquad (\text{ for } \delta_l(p) = 0 \text{ for } l \geq 1\ )$$

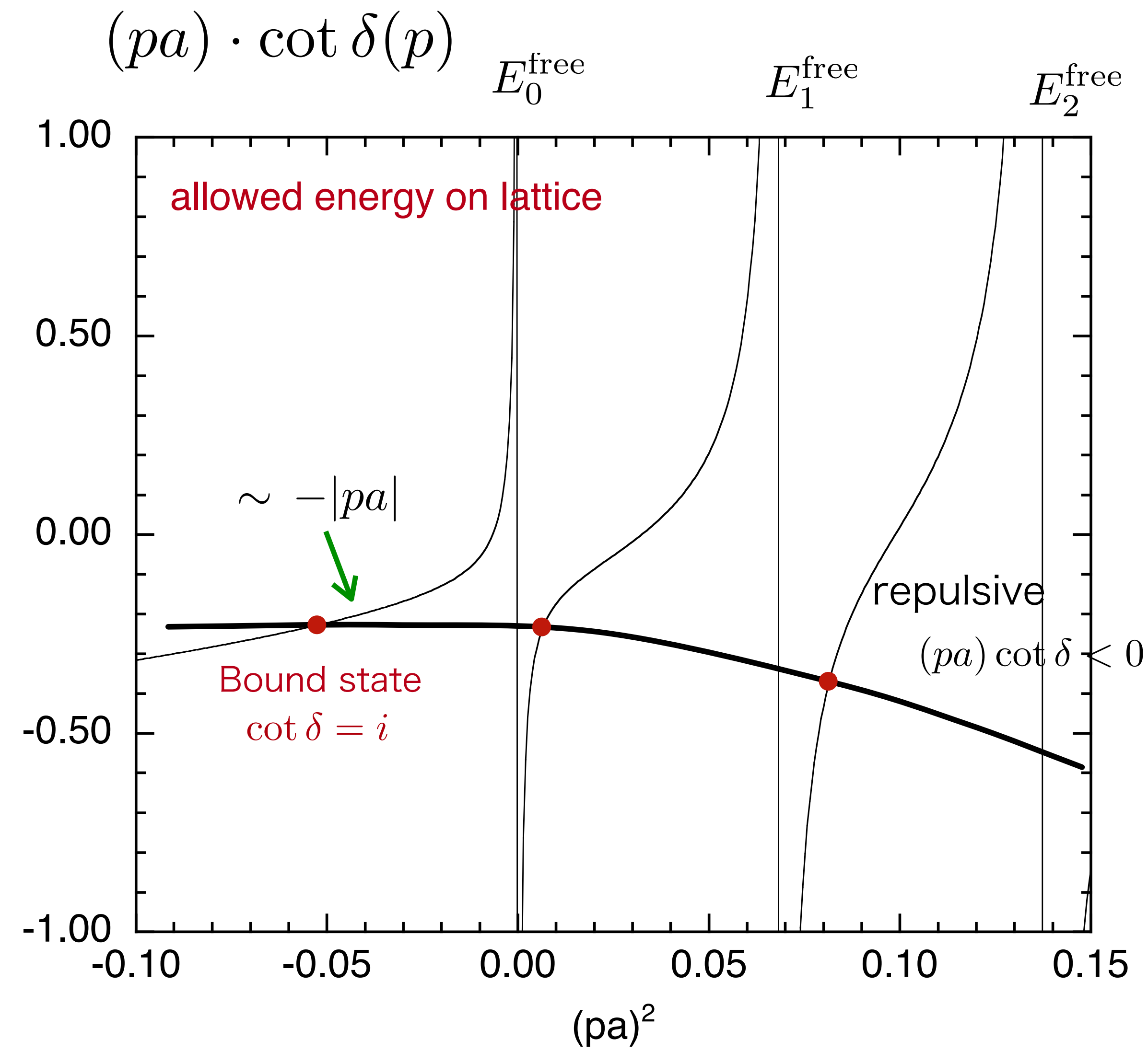
$$= \alpha_0(p)\cdot j_0(px)\ +\ \beta_0(p)\cdot n_0(px)\ +\ \left\{ j_l(px)\ ;\ l \geq 1 \right\}$$

$$p\cdot\cot\delta(p) = p\frac{\alpha_0(p)}{\beta_0(p)} = \frac{2}{\sqrt{\pi}L}\cdot\frac{1}{\sqrt{4\pi}}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{1}{n^2 - q^2} \qquad (\ q = 2\pi/L\cdot p\ )$$

: Lüscher's finite size formula

at fixed volume ($L/a = 24$)





$$\Delta E = E - E^{\mathrm{free}}$$

attractive $\Delta E < 0$
repulsive $\Delta E > 0$

$$|\Delta E| \propto 1/V$$

in case of strong attractive int.

Energy of ground state :

$$E_{-1} = E_B + \mathrm{e}^{-L}$$

Extension of finite size formula

- elastic SC.

$$A(\mathbf{p}_A) + B(\mathbf{p}_b) \quad , \; m_A \neq m_B \quad , \; \mathbf{p}_A + \mathbf{p}_b \neq 0$$

$$A \; , B \;\; : \;\; \text{hadron with spin}$$

K. Rummukainen and S.A.Gottlieb, NPB450(1995)397.

X. Feng, K. Jansen, D. Renner, PoS(Lat2010)104.

J.J. Dudek, R.G. Edward, C.E. Thomas, RD86(2012)034031.

Z.Fu, PRD85(2012)014506.

L.Leskovec and S. Prelovsek, PRD85(2012)114507.

M.Göckeler et al, PRD86(2012)094513.

M.Döring et al, EPJA48(2012)114.

N.Li and C.Liu, PRD87(2013)114507.

- multi SC. state (ex $\pi\pi$ + KK for $I=0$)

C. Liu, X. Feng, S. He, Int.J.Mod.Phys. A21(2006)847.

M. Lang. et.al., PLB681(09)439.

HSC coll., PRL113(2014)182001; PRD91(2015)054008.

- three particles (spin less)

M. Hansen and S.R. Sharpe, PRD90(2014)116003.

Extraction of energy of excited state

$\mathcal{O}(t)$: op. with some quantum number

$$G(t) = \langle 0|\, \mathcal{O}^\dagger(t)\, \mathcal{O}(t_0)\, |0\rangle = \sum_n A_n \cdot \mathrm{e}^{-E_n\cdot(t-t_0)} \qquad \text{for } t >> t_0$$

In principle, E_n of excited stats can be extracted by multi exponential fitting.

But it is practically very difficult.

$\mathcal{O}_i \ (i = 1, 2, \cdots N)$: ops. with some quantum number

assuming the lowest *N* states dominate correlation function.

$$G_{ij}(t) = \langle 0|\, \mathcal{O}_i^\dagger(t)\, \mathcal{O}_j(t_0)\, |0\rangle$$

$$= \sum_{n=0}^{N-1} V_{in}^\dagger \cdot \lambda_n(t-t_0) \cdot V_{nj} \quad = \Big[V^\dagger \cdot \lambda(t-t_0) \cdot V \Big]_{ij} \qquad \text{for } t >> t_0$$

$$V_{nj} = \langle n|\mathcal{O}_j|0\rangle \ , \ \ \lambda_n(t) = \exp(-E_n t)$$

so,

$$\Big[G(t_R)^{-1} G(t) \Big]_{n\text{-th eigenvalue}} = \lambda_n(t - t_R)$$

We can extract E_n by single exponential fitting.

: Variational method M. Lüscher and U. Wolff, NPB339(1990)222.

ex) ρ meson $\;(\, I = 1\,,\; J^{PC} = 1^{--}\,)$

- $\mathcal{O}^{\pi\pi}(\mathbf{p}) = \pi^+(\mathbf{p})\pi^-(-\mathbf{p}) - \pi^+(-\mathbf{p})\pi^-(\mathbf{p})$
- $\rho_j = \bar{u}\gamma_j u - \bar{d}\gamma_j d$

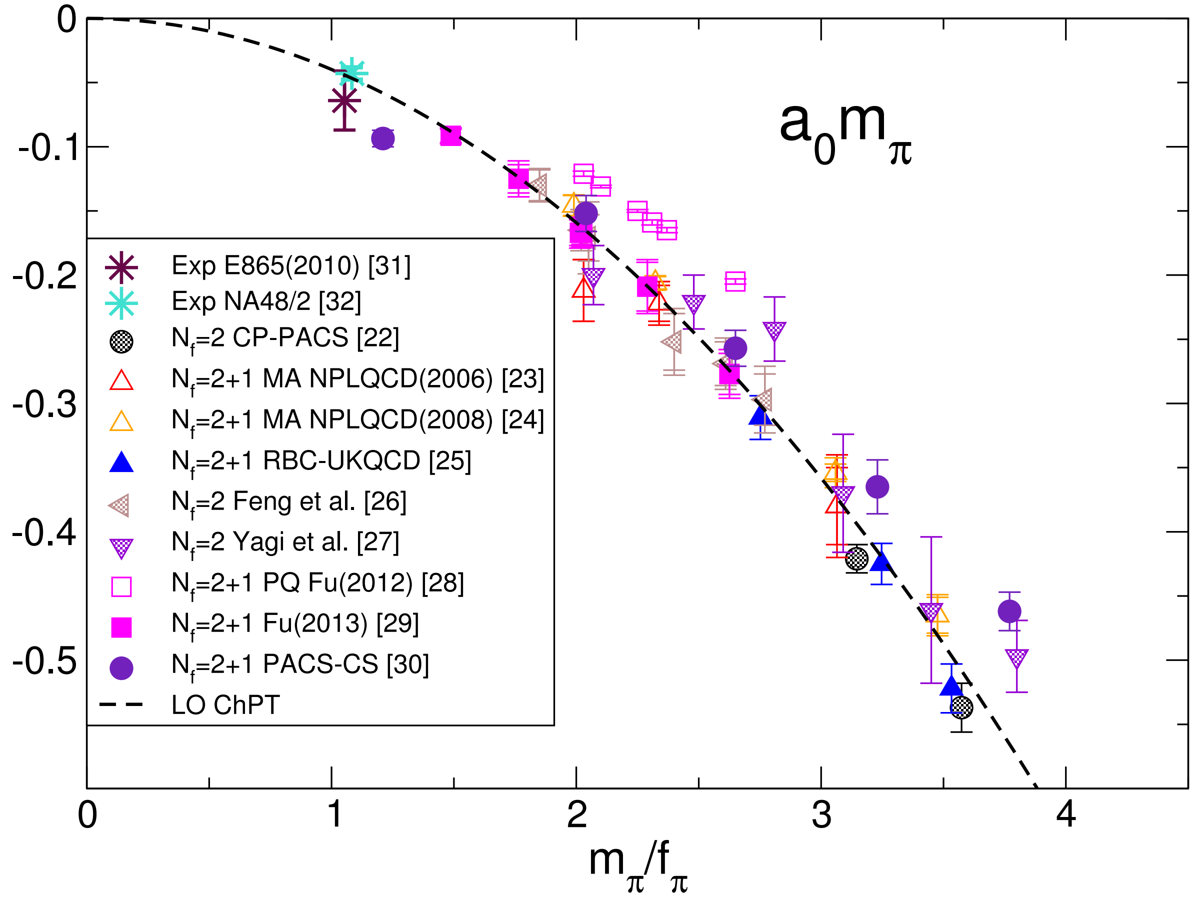
ex) $D_{s0}^*(2317)\;(\, I = 0\,,\; J^P = 0^+\,)$ (bound state)

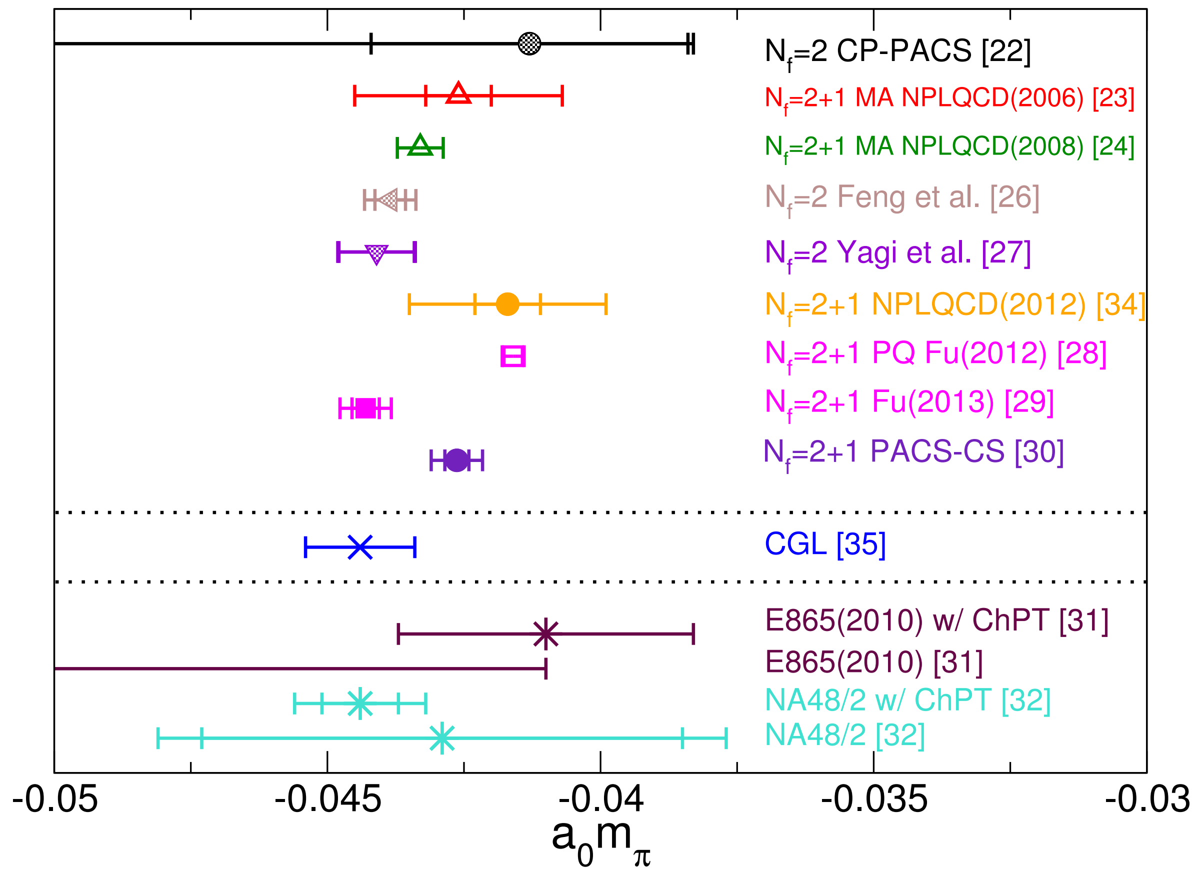
- $\mathcal{O}^{DK}(\mathbf{p}) = D^+(\mathbf{p})K^-(-\mathbf{p}) + D^-(\mathbf{p})K^+(-\mathbf{p})$
- $M = \bar{s}c$

3. Recent progress

(1) Scattering length of *I=2* S-wave $\pi\pi$

Talk by Yamazaki at Lat.2015 (PoS(Lattice2015)009)





$$\langle \pi\pi(t)\pi\pi(0)\rangle = A \cdot \exp(-Et)$$

for large t region

$$E = 2\sqrt{m_\pi^2 + p^2}$$

$$a_0 \sim \tan\delta(p)/p$$

$f_\pi = f_\pi(m_\pi)$ from lattice cal.

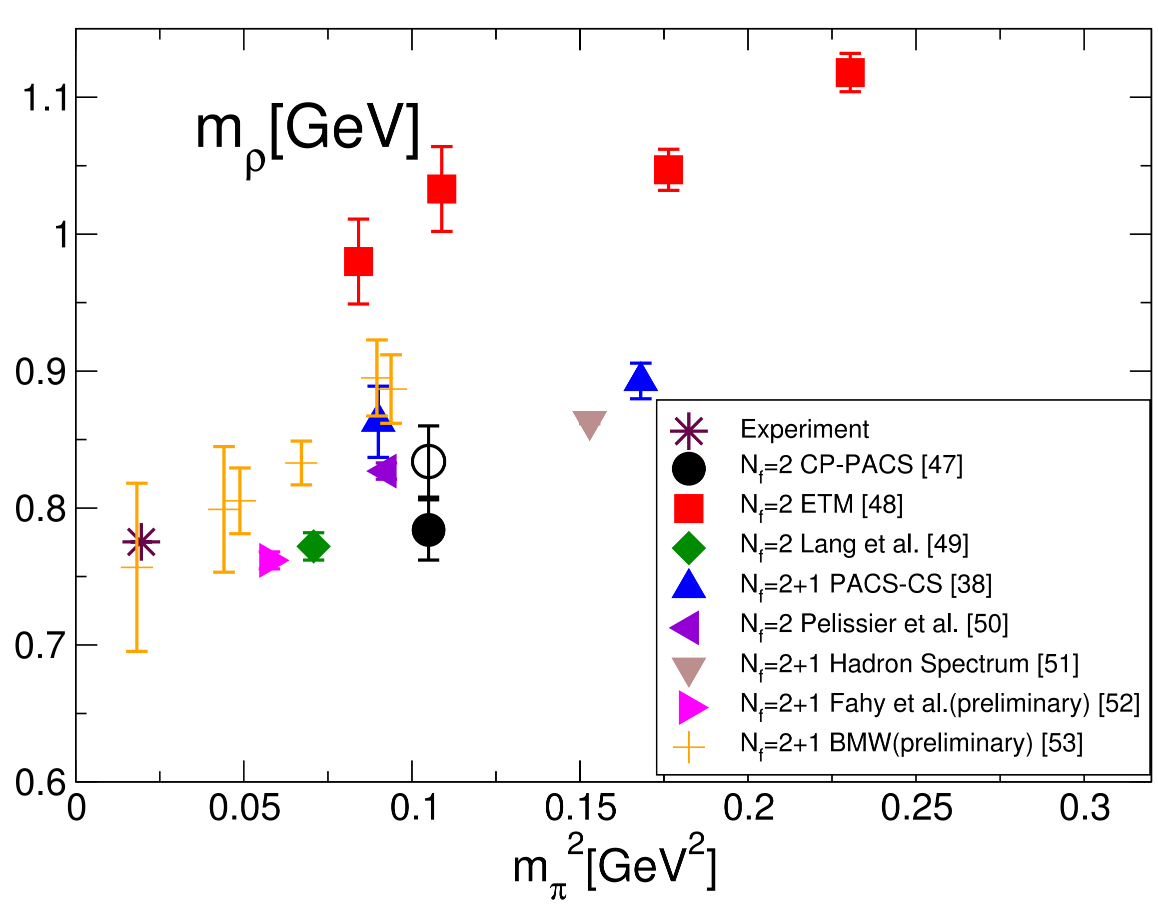
$$\text{line} = -m_\pi^2/(8\pi f_\pi^2(m_\pi))$$

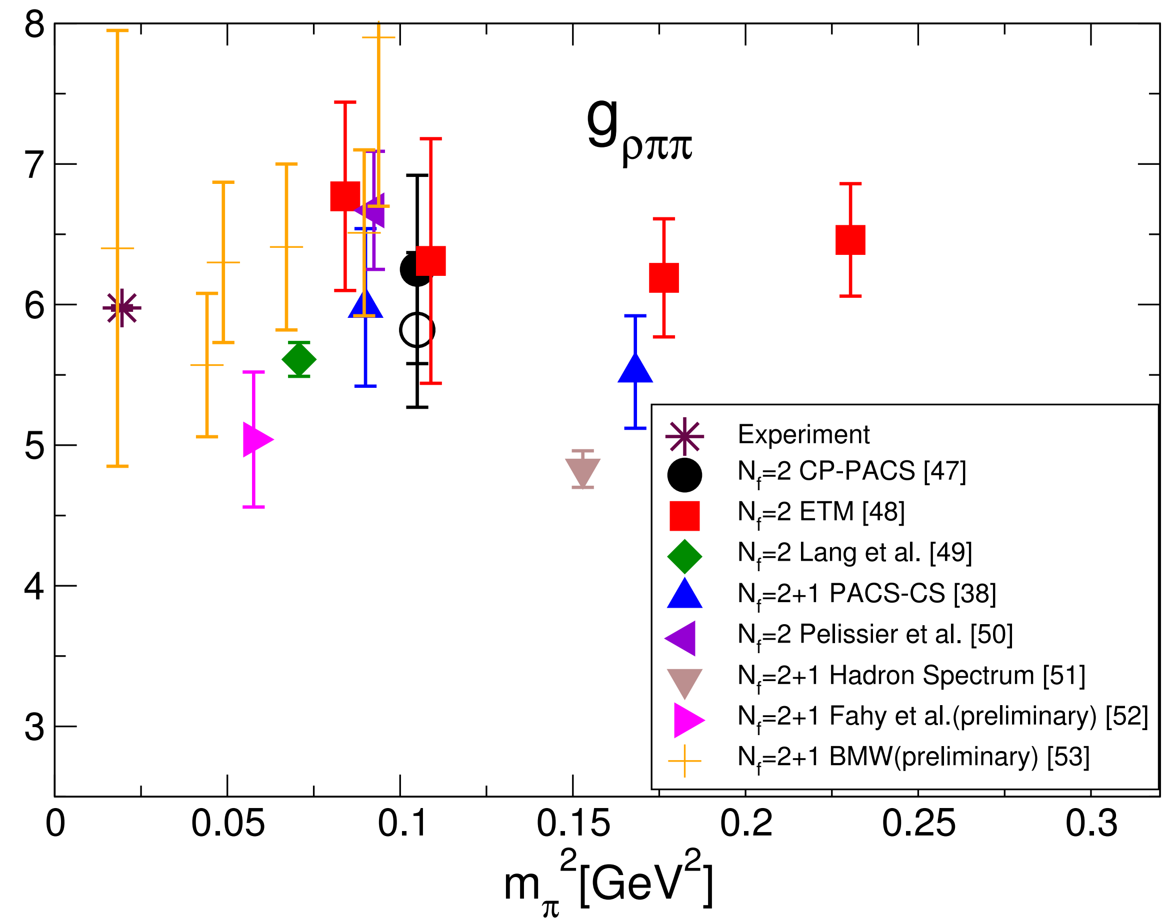
: LO CHPT with $f_\pi(m_\pi)$

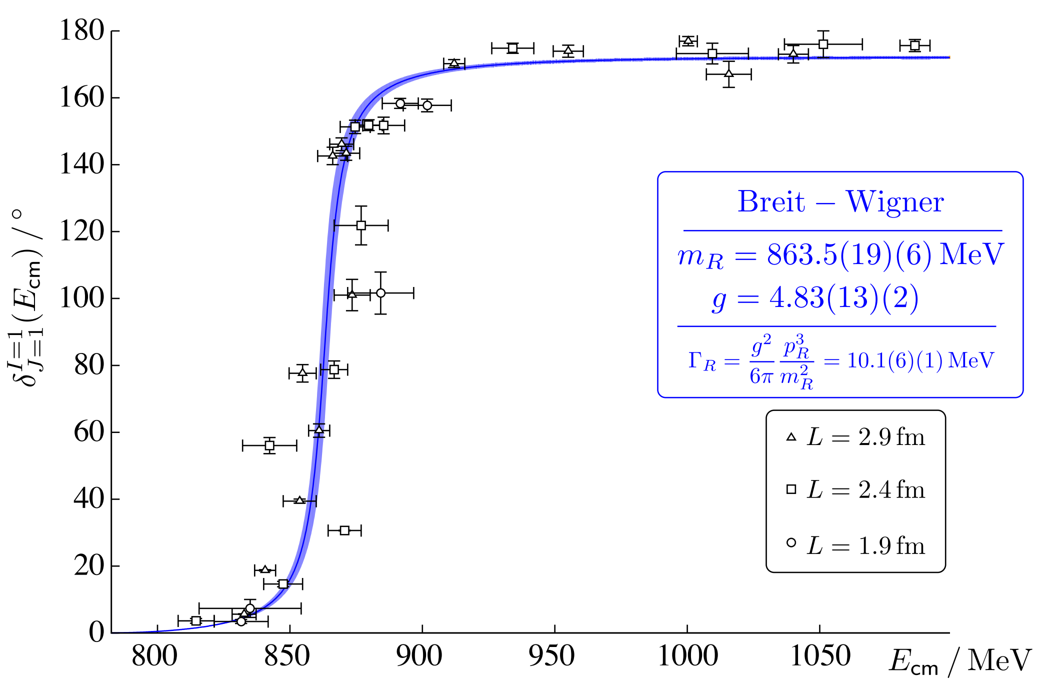
all results are on LO CHPT with $f_\pi(m_\pi)$
and consistent with expt.

(2) ρ meson decay ($I=1$ P-wave $\pi\pi$)

Talk by Yamazaki at Lat.2015 (PoS(Lattice2015)009)







HSC coll.,. PRD87(2013)034505.

exp. : $\Gamma = \dfrac{g_{\rho\pi\pi}^2}{6\pi}\dfrac{p_\rho^3}{m_\rho^2} = 146.4(1.1)\,\mathrm{MeV}$

$$g_{\rho\pi\pi} = 5.878(22)$$

$$m_\rho = 775.5(0.4)\,\mathrm{MeV}$$

$$\frac{p^3}{\tan\delta(p)}/\sqrt{s} = \frac{6\pi}{g_{\rho\pi\pi}^2}\cdot(m_\rho^2 - s)$$

almost consistent with expt.

(3) $D_0^*(2400)$ and $D_1(2420, 2430)$ ($(D, D^*) + \pi$, S-wave)

neutral D meson : $D = \bar{c}\Gamma u$

	J^P	Γ	
$D(1864)$	0^-	γ_5	stable
$D^*(2010)$	1^-	γ_j	$D\pi$, $\Gamma = 96\,\mathrm{KeV}$
$D_0^*(2400)$	0^+	I	$D\pi$, $\Gamma = 267\,\mathrm{MeV}$
$D_1(2420)$	1^+	$\gamma_j\gamma_5$	$D^*\pi$, $\Gamma = 27\,\mathrm{MeV}$
$D_1(2430)$	1^+	$\gamma_j\gamma_5$	$D^*\pi$, $\Gamma = 384\,\mathrm{MeV}$

total angular mom. : $J = j_u + s_c$

j_u : total angular mom. of u quark

s_c : spin of c quark

in heavy charm limit,

j_u is good quantum number.

expectation form heavy quark sym. :

$j_u^P = 1/2^-$, $j^P = (\,0^-\,,\,1^-\,) = (D(1864)\,,\,D^*(2010)\,)$

$j_u^P = 1/2^+$, $j^P = (\,0^+\,,\,1^+\,) = (D_0^*(2400)\,,\,D_1(2430)\,)$: S-wave decay [broad]

$j_u^P = 3/2^+$, $j^P = (\,1^+\,,\,2^+\,) = (D_1(2420)\,,\,D_2^*(2460)\,)$: D-wave decay [narrow]

(in heavy charm limit)

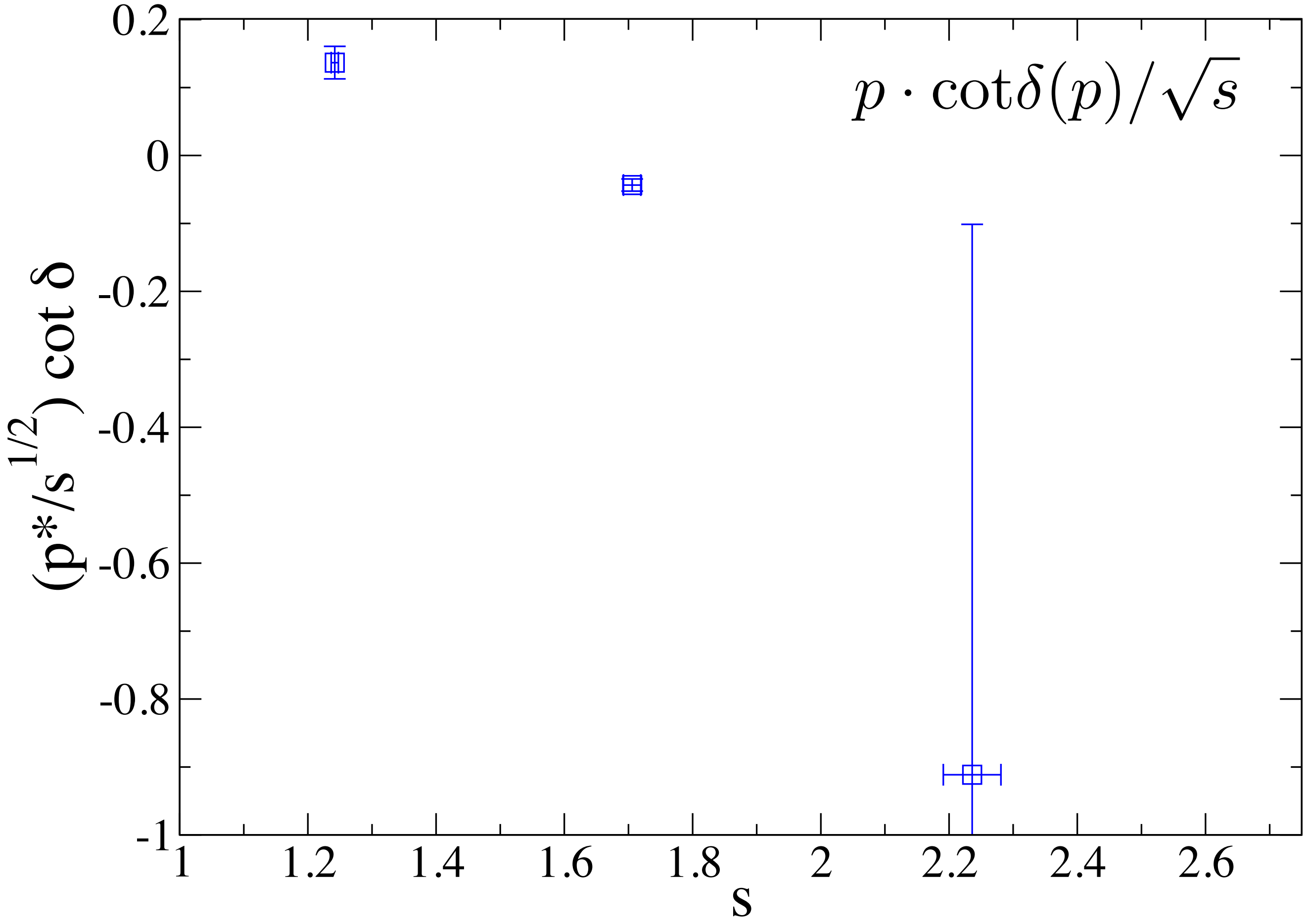
$D_0^*(2400)$ and $D_1(2420, 2430)$ by lattice QCD

D. Mohler, S. Prelovsek, R.M. Woloshyn, PRD87(2013)034501.

$N_L^3 \times N_T$	κ_l	β	a[fm]	L[fm]	#configs	m_π[MeV]
$16^3 \times 32$	0.1283	7.1	0.1239(13)	1.98	280/279	266(3)(3)

$D_0^*(2400)$

SC. phase shift of $D + \pi$, S-wave



$$p \cdot \cot \delta(p)/\sqrt{s} = (6\pi)/g^2 \cdot (m_R^2 - s)$$

$$g = 2.55(21)(03)\,\mathrm{GeV}$$

$$m(D_0^*) - M_{\overline{1S}} = 350.8(20.2)(3.7)\,\mathrm{MeV}$$

$$M_{\overline{1S}} = (M(D) + 2M(D^*))/4$$

Expt :

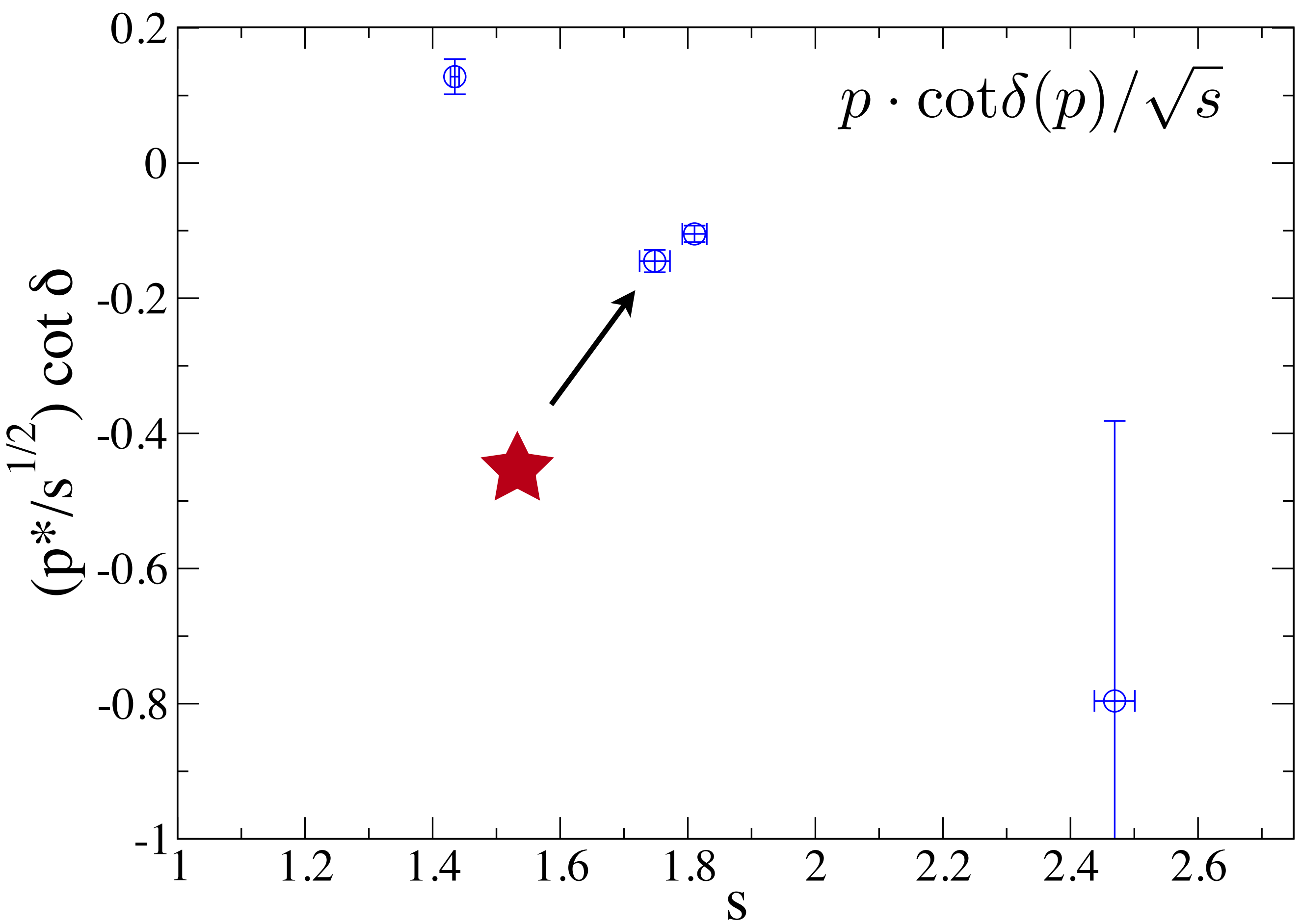
$$\Gamma = 267(40)\,\mathrm{MeV} \quad \to \quad g \leq 1.92(14)\,\mathrm{GeV}$$

$$m(D_0^*) - M_{\overline{1S}} = 347(2)\,\mathrm{MeV}$$

almost consistent with expt.

$D_1\,(2420, 2430)$

SC. phase shift of $D^* + \pi$, S-wave



★ : large overlap with $\bar{c}\gamma_j\gamma_5 u$ op.
small overlap with $D^* + \pi$ (S-wave) op.

maybe $D_1(2420)(\Gamma = 27\,\mathrm{MeV})$
: very narrow

omitting this state,
they analyze as one resonance with

$$p \cdot \cot\delta(p)/\sqrt{s} = (6\pi)/g^2 \cdot (m_R^2 - s)$$

Expt :

$g = 2.01(15)(02)\,\mathrm{GeV}$	$\Gamma = 384(107)\,\mathrm{MeV} \;\rightarrow\; g \le 2.50(40)\,\mathrm{GeV}$
$m(D_1(2430)) - M_{\overline{1S}} = 380.7(20.0)(4.0)\,\mathrm{MeV}$	$456(40)\,\mathrm{MeV}$
★ $m(D_1(2420)) - M_{\overline{1S}} = 448.77(14.1)(4.7)\,\mathrm{MeV}$	$449.9(0.6)\,\mathrm{MeV}$

almost consistent with expt.

(4) $D^*_{s0}(2317)$ ($D+K$, S-wave , bound state)

	J^P		Mass	
K	0^-	$\bar{s}\gamma_5 u$	498	
D	0^-	$\bar{u}\gamma_5 c$	1864	
D^*_{s0}	0^+	$\bar{s}c$	2317	$<$ ($K+D=2362$)

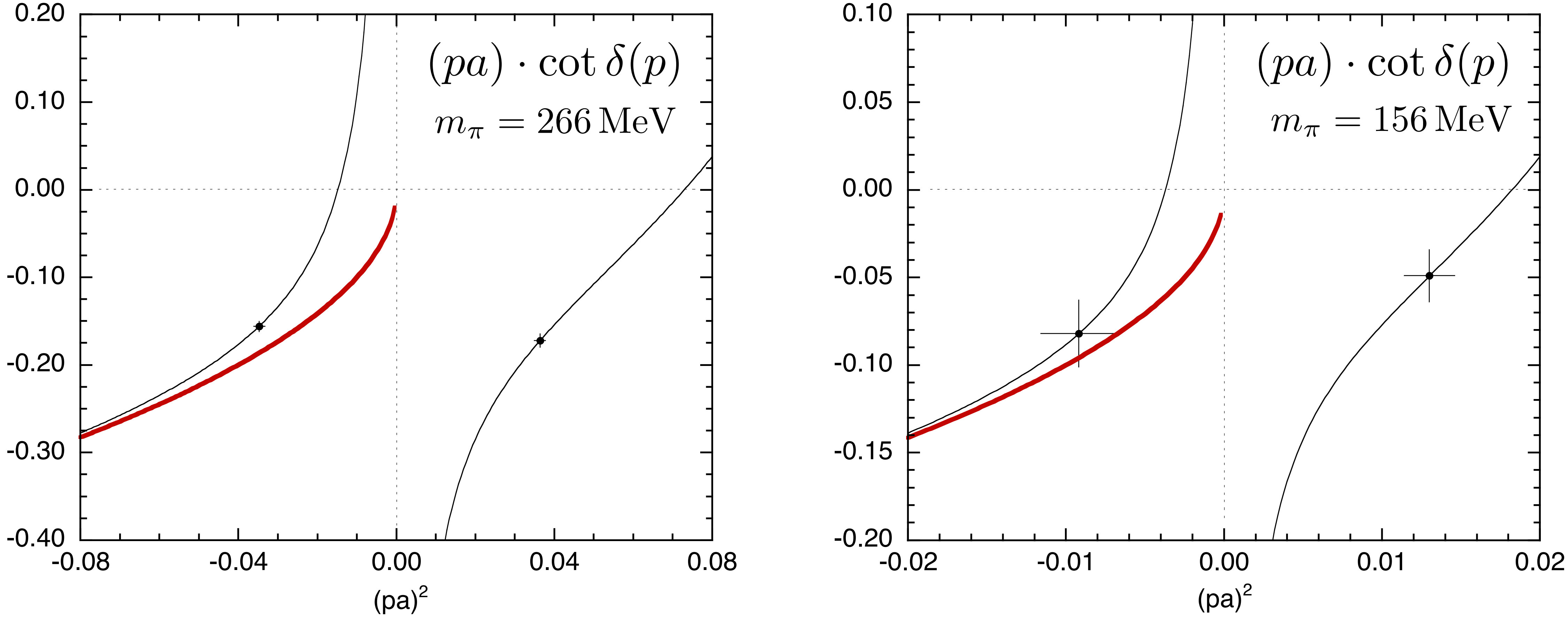
This state should be found
as bound state of D + K scattering system.

D. Mohler, S. Prelovsek, R.M. Woloshyn,
PRL111(2013)222001, PRD90(2014)034510.

ID	$N_L^3 \times N_T$	N_f	a [fm]	L [fm]	No. configs	m_π [MeV]	m_K [MeV]
(1)	$16^3 \times 32$	2	0.1239(13)	1.98	279	266(3)(3)	552(2)(6)
(2)	$32^3 \times 64$	2+1	0.0907(13)	2.90	196	156(7)(2)	504(1)(7)

They also study $D_{s1}(2536)$, $D^*_{s2}(2573)$

SC. phase shift of $D + K$, S-wave



black line : finite size formula

red line $= -|pa|$: corresponding to $\cot\delta = i$

ground state is near red line => bound state $D_{s0}^*(2317)$

m_π	$E_0 - M_{\overline{1S}}$
$266\,\mathrm{MeV}$	$254.4(4.3)(2.3)\,\mathrm{MeV}$
$156\,\mathrm{MeV}$	$245(15)(4)\,\mathrm{MeV}$
Expt.	$241.45(60)\,\mathrm{MeV}$

consistent with expt.

(5) *NN* scattering by two methods

finite size method vs. HAL potential method

HAL potential method

HAL coll., PRL99(2007)02201, PTP123(2010)89.

(1) Calculation of BS wave function by the lattice QCD.

$$\Phi(\mathbf{x}) = \langle 0|\, N(\mathbf{x}/2)\, N(-\mathbf{x}/2)\, |NN; E_p\,\rangle \qquad E_p = 2\sqrt{m_N^2 + p^2}$$

(2) Extraction of "potential" *V* from wave function by Schrödinger eq. at energy E_p .

$(\ \Phi(\mathbf{x}) \to V(\mathbf{x}) \ )$

$$-\frac{\nabla^2}{m_N}\Phi(\mathbf{x}) + V(\mathbf{x})\Phi(\mathbf{x}) = E_p\Phi(\mathbf{x})$$

[More sophisticated method is recently used.]

(3) Solving Schrödinger eq. with the potential *V* at general energy E_k .

$(\ V(\mathbf{x}) \to \Psi(\mathbf{x};k) \ )$

$$-\frac{\nabla^2}{m_N}\Psi(\mathbf{x};k) + V(\mathbf{x})\Psi(\mathbf{x};k) = E_k\Psi(\mathbf{x};k) \qquad E_k = 2\sqrt{m_N^2 + k^2}$$

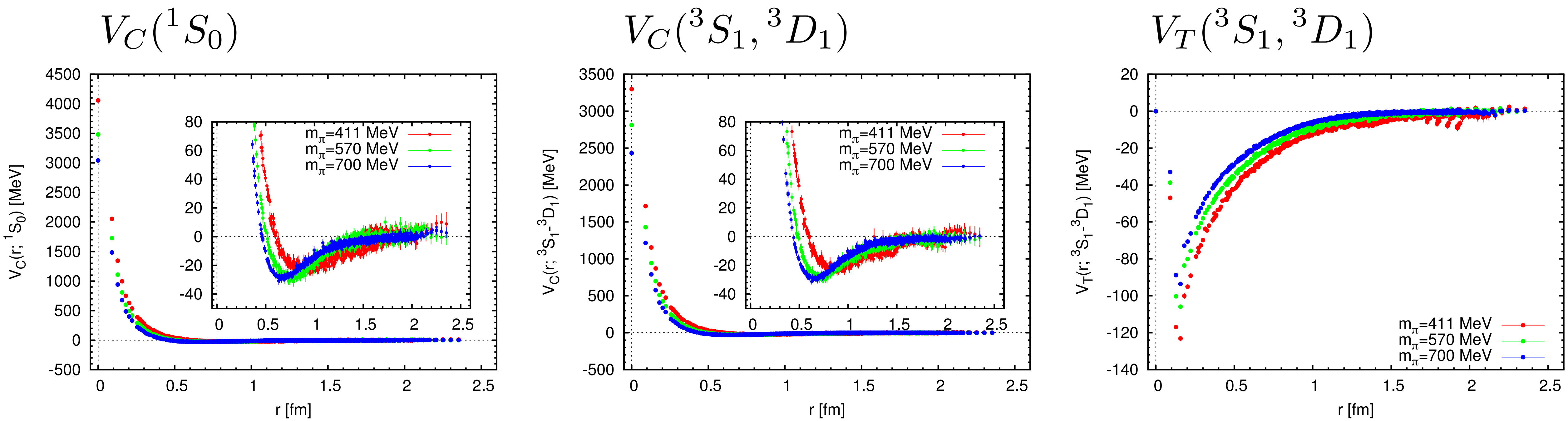
$\Psi(\mathbf{x};k)$ => SC. phase shift at general energy E_k

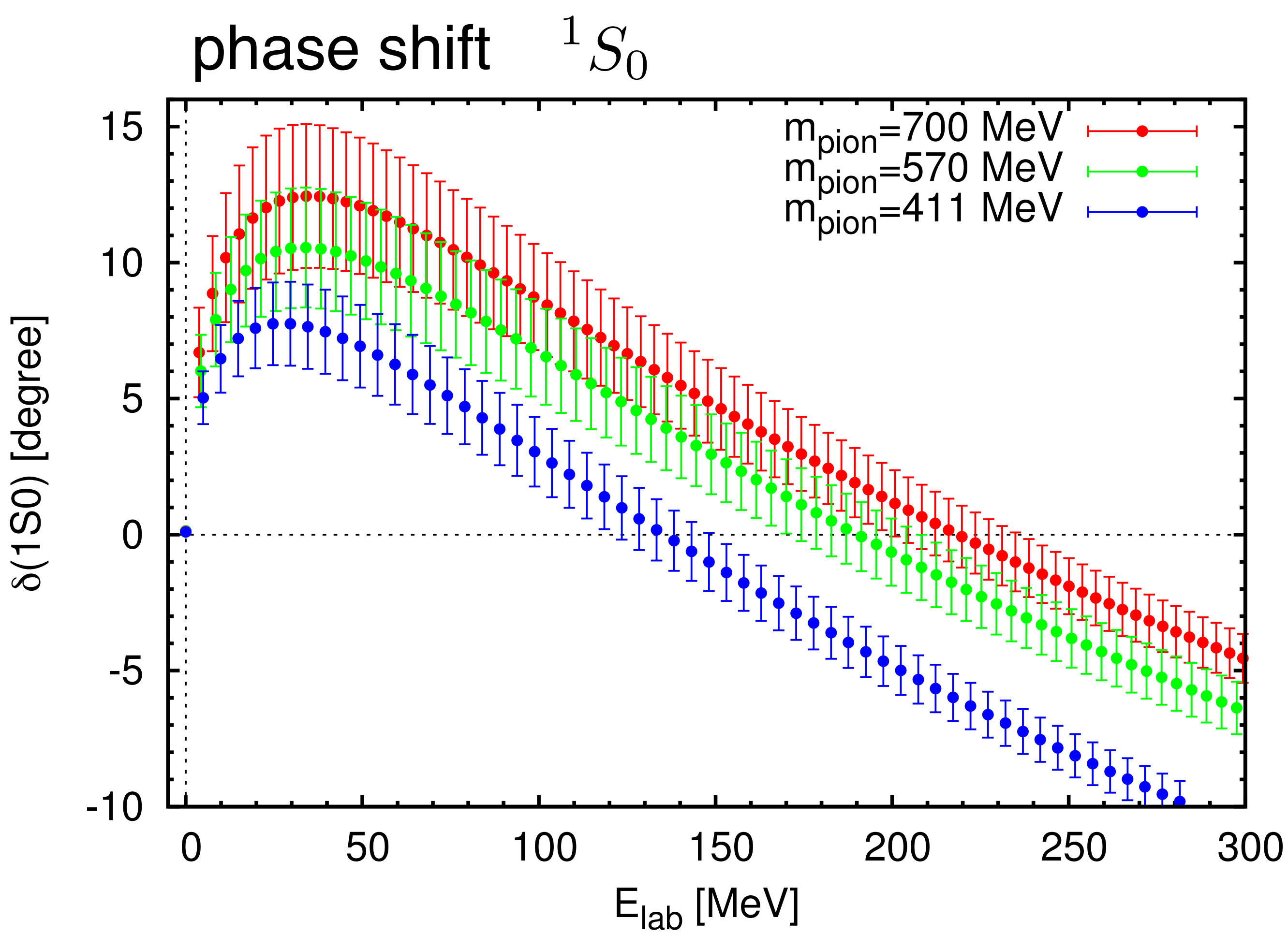
Works by HAL coll.

- NN potential
  N. Ishii , S. Aoki, T. Hatsuda, PRL99(2007)02201; HAL coll., PoS CD12(2013)025.
- Energy dependence of NN potential K. Murano et al, PTP125(2011)1225.
- LS force in odd parity sectors HAL coll., PLB735(2014)19.
- Hyperon potential HAL coll., Int.J.Mod.Phys.E19(2010)2442.
- ΩΩ potential HAL coll., PoS(Lattice 2014)232; arXiv:1503.03189.
- NΩ potential HAL coll., NPA928(2014)89.
- NNN potential HAL coll., PTP127(2012)723.
- H dibaryon HAL coll., PRL106(2011)162002; NPA88(2012)28.
- Charmed meson system HAL coll., PLB729(2014)85.
- Couple channel *see Parallel-B 27-1 by Sasaki.*
- Z_c^+ *see Parallel-B 27-2 by Ikeda.*

......

HAL coll., PoS CD12(2013)025.





much smaller than expt.

No bound state in both $^1S_0$ and $^3S_0$.

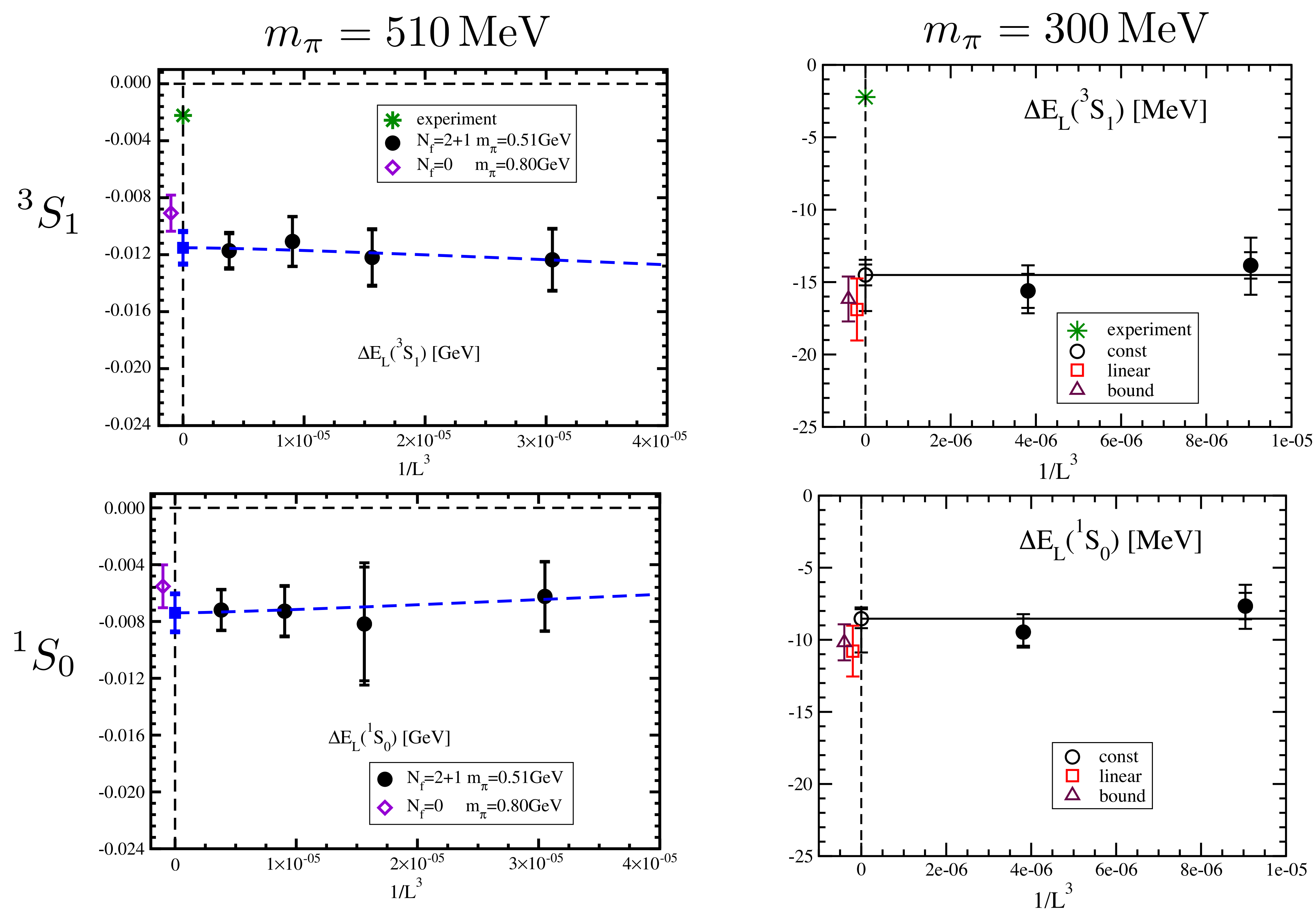
: inconsistent with expt.

[due to large quark mass ?]

Binding energy of *NN* ground state

$$\Delta E = E_0 - 2 \cdot m_N$$

T. Yamazaki, K.-I. Ishikawa, Y. Kuramashi, A. Ukawa, PRD86(2012)074514; arXiv:1502.04182.



Bound state in both channels !!

: inconsistent with Expt. and HAL coll. !!

Reason ?

[Consistency is seen in *I=2 ππ* system.]

T. Kurth, N. Ishii, T. Doi, S. Aoki, T. Hatsuda, JHEP1312(2013)015.

4. Summary

Analysis of scattering phase shift give us
a lot of information of hadron interactions.

In recent the lattice calculation,
a lot of studies of resonance and bound state
form the phase shift are carried out.

Next step of the lattice calculations :

- Solving the problem in *NN* scattering system.

  [Finite size method vs. HAL potential method]

- Calculations at physical quark mass.
- Application to other systems (including the baryon).

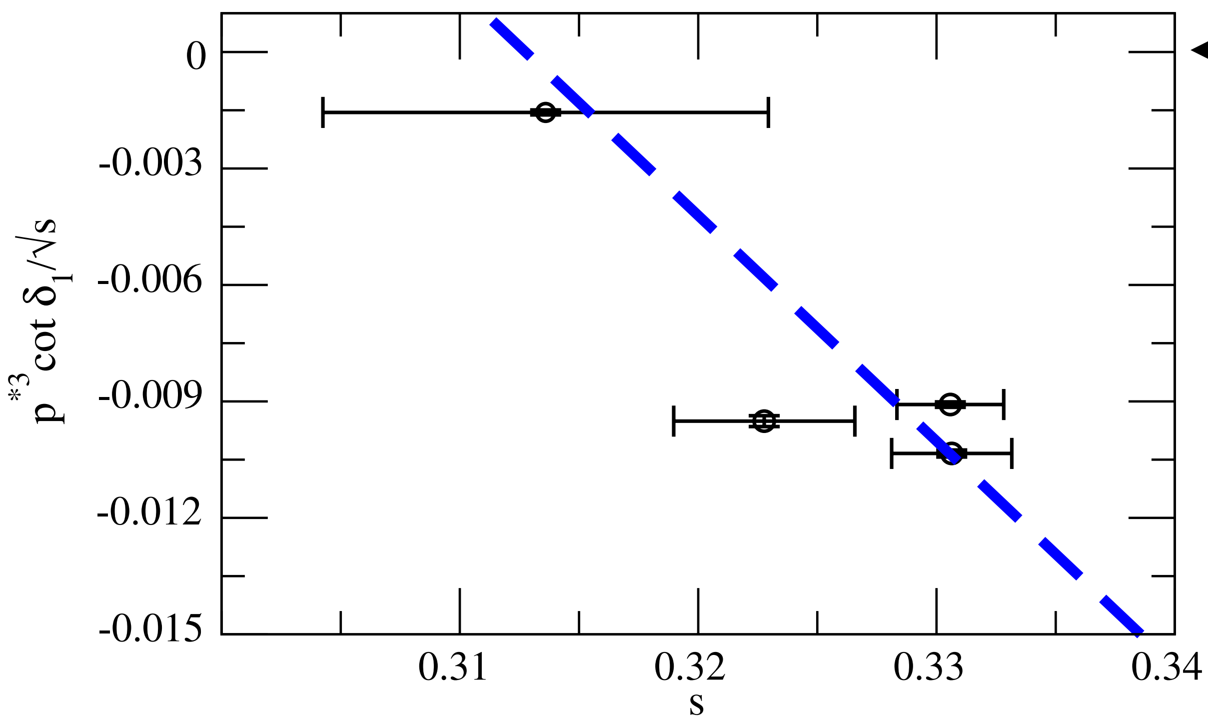
Thanks for your attention.

Back up

$K^*(892)$ ($K + \pi$, I=1/2, P-wave)

(1) S. Prelovsek, L. Leskove, C.B. Lang, D. Mohler, PRD88(2013)054508.

$$m_\pi = 266\,\mathrm{MeV}$$



← resonance

far from resonance

=> large error for decay width

$$\frac{p^3}{\tan\delta(p)}/\sqrt{s} = \frac{6\pi}{g^2}\cdot(m^2 - s)$$

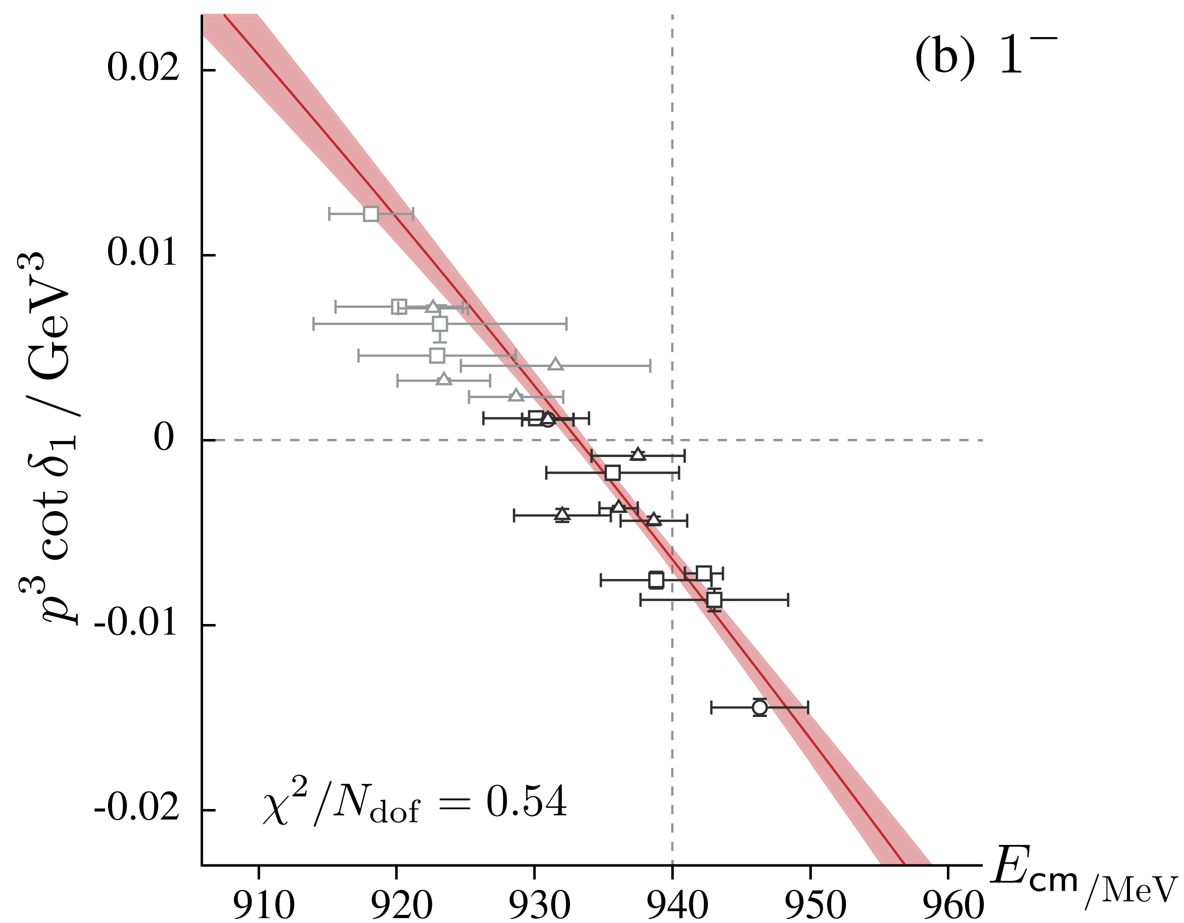
	exp :
$m_{K^*} = 891(14)\,\mathrm{MeV}$	$891.66(26)\,\mathrm{MeV}$
$g = 5.7(1.6)$	$5.72(6)$

consistent with expt.

K^*(892) ($K + \pi$, I=1/2, P-wave)

(2) HSC coll., PRL113(2014)182001; PRD91(2015)054008.

$$m_\pi = 391\,\mathrm{MeV}$$



$$\frac{p^3}{\tan\delta(p)} = \frac{6\pi}{g^2}\cdot\sqrt{s}(m^2 - s)$$

$m_{K^*} < m_K + m_\pi$

: bound state

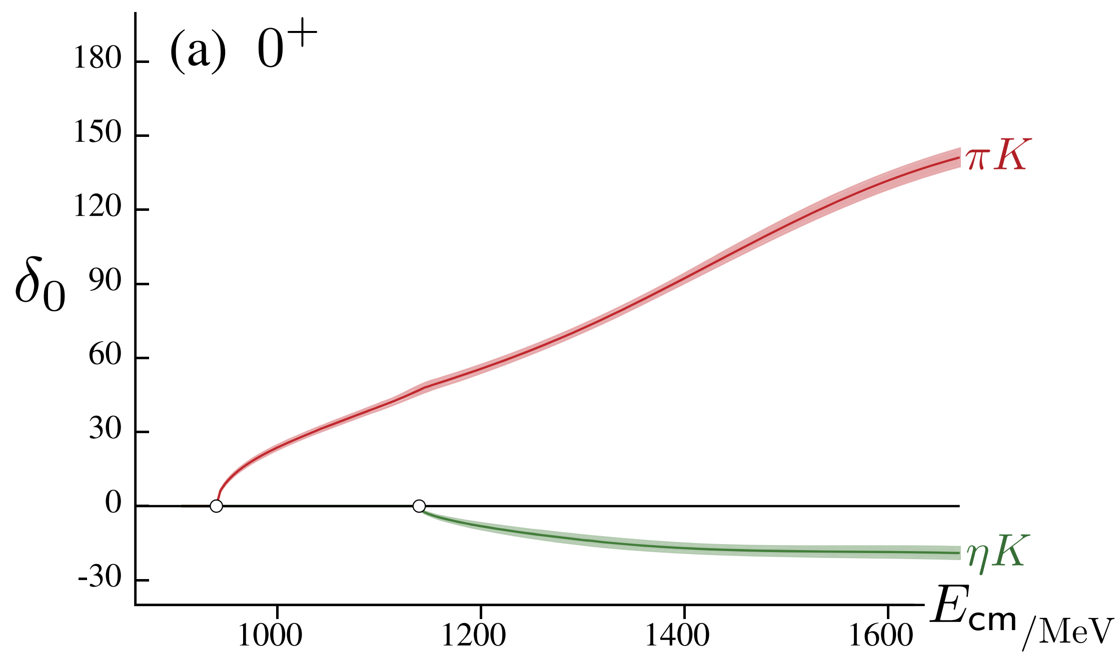
un-physical kinematics.

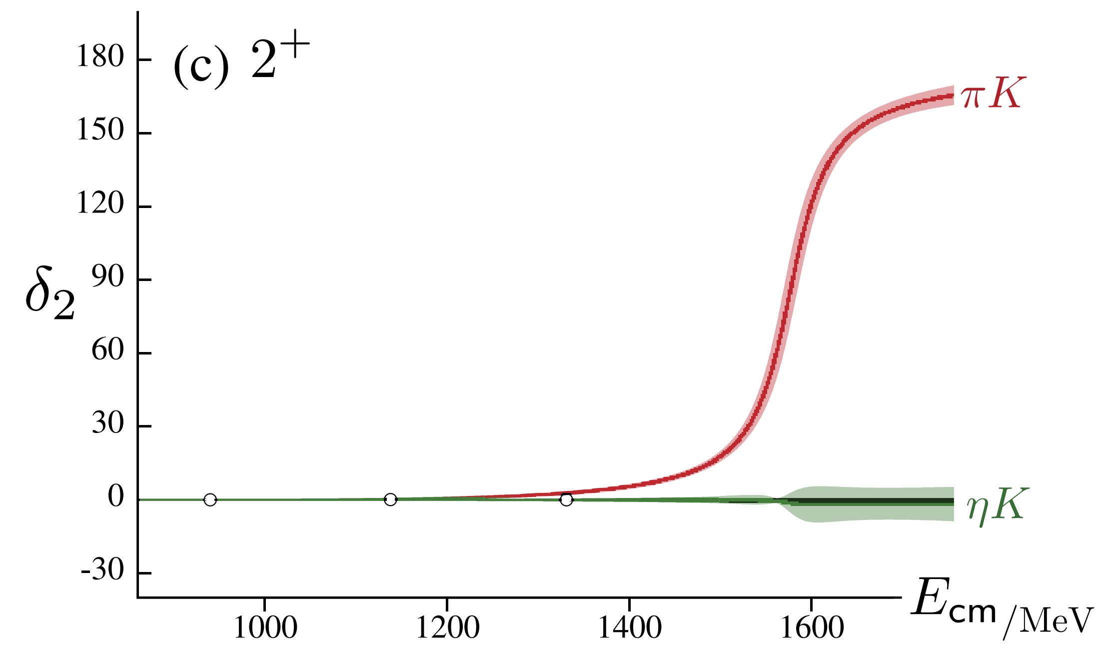
quark mass is too large.

$K + \pi$, I=1/2, S,D-wave

HSC coll., PRL113(2014)182001; PRD91(2015)054008.

$$m_\pi = 391\,\mathrm{MeV}$$





$$t_{ij}^{-1}(s) = \frac{1}{(2p_i)^J} K_{ij}^{-1}(s) \frac{1}{(2p_j)^J} + I_{ij}(s),$$

$$K_{ij}(s) = \sum_p \frac{g_i^{(p)} g_j^{(p)}}{m_p^2 - s} + \sum_n \gamma_{ij}^{(n)} s^n,$$

$$\det[\delta_{ij}\delta_{JJ'} + i\rho_i\, t_{ij}^{(J)}(E_{\mathrm{c.m.}})(\delta_{JJ'} + i\mathcal{M}_{JJ'}^{\vec{P}\Lambda}(p_i L))] = 0,$$

$$m_\kappa < m_K + m_\pi$$

: bound state

un-physical kinematics.

quark mass is too large.

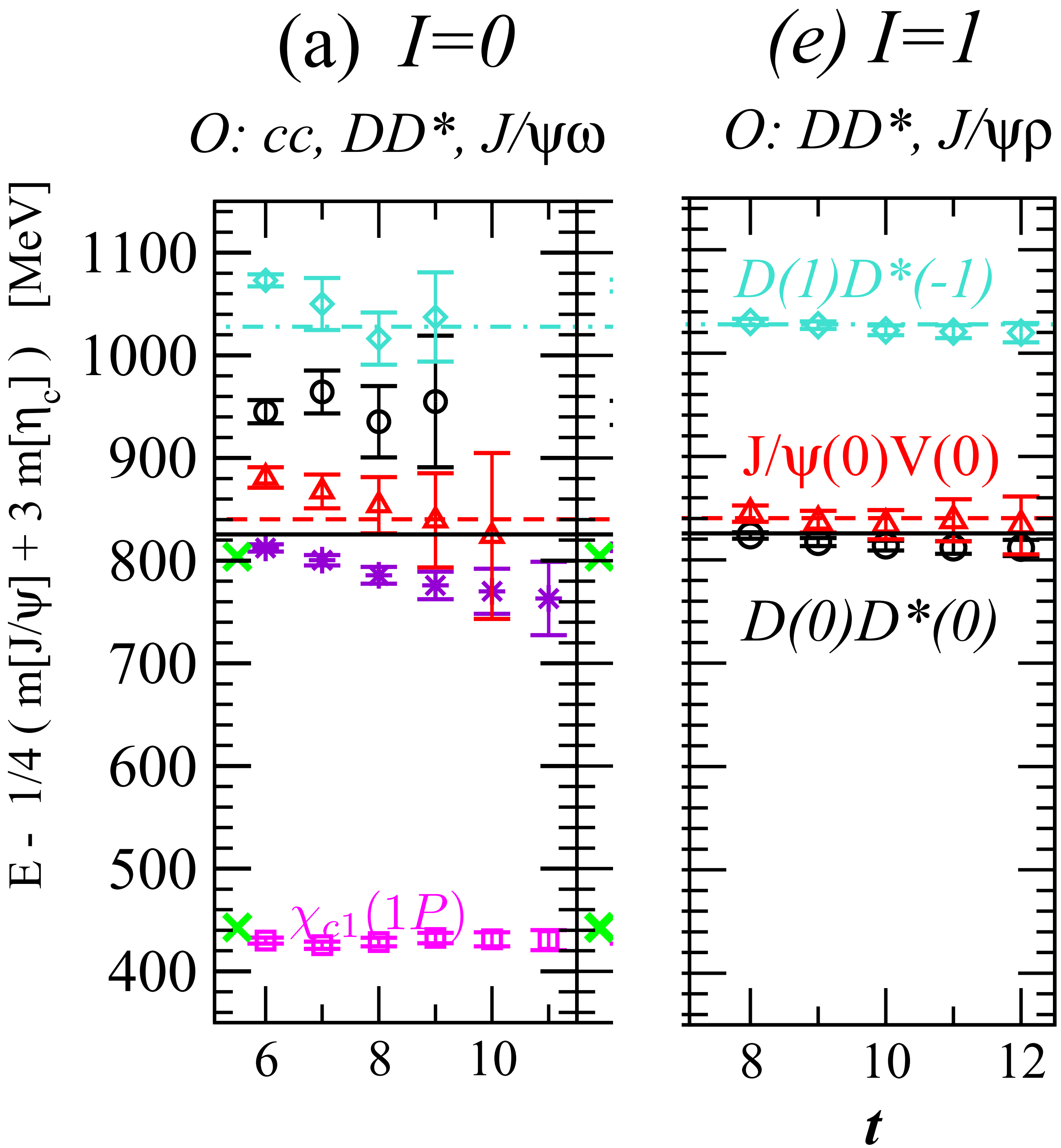
X(3872)

S. prelovsek and L. Leskovec, PRL111(2013)192001.

$$\mathrm{op} = \bar{c}c\,,\ DD^*\,,\ J/\Psi V \quad (\ J^{PC} = 1^{++}\ )$$

$$m_\pi = 266\,\mathrm{MeV}$$

$$\log(\lambda_n(t)/\lambda_n(t+1)) \sim E_n$$



one additional state is found in *I=0* system

corresponding to X(3872)

$X(3872)$	$m_X - \frac{1}{4}(m_{\eta_c} + 3m_{J/\psi})$	$m_X - (m_{D^0} + m_{D^{0*}})$
Lattice$^{L\to\infty}$	815 ± 7 MeV	-11 ± 7 MeV
Experiment	804 ± 1 MeV	-0.14 ± 0.22 MeV

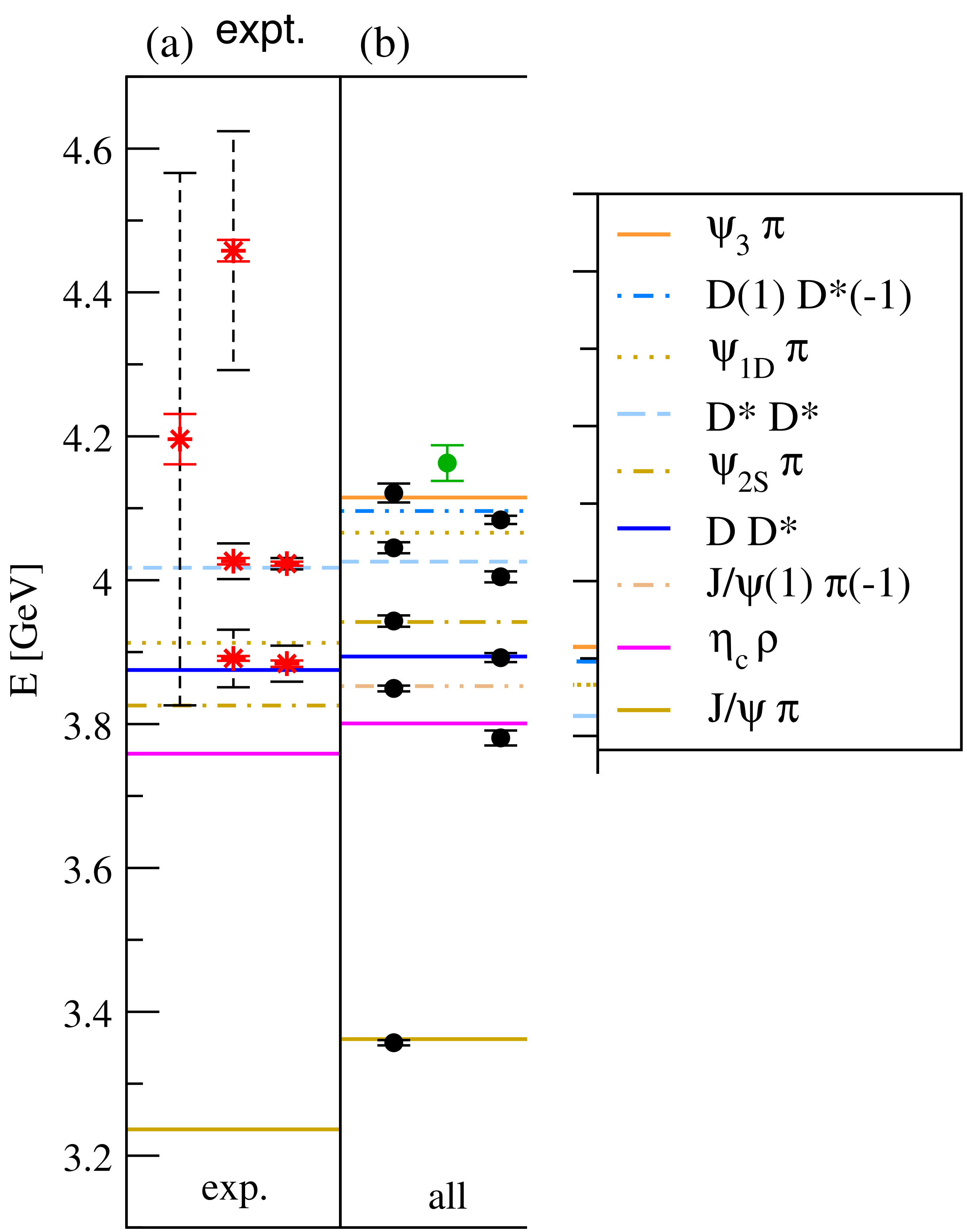
(assuming width is narrow)

Z_c^+ (3900)

S. prelovsek, C.B. Lang, L. Leskovec, D. Mohler, PRD91(2015)014504.

$\bar{c}c\bar{d}u\ ,\ I^G(J^{PC}) = 1^+(1^{+-})$

$m_\pi = 266\,\mathrm{MeV}$



ops :

$$J/\psi(0)\pi(0),\ \eta_c(0)\rho(0),\ J/\psi(1)\pi(-1),\ D(0)\bar{D}^*(0),$$
$$\psi_{2S}(0)\pi(0),\ D^*(0)\bar{D}^*(0), \psi_{1D}(0)\pi(0),\ \eta_c(1)\rho(-1),$$
$$D(1)\bar{D}^*(-1),\ \psi_3(0)\pi(0),\ J/\psi(2)\pi(-2),\ D^*(1)\bar{D}^*(-1)$$
$$D(2)\bar{D}^*(-2) \qquad (2)$$

No additional state is found for m < 4200MeV

Comments for multi SC. system

ex) $I=0$ $\pi\pi$

at allowed E

infinite volume

$\langle 0|\pi\pi(\mathbf{x})|\pi\pi;\mathbf{k}\rangle$ $\quad$ $\langle 0|K\bar{K}(\mathbf{x})|\pi\pi;\mathbf{k}\rangle$

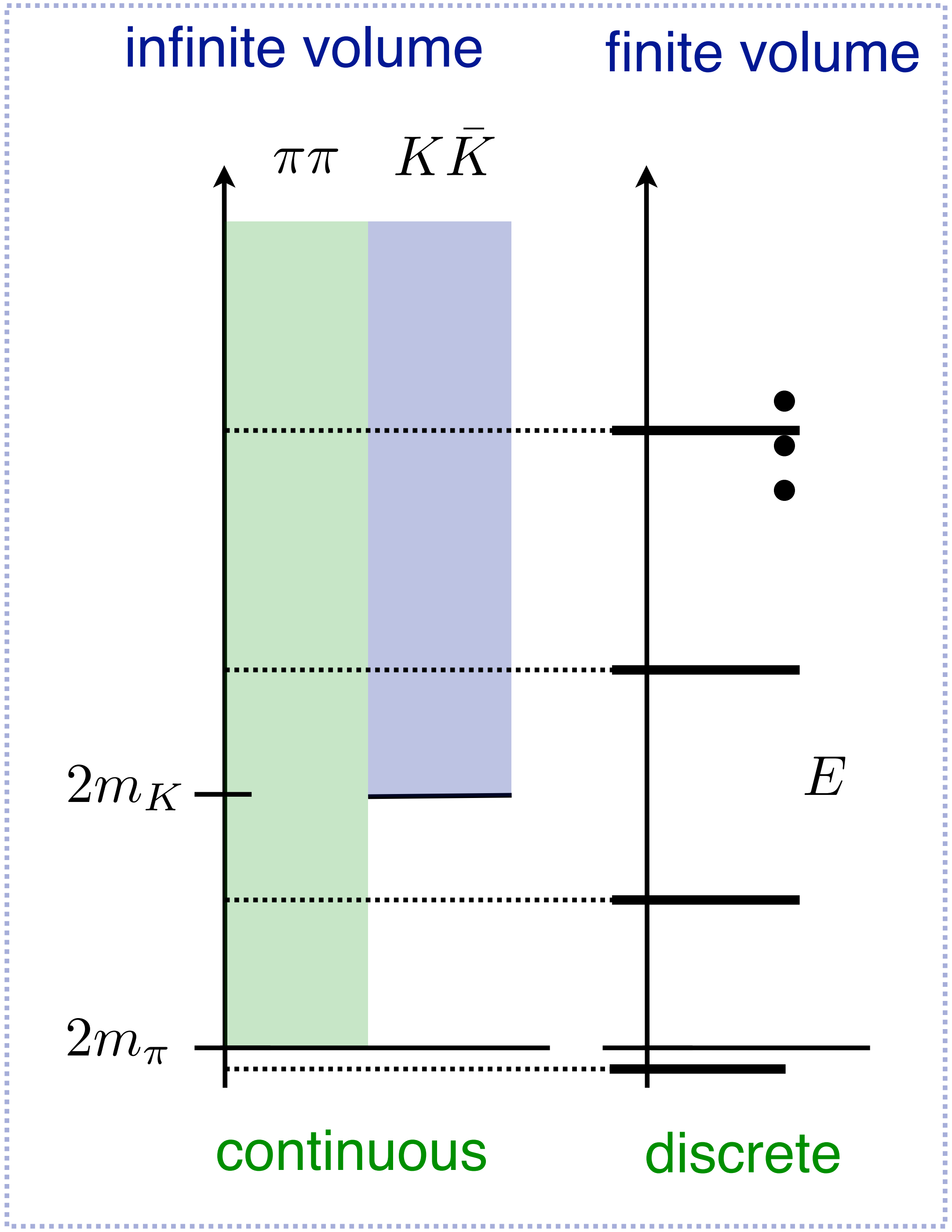
$\langle 0|\pi\pi(\mathbf{x})|K\bar{K};\mathbf{k}\rangle$ $\quad$ $\langle 0|K\bar{K}(\mathbf{x})|K\bar{K};\mathbf{k}\rangle$

physics : $\delta_{\pi\pi}(E), \delta_{K\bar{K}}(E), \eta(E)$

finite volume

$$\langle 0|\pi\pi(\mathbf{x})|E\rangle = A\cdot\langle 0|\pi\pi(\mathbf{x})|\pi\pi\rangle + B\cdot\langle 0|\pi\pi(\mathbf{x})|K\bar{K}\rangle$$

$$\langle 0|K\bar{K}(\mathbf{x})|E\rangle = C\cdot\langle 0|K\bar{K}(\mathbf{x})|\pi\pi\rangle + D\cdot\langle 0|K\bar{K}(\mathbf{x})|K\bar{K}\rangle$$



Allowed energy :

$$f(\delta_{\pi\pi}(E), \delta_{K\bar{K}}(E), \eta(E), E) = 0$$

: one relation

We can not get $\delta_{\pi\pi}(E), \delta_{K\bar{K}}(E), \eta(E)$ individually from E

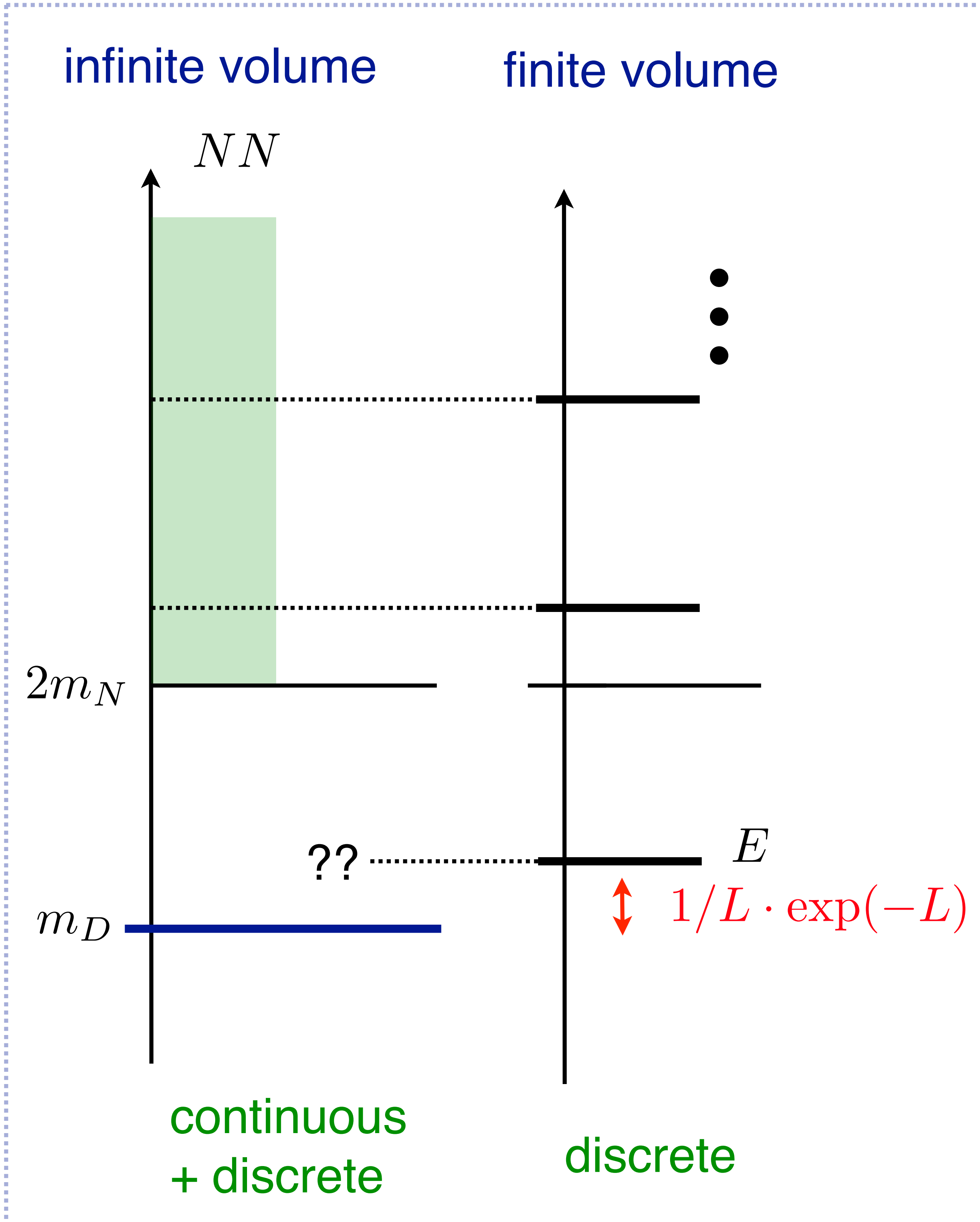
Parameterization of the scattering amplitudes.

E of a lot of eigenstates.

=> hadron interaction

2) SC. state + Bound sate

ex) Two nucleon $^3S_1$ (Deuteron bound state)



at allowed E

$$E = 2 \cdot \sqrt{m_N^2 + (i\kappa)^2} < 2m_N$$

$$\kappa \in \mathbb{R}$$

$$\phi(\mathbf{x}; i\kappa) = \sum_{lm} C_{lm} \cdot \phi_\infty^{lm}(x; i\kappa)$$

: satisfies p.BC.

→ Lüscher's formula

: relation between

E and $\delta_{NN}(i\kappa)$

for $L \to \infty$

$E \to m_D$

$\tan \delta_{NN}(i\kappa) \to -i$