Role of instanton size distributions for dynamical chiral symmetry breaking

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We study the properties of SU(N_f) light quarks in the multi-instanton vacuum of QCD. We formulate the dynamical symmetry breaking for light quarks of various numbers of flavor N_f with the inclusion of the instanton size distribution. We find that the quark mass function increases rapidly toward small Euclidean momenta when the finite size distribution is introduced with a power law fall-off of large instanton size, ρ^{-n} (n>3). We observe a confining feature of light quarks for small fall-off parameter n and for large packing fraction ρ/R in the behavior of the quark mass function extrapolated to the timelike region.

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I. INTRODUCTION

The instanton is a classical solution of the Euclidean Yang-Mills equation [1–3]. It is natural to imagine that the QCD vacuum is filled with instantons. In fact, recent lattice QCD calculations based on cooling procedures indicate clearly the existence of instantons in the QCD vacuum [4–6]. Hence, large scale features of the vacuum may be characterized by nonperturbative effects of instantons. Actually, instantons are related directly to the gluon condensate and the topological susceptibility, which is necessary to solve the U_{A}(1) problem. With the development of the instanton model of the QCD vacuum, many studies have attempted to describe the light hadron properties in terms of instantons [7,8].

One of the most important nonperturbative phenomena is chiral symmetry breaking in the QCD vacuum. Banks and Casher suggested that the chiral condensate, being the order parameter of the chiral symmetry breaking, is characterized by the value of the fermion spectral density at zero energy [9]. The massless Dirac operator has a fermion zero mode around the instanton background, which was found by ’t Hooft [2]. However, such an exact zero mode contribution to the chiral condensate vanishes in the thermodynamic limit [9,10]. Therefore, it is necessary to consider the spectral density in the instanton and anti-instanton medium. Here, the overlap integral between the positive and negative chirality fermion zero modes do contribute a finite value of spectral density at the zero energy. The chiral properties as the quark condensate and the constituent quark mass are provided by the information on the average size of instantons and their density [8]. The work done by Diakonov and Petrov, on the instanton interactions introducing the quantum fluctuations around multi-instantons, provided the repulsive force between instanton and anti-instanton, which suppressed the appearance of large size instantons [8].

The color confinement is the other important phenomenon in hadron physics. This mechanism has been discussed by the dual superconductor picture, which is conjectured by several authors [11–13]. Taking the Abelian gauge ’t Hooft verified the mechanism of the appearance of the color magnetic monopole [14]. The condensation of color monopoles induces the dual-Meissner effect to repel the color electric field in the QCD vacuum. The dual-Meissner effect provides the linearly rising potential between a static quark and antiquark. These facts are naturally expressed in the dual Ginzburg-Landau (DGL) theory [15,16]. In the DGL theory it is shown that the chiral symmetry breaking occurs if the monopole condenses [17]. These findings are supported by lattice QCD calculations [18,19].

Recently, it was found that there is a one-to-one correspondence between an instanton and a magnetic monopole in the Abelian gauge [20–24]. Therefore, instantons may be important for the promotion of long monopole loops as a signal of monopole condensation. These correlations motivated numerical works to clarify monopole feature in the multi-instanton system instead of the QCD vacuum by taking the maximally Abelian gauge [25]. In this calculation, the distribution of large size instantons is considered by assuming the size distribution falling-off with ρ^{-n} (n>3) in the infrared region. An appearance of highly complicated monopole loops at a high instanton density in Ref. [25] indicates that instantons play an essential role in the promotion of a global network of monopoles. More quantitative study for color confinement was performed by measuring the Wilson loop of the multi-instanton vacuum [26]. The multi-instanton system with the size distribution provides a linear confining static quark potential in the physically interesting region with about a half value of the experimental string tension [26]. There is an interesting statement made by Diakonov on the color confinement. The infrared behavior of instantons may be necessary to consider for the cause of confinement [27], while the features of chiral symmetry breaking depend only on the average instanton size. In Ref. [28] Diakonov and Petrov calculate the Wilson loop in an ensemble of instantons, and show that its infrared behavior for large quark-antiquark separations depend crucially on the effective distribution of instantons in their sizes. They mention that one gets a linear confinement potential if this distribution falls off as the cubic power of the instanton sizes. In this calculation, these authors assumed the size distribution of instantons and anti-instantons as ρ^{b-5} with b = \frac{1}{11}N_c - \frac{2}{3}N_f for small size instanton and ρ^{-n} (power law fall-off) for large size instanton.

These recent works motivated us to study the properties...
of SU($N_f$) light quarks in the background of multi-instantons. Thorough works were performed by Diakonov and Petrov with the assumption of the multi-instanton vacuum with a fixed size. Hence, we would like to introduce the instanton size distribution to the framework of Diakonov and Petrov.

This paper is organized as follows. In Sec. II, we review the Diakonov-Petrov (DP) formalism for the treatment of SU($N_f$) light quarks in the multi-instanton vacuum. We introduce the instanton size distribution to the DP formalism. In Sec. II A, we work out the case of one flavor quark system with the inclusion of the size distribution. In Sec. II B, we then formulate the case of an $N_f$ flavor light quark case using the bosonization method. In Sec. III, we give numerical results of the dynamical quark mass with various fall off of the instanton size distribution. In Sec. IV, we compare results in the Schwinger-Dyson approach to the dynamical quark mass with the one given in the previous section. In Section V, we show physical quantities, the quark condensate and the pion decay constant. Sec. VI is devoted to the summary of this paper.

II. EFFECTIVE ACTION OF LIGHT QUARKS IN THE MULTI-INSTANTON VACUUM

We study the behaviors of the light quarks in the multi-instanton vacuum of QCD. The QCD vacuum is assumed as an ensemble of instantons and anti-instantons. Diakonov et al. developed an effective theory for light quarks based on the multi-instanton vacuum of QCD [8,29]. In this section we will briefly review the work for further development.

In order to construct the effective fermion action, we try first to get the fermion propagator. In the background of the multi-instanton configuration, the fermion Dirac operator is given as

$$D_{\mu}(\xi) = \partial_{\mu} - iA_{\mu}(x;\xi),$$

$$A(x;\xi) = \sum_{A=1}^{N_+} A_+(x;\xi^A) + \sum_{A=1}^{N_-} A_-(x;\tilde{\xi}^A),$$

where $A_\pm$ are the fields of individual $I$'s and $\bar{I}$'s, in the singular gauge. Here, $\xi^A$ and $\tilde{\xi}^A$ denote the set of collective coordinates of the $I$th instanton and $\bar{I}$th anti-instanton, respectively. In the presence of a single instanton, the fermion propagator has a singularity in the chiral limit due to the zero mode [2].

The authors of Ref. [29] approximate it as the sum of the free propagator and the explicit contribution of the zero mode:

$$[-iD(\xi_{I\bar{I}}) + m_f]^{-1}_{\text{inst}} \approx (-i\partial)^{-1},$$

$$\Phi_{\pm}(x;\xi_{I\bar{I}})\Phi_{\pm}^+(y;\xi_{I\bar{I}})\frac{1}{m_f},$$

(2)

where $\Phi_{\pm}(x;\xi_{I\bar{I}})$ are the wave functions of the fermion zero mode in the background of one (anti-)instanton with the collective coordinates $\xi_{I\bar{I}}$. The idea of the approximation, Eq. (2), is based on the eigen-state representation of the fermion Dirac operator. Using the eigen-states of the operator, 

$$-iD(\xi)\Phi_n(x;\xi) = \lambda_n\Phi_n(x;\xi),$$

the propagator is expressed as

$$[-iD(\xi_{I\bar{I}}) + m_f]^{-1}_{\text{inst}} = \sum_{n=0}^{N_f} \frac{\Phi_{\pm}(x;\xi_{I\bar{I}})\Phi^+_{\pm}(y;\xi_{I\bar{I}})}{\lambda_n + m_f} + \frac{\Phi_{\pm}(x;\xi_{I\bar{I}})\Phi^+_{\pm}(y;\xi_{I\bar{I}})}{m_f},$$

(3)

In the chiral limit, $m_f\rightarrow 0$, the second term on the right-hand-side (RHS) dominates around the instanton. Hence, the authors of Ref. [29] substitute the free propagator for the first term on the RHS as a good approximation.

We find then that Eq. (2) is given by an approximate fermion action:

$$\exp(-\tilde{S}^{(\tilde{I})}([\tilde{\psi},\psi])) \approx \exp\left(-\sum_{I=0}^{N_f} d^4x\tilde{\psi}_f(-i\partial)\psi_f\right)$$

$$\times \prod_{I=0}^{N_f} \left(1 + \frac{1}{m_f}V^{g^I}_{\pm}([\tilde{\psi}_f,\psi_f])\right),$$

(4)

with

$$V^{g^I}_{\pm}([\tilde{\psi}_f,\psi_f]) = \int d^4x\tilde{\psi}_f(-i\partial)\Phi_{\pm}(x;\xi_{I\bar{I}})$$

$$\times \int d^4y\Phi^+(y;\xi_{I\bar{I}})(-i\partial)\psi_f(y).$$

(5)

We obtain the expression, Eq. (2), for the quark propagator following the standard procedure of getting the propagator from this action. Here we write down explicitly the expression of $\Phi_{\pm}(x;\xi_{I\bar{I}})\Phi^+_{\pm}(y;\xi_{I\bar{I}})$ to show their dependence on the collective coordinates [2,8],

$$\frac{1}{\pi} \frac{1}{(x^2 + \rho^2)^{1/2}} \frac{\rho}{\sqrt{x^2}}.$$

(7)

The collective coordinates, $z$, $U$, and $\rho$, represent the center of (anti-)instanton, the color, orientation, and the size, respectively.

Assuming the diluteness of the instanton medium, we obtain the fermion action in the background of a multi-instanton configuration by the product of contributions from each flavor and (anti-)instanton. Then the averaged fermion determinant is obtained as
We note here that the saddle point in $l$ we employ an ansatz.

For instance, in the case of $N_f$ flavors, $N_f V_{\lambda=1}$.

we have taken the flavor symmetry, $m_1 = \ldots = m_{N_f} = m$. The instanton size distribution is denoted by $f(\rho)$ and we employ an ansatz [25],

$$f(\rho) = f_n^{(N_f)}(\rho) = \frac{\rho^{N_f}}{(\rho^2 \rho_1)^a (\rho^2 \rho_2)^{-b}}$$

with $b^N_{c,2} N_f$. We fix the parameters, $\rho_1$ and $\rho_2$, by the conditions,

$$\int_0^\infty d\rho f(\rho) = 1, \quad \int_0^\infty d\rho f(\rho) \rho = \rho_b.$$  \hspace{1cm} (11)

For instance, in the case of $N_f = 3$, $N_f = 2$, and $n = 3$, those parameters are given by $\rho_1 / \rho_b = 1.101$ and $\rho_2 / \rho_b = 0.659$.

To proceed we expand the $N_f$ products (the large parentheses) in Eq. (9) and get a series of terms characterized by the power of $1/m$. Then we perform the integration over the color space rotation variables, $f dU_{H_{\lambda}}$, to leading order in $1/N_c$ [29]. For instance, the term of order $O(1/mN_f)$ consists of $\Gamma_N f_n^{(N_f)}[\bar{\psi}_f, \psi_f]$ and for this term the integral over the color space rotation variables leads,

$$\frac{1}{mN_f} \int dU_{H_{\lambda}} \prod_{f=1}^{N_f} \Gamma_N^{(l^{(f)}_{\lambda})}[\bar{\psi}_f, \psi_f] = \det_{N_f}^{(z_{\lambda})} J_{\lambda f}^{(z_{\lambda})}(l^{(f)}_{\lambda}),$$

(12)

where the currents, $J_{\lambda f}(z, \rho)$, are color singlets and $N_f \times N_f$ matrices in flavor,

$$J_{\lambda f}^{(z_{\lambda})}(z, \rho) = \frac{4\pi^2}{mN_c} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \times e^{-i(k-l)\rho} F(|l| \rho) \times \left\{ \bar{\psi}_f(k) \frac{1 + \gamma^5}{2} \psi_f(l) \right\}.$$  \hspace{1cm} (13)

In Eq. (12) the notation $\det_{N_f}$ means a determinant of the matrix $J_{\lambda f}$ in the flavor space. $F(|l| \rho) \rho^2 F(|l| \rho)$ in Eq. (13) is a form factor originating from the wave function of the zero mode in momentum space,

$$2\pi \rho^2 \frac{k^4}{|l|^2} F(|l| \rho) = \int d^4x e^{ik \cdot x} k^4 \Phi(x, \rho).$$  \hspace{1cm} (14)

The explicit form of $F(|l| \rho)$ is given as

$$F(|l| \rho) = 2 \left\{ t [I(t)K_1(t) - I_1(t)K_0(t)] - I_1(t)K_1(t) \right\} \quad \text{with} \quad t = \frac{1}{2} |l| \rho.$$  \hspace{1cm} (15)

where $I$ and $K$ are modified Bessel functions of the first and second kinds, respectively. The single-instanton averages, Eq. (9), are finally expressed as [29]

$$W_{\lambda} = \int \frac{d^4z}{V} d\rho f(\rho) d\rho [J_{\lambda f}(z, \rho) + 1],$$

(16)

where the determinants are over flavor indices.

After these preparations, we would like to come back to the averaged fermion determinant, $\bar{\det}_{N_f}$ in Eq. (8). In the thermodynamic limit, $N \rightarrow \infty$, $V \rightarrow \infty$, with $N / V$ fixed, we use the formula [30]

$$(ab)^N = \int \frac{d\lambda}{2\pi} \exp \left\{ N \log \frac{aN}{\lambda} - N + \lambda b \right\},$$

(17)

which is valid for large $N$. Hence, we can write Eq. (8) as

$$\bar{\det}_{N_f} = \int \frac{d^4z}{V} \det [\bar{\psi}_f(-i\bar{\psi}_f)] \int \frac{d\lambda_+}{2\pi} \frac{d\lambda_-}{2\pi} \exp \left\{ N \log \frac{N_{\lambda_+}}{\lambda_+ V} - N_{\lambda_-} + \lambda_+ \right\} d^4z d\rho f(\rho)$$

$$\times \det [J_+(z, \rho) + 1] + [\cdots - \cdots].$$  \hspace{1cm} (18)

We note here that the saddle point in $\lambda_\pm$ in the thermodynamic limit leads to the original expression for $\bar{\det}_{N_f}$ in Eq. (8).

A. $N_f = 1$ case

In order to calculate the averaged fermion determinant in Eq. (18), we have to perform integration over the fermion fields. This integration can be done for the case of one flavor, $N_f = 1$,
The saddle point of \( \mathcal{W}(\lambda_+, \lambda_-) \) in the variables \( \lambda_\pm \) is given by the variation,

\[
\frac{\partial}{\partial \lambda_\pm} \mathcal{W}(\lambda_+, \lambda_-) = 0.
\]

In order to write down Eq. (21) explicitly, we would like to redefine the variables \( \lambda_\pm \) as

\[
\lambda_\pm = \frac{mN_f}{4\pi^2\rho^2} \mu_\pm.
\]

With the help of this, we obtain the condition to fix the saddle point values of \( \lambda_\pm \) (corresponding to \( \mu_\pm \)),

\[
\frac{V}{2N_\pm} \left\{ 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M_+(k^2)M_-(k^2)}{k^2 + M_+(k^2)M_-(k^2)} + \frac{mN_c}{2\pi^2\rho^2} \mu_\pm \right\} = 1,
\]

where we have defined

\[
M_\pm(k^2) = \mu_\pm \int d\rho f_n^{(1)}(\rho) \left( \frac{\rho}{\rho^2} \right)^2 F^2(|k|\rho).
\]

B. Bosonization method: The case of \( \Nf \) larger than one

In the case of \( \Nf = 1 \), the integration over the fermion fields can be done exactly as shown in the previous section. However, for the case of multiple flavors, \( \Nf \geq 2 \), the instanton-fermion vertex becomes \( 2\Nf \)-fermionic interaction. To carry out the integral over the fermion fields, we introduce auxiliary boson fields to linearize the exponent of Eq. (18). This can be done using the formula [29]

\[
\exp(\lambda \det[J]) = \int d\mathcal{B} \exp[-(N_f-1)\lambda^{-1/(N_f-1)}(\det \mathcal{B})^{1/(N_f-1)}] + \text{tr}(\mathcal{B}J),
\]

which holds in the saddle point approximation. Here, \( \mathcal{B} \) is a Hermitian \( \Nf \times \Nf \) matrix variable, which contains scalar and pseudoscalar bosons with flavor singlet and multiplet, triplet for \( \Nf = 2 \), octet for \( \Nf = 3 \), and so on. Using Eq. (25), we rewrite Eq. (18) as

\[
\overline{\text{Det}}_{N_\pm} = \int D\bar{\psi} D\psi \exp \int d^4x \left\{ -\sum_f N_f \bar{\psi}_f(-i\gamma^5 d)_f \psi_f \right\} \int \frac{d\lambda_+}{2\pi} \frac{d\lambda_-}{2\pi} \exp \left\{ N_+ \log \left( \frac{N_+}{\lambda_+ V} \right) - N_+ + [\rightarrow +] \right\}
\]

\[
\times \int D\mathcal{B}_+ D\mathcal{B}_- \exp \int d^4x \int d\rho \left\{ -(N_f-1)\lambda f_n^{(N_f)}(\rho)^{-1/(N_f-1)}(\det \mathcal{B}_+(x,\rho))^{1/(N_f-1)} \right.
\]

\[
+ \text{tr}(\mathcal{B}_+(x,\rho)(J_+(x,\rho) + 1)) [\rightarrow +] \right\}.
\]

The saddle point conditions of \( \lambda_\pm \) are given as

\[
\lambda_\pm^{1/(N_f-1)} = \frac{1}{N_\pm} \int d^4x \int d\rho f_n^{(N_f)}(\rho)^{-1/(N_f-1)}(\det \mathcal{B}_+(x,\rho))^{1/(N_f-1)}.
\]
\[ \text{Det}_{\mathcal{N}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ \frac{N_f}{2} \bar{\psi}_J (-i\partial) \psi_J \right\} \]
\[ \times \int \mathcal{D}B_+ \mathcal{D}B_- \exp \left\{ \int d^4x \left[ \rho \text{tr}(B_+(x,\rho)(J_+(x,\rho)+1)) + [\cdots] \right] \right\} \]
\[ \times \exp \left[ -N_+(N_f-1) \log \left( \int d^4x \int d\rho f_n^{(N_f)}(\rho)^{1/(N_f-1)}(\text{det} B_+(x,\rho))^{1/(N_f-1)} + [\cdots] \right) \right]. \tag{28} \]

We assume that the diagonal elements of \( B \) are dominant and \( x \) independent, and factorize the instanton size dependence of \( B \) as
\[ B_\pm(x,\rho) = f_n^{(N_f)}(\rho) \text{diag}(B_{\pm 1}(\rho), \ldots, B_{\pm N_f}(\rho)). \tag{29} \]

We search for a minimum point of the exponent of Eq. (28),
\[ \frac{\partial}{\partial B_{\pm f}(\rho)} \mathbb{W}(B_{+ f}(\rho), B_{- f}(\rho)) = 0, \tag{30} \]
where
\[ \mathbb{W}(B_{+ f}(\rho), B_{- f}(\rho)) = V \sum_f \int d\rho f_n^{(N_f)}(\rho) B_{+ f}(\rho) - N_+(N_f-1) \log \left( \int d\rho f_n^{(N_f)}(\rho) \left( \prod f_n^{(N_f)}(\rho) \right)^{1/(N_f-1)} \right) + [\cdots] \]
\[ + \sum_f \text{Sp} \log \left( \frac{k + \int d\rho f_n^{(N_f)}(\rho) \left[ B_{+ f}(\rho) + B_{- f}(\rho) \right] - \left[ B_{+ f}(\rho) - B_{- f}(\rho) \right] \gamma^5 \frac{\pi^2 \rho^2}{mN_c} \left( \rho \right)^2 F^2(|k|\rho)}{k} \right). \tag{31} \]

We shall write the condition Eq. (30) explicitly below. In order to do this, we would like to redefine the variables \( \lambda_\pm \) by \( \mu_\pm^{(N_f)} \) and \( B_{\pm f} \) by \( b_{\pm f} \) as
\[ \lambda_\pm = \left( \frac{V}{N_\pm} \right)^{N_f-1} \left( \frac{mN_c}{4\pi^2 \rho^2} \right)^{N_f} \]
\[ B_{\pm f}(\rho) = \frac{mN_c}{4\pi^2 \rho^2} b_{\pm f}(\rho). \tag{32} \]

Introducing \( M_{\pm f}^{(N_f)} \) for convenience,
\[ M_{\pm f}^{(N_f)}(k^2) = \int d\rho f_n^{(N_f)}(\rho) b_{\pm f}(\rho) \left( \frac{\rho}{\sqrt{\rho}} \right)^2 F^2(|k|\rho), \tag{33} \]
we can write the equation corresponding to Eq. (30) as
\[ \frac{V}{2N_\pm} \int \frac{4N_c}{(2\pi)^4} \frac{M_{\pm f}^{(N_f)}(k^2)}{k^2 + M_{\pm f}^{(N_f)}(k^2)} M_{\mp f}^{(N_f)}(k^2) \]
\[ + \frac{mN_c}{2\pi^2 \rho^2} \int d\rho f_n^{(N_f)}(\rho) b_{\mp f}(\rho) = 1. \tag{34} \]

On the other hand, Eq. (27) is rewritten as
\[ \frac{\partial}{\partial f_n^{(N_f)}(\rho)} \mathbb{W} = \mu_\pm^{(N_f)} = \mu_\pm \int d\rho f_n^{(N_f)}(\rho) \left( \frac{\rho}{\sqrt{\rho}} \right)^2 F^2(|k|\rho). \tag{35} \]

In the case of \( N_f = 1 \) the dynamical quark mass is given by Eq. (24) which corresponds to the general mass function, Eq. (33), with \( b_{\pm f}(\rho) = \mu_\pm f_n^{(1)}(\rho)/f_n^{(N_f)}(\rho) \). As found in Eq. (34) clearly, \( b_{\pm f}(\rho) \) given by \( \mu_\pm f_n^{(1)}(\rho)/f_n^{(N_f)}(\rho) \) is a solution of Eq. (34). In the large \( \rho \) region the ratio \( f_n^{(1)}(\rho)/f_n^{(N_f)}(\rho) \) is going to 1 as \( \rho \) is increased and in the small \( \rho \) region \( f_n^{(1)}(\rho)/f_n^{(N_f)}(\rho) \sim \rho^{(N_f-1)} \). Finally we obtain the saddle point condition for general \( N_f \) expressed as
\[ \mu_\pm^{(N_f)} = \mu_\pm \int d\rho f_n^{(N_f)}(\rho) \left( \frac{f_n^{(1)}(\rho)}{f_n^{(N_f)}(\rho)} \right)^{N_f/(N_f-1)(N_f-1)/N_f}. \tag{36} \]

The mass function is given by
\[ M_{\pm f}^{(N_f)}(k^2) = M_{\pm}(k^2) = \mu_\pm \int d\rho f_n^{(1)}(\rho) \left( \frac{\rho}{\sqrt{\rho}} \right)^2 F^2(|k|\rho). \tag{37} \]

Here \( \mu_\pm \) is the solution of the equation.
We note that the mass function $M_6(k^2)$ has a singularity at $k^2=0$, if the size distribution $f(r)$ decreases as $r^{-3}$, or more weakly, in the large $r$ region. Hence, we must restrict ourselves in the region $n>3$, where $n$ is the power of the distribution function $f(r)$ at large $r$. $\rho^{-n}$.

III. NUMERICAL RESULTS FOR THE MASS FUNCTION

The free parameters are the average size of instanton, $\bar{r}$, the packing fraction, $\bar{\rho}/\bar{R}$ with $\bar{R}^4=V(N_++N_-)$, and the instanton size distribution function, $f(r)$. In Fig. 1 we show the results of mass functions with several size distributions characterized by $n$ of Eq. (10). All the calculations are done with $N_+=3$ and $N_+=N_-$ in the chiral limit $m=0$. We fix the average size of instanton, $\bar{r}=1/3$ fm. The dashed curves correspond to the results with the fixed instanton size [8].

In the infrared region, $|k|<300$ MeV, the mass function depends largely on the instanton size distribution, while in the large momentum region, $|k|>300$ MeV, the difference is not much. With the distribution function, $n=3.1$, the mass function has quite a peculiar behavior. As mentioned before, with $n=3$ the mass function has a singularity at $|k|=0$. Diankonov mentioned in Ref. [27] that it corresponds, in a sense, to the confinement of light quarks. The large size instantons make large contributions to the behavior of light quarks at small momenta.

Several authors discuss the extrapolation of the mass

FIG. 1. The dynamical quark mass, $M(k^2)$, as a function of the momentum in the Euclidean space, $|k|$. They are shown in unit of MeV. The dashed curve is the mass function for the case of the fixed instanton size, $\delta(\rho-\bar{\rho})$. The three curves are those with the fall-off of the large instanton size distribution in the form of $f(\rho)\sim\rho^{-n}$ with $n=5$, 4 and 3.1 from the bottom. Figure (a) is the case for the packing fraction $\bar{\rho}/\bar{R}=0.2$, (b) for $\bar{\rho}/\bar{R}=0.25$ and (c) for $\bar{\rho}/\bar{R}=0.3$.

\[
\frac{V}{2N_\pm}\left\{4N_c \int_0^{k_\pm} \frac{dk}{(2\pi)^2} \frac{M_+(k^2)M_-(k^2)}{k^2+M_+(k^2)M_-(k^2)} + \frac{mN_c}{2\pi^2} \right\} = 1.
\]

FIG. 2. The quark mass function extrapolated to the timelike region; i.e., the negative Euclidean squared momentum, as a function of the squared momentum. Figure (a) is the case for the packing fraction $\bar{\rho}/\bar{R}=0.2$, (b) for $\bar{\rho}/\bar{R}=0.25$ and (c) for $\bar{\rho}/\bar{R}=0.3$. The curve, $M(k^2)=-k^2$, is shown by the solid line. The straight extrapolation is made by using the derivative at $k^2=0$. 

We have calculated the mass function $M_\pm(k^2)$ by the numerical integration of the mass function on the $k^2$ axis.

\[
M_\pm(k^2) = \frac{1}{2\pi^2} \int_0^{k_\pm} \frac{dk}{k^2} M(k^2) = \frac{1}{2\pi^2} \int_0^{k_\pm} \frac{dk}{k^2} \left( \rho_{\pm}(\pm k^2) \right)
\]

\[
\rho_{\pm}(\pm k^2) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{k^2}{\rho_{\pm}} \right) \right\}
\]
dashed curve is that for the case of the fixed instanton size. The subsequent solid curves are those for various instanton size distributions specified by the power $n$ of $\rho^{-n}$ with $n = 5, 4,$ and $3.1$ from the bottom. $M(0)$ increases rapidly as $n$ approaches $n \sim 3$ for a fixed packing fraction, $\bar{\rho}/\bar{R}$.

**IV. SCHWINGER-DYSON APPROACH TO THE DYNAMICAL QUARK MASS**

In the previous sections we have considered the saddle point of the effective action using the bosonization formalism. In that formalism we have employed the approximation where the boson field is constrained on the diagonal elements in the flavor space, Eq. (29). Using the obtained saddle point values we calculate physical quantities. In order to calculate physical quantities as the quark condensate, $(\bar{\psi}\psi)$, and the pion decay constant, $f_\pi$, we should know the quark propagator. The quark propagator is given by

$$S(q) = \int d^4x d^4y \langle 0 |(\bar{\psi}(x)\psi(y))|0\rangle e^{-i q \cdot (x - y)}.$$  \hspace{1cm} (39)

We employ the Schwinger-Dyson approach to the dynamical quark mass,

$$S(q) = S_0(q) + S_0(q)K(q)S(q),$$  \hspace{1cm} (40)

where $K$ corresponds to the kernel contributed by one instanton and anti-instanton in leading of $1/N_c$. In general there are other contributions to the kernel, consisting of multi-instanton vertices. Each vertex is proportional to $(1/N_c)^N$. Hence, in the large $N_c$ limit, the contributions from the multi-instanton vertices are suppressed. Moreover, the contributions from one instanton and anti-instanton with $(N_f - 1)$ quark loops are dominant. We employ the kernel contributed by one instanton and anti-instanton with $(N_f - 1)$ quark loops, in the large $N_c$ limit.

As usual, Eq. (40) is rewritten as

$$S^{-1}(q) = S_0^{-1}(q) - K(q).$$  \hspace{1cm} (41)

We shall write down the explicit formula for $S^{-1}(q)$,

$$S^{-1}(q) = 4 + \frac{1 - \gamma^5}{2} \mu_+^{(N_f)} \int d\rho f_n^{(N_f)}(\rho) \left( \frac{\rho}{\bar{\rho}} \right)^2 F^2(|q|\rho) \left( \frac{\mu_+^{(N_f)}}{2N_c} \right)^2 \left( \frac{d^4 k}{(2\pi)^4} \right)^2 F^2(|k|\rho) \times \text{Tr} \left[ 1 - \gamma^5 \frac{S(k)}{2\pi^2 \rho^2} \right] + \frac{mN_c}{2\pi^2 \rho^2} \right]^{N_f - 1} + \frac{1 + \gamma^5}{2} \mu_-^{(N_f)} \int d\rho f_n^{(N_f)}(\rho) \left( \frac{\rho}{\bar{\rho}} \right)^2 F^2(|q|\rho) \times \left( \frac{\mu_-^{(N_f)}}{2N_c} \right)^2 \left( \frac{d^4 k}{(2\pi)^4} \right)^2 F^2(|k|\rho) \text{Tr} \left[ 1 + \gamma^5 \frac{S(k)}{2\pi^2 \rho^2} \right] + \frac{mN_c}{2\pi^2 \rho^2} \right]^{N_f - 1}.$$  \hspace{1cm} (42)

Here $\mu_\pm^{(N_f)}$ are given by Eq. (36). We parametrize the quark propagator,

$$S^{-1}(q) = 4 + \frac{1 - \gamma^5}{2} M_+^{(N_f)}(q^2) + \frac{1 + \gamma^5}{2} M_-^{(N_f)}(q^2).$$  \hspace{1cm} (43)
which is required by the Lorentz invariance.

After some manipulations we find that

$$\mathcal{M}^{(N_f)}(q^2) = \mu^{(N_f)} \int d\rho f_n^{(N_f)}(\rho) \left( \frac{\rho}{\rho} \right)^2 F^2(|q|\rho) \times [P_\pm(\rho)]^{N_f-1},$$

where

$$P_\pm(\rho) = \mu^{(N_f)} \left( \frac{V}{2N^2} \right) \left( \frac{4N_c}{(2\pi)^4} \right)^{\frac{2}{N_f}} \times \frac{1}{\rho^2} \left( \frac{M^{(N_f)}(k^2)}{k^2 + \mathcal{M}^{(N_f)}(k^2)} \right)^{N_f} \left( \frac{1}{1 + \mathcal{M}^{(N_f)}(k^2)} \right)^{N_f} \frac{mN_c}{2\pi^2 \rho^2} \mu^{(N_f)} \int d\rho f_n^{(N_f)}(\rho) P_\pm(\rho).$$

In order to compare $\mathcal{M}^{(N_f)}(q^2)$ to the mass functions obtained in the bosonization procedure, Eq. (37), we define variables $P_\pm(\rho)$ as

$$P_\pm(\rho) = [P_\pm(\rho)]^{N_f-1}. \quad (46)$$

Using the variables $P_\pm(\rho)$ we rewrite Eqs. (44), (45) as

$$\mathcal{M}^{(N_f)}(q^2) = \mu^{(N_f)} \int d\rho f_n^{(N_f)}(\rho) \left( \frac{\rho}{\rho} \right)^2 F^2(|q|\rho) P_\pm(\rho), \quad (47)$$

and

$$\int d\rho f_n^{(N_f)}(\rho) \left( P_\pm(\rho) \right)^{N_f/(N_f-1)}$$

$$ = \frac{V}{2N^2} \left( \frac{4N_c}{(2\pi)^4} \right)^{\frac{2}{N_f}} \times \frac{1}{\rho^2} \left( \frac{M^{(N_f)}(k^2)}{k^2 + \mathcal{M}^{(N_f)}(k^2)} \right)^{N_f} \left( \frac{1}{1 + \mathcal{M}^{(N_f)}(k^2)} \right)^{N_f} \frac{mN_c}{2\pi^2 \rho^2} \mu^{(N_f)} \left( \frac{1}{1 + \mathcal{M}^{(N_f)}(k^2)} \right)^{N_f} \int d\rho f_n^{(N_f)}(\rho) P_\pm(\rho). \quad (48)$$

Obviously

$$P_\pm(\rho) = \frac{\mu \int f_n^{(1)}(\rho)}{\mu \int f_n^{(N_f)}(\rho)}$$

satisfy Eq. (48). With this condition the mass functions, Eq. (47), are given by the same equations as those in the bosonization procedure. This suggests that the approximation for the boson field, Eq. (29), corresponds to the Schwinger-Dyson approach to the dynamical quark mass in the large-$N_c$ limit.

**V. QUARK CONDENSATE AND PION DECAY CONSTANT**

Using the obtained dynamical quark masses depending on the packing fraction and the size distribution function, we calculate the vacuum quark condensate and the pion decay constant. In Fig. 4 we show the results of the quark condensate as a function of the packing fraction. The quark condensate is given by

$$\langle \bar{\psi} \psi \rangle = -4N_c \int \frac{dk^2}{(2\pi)^4} \frac{M(k^2)}{k^2 + M^2(k^2)}.$$

A reasonable value is obtained for the packing fraction of $\bar{\rho}/R=0.3$ for the case of $f(\rho) = \delta(\rho - \bar{\rho})$ as well known [8]. The quark condensate increases slightly with the use of the size distribution. It increases monotonically with the packing fraction, $\bar{\rho}/R$.

For the pion decay constant we employ the Pagels-Stokar formula [32]

$$f_\pi = \frac{N_c}{(2\pi)^2} \int_0^\infty dk^2 \frac{k^2}{\{k^2 + M^2(k^2)\}^2} \times \left( M^2(k^2) - \frac{k^2}{2} \frac{dM^2(k^2)}{dk^2} M(k^2) \right).$$

The results are shown in Fig. 5. The pion decay constant is
very insensitive to the size distribution. It increases monotonically with the packing fraction, $\bar{\rho}/R$, like the quark condensate presented above.

VI. SUMMARY

We have studied the properties of light SU($N_f$) flavor quarks in the multi-instanton vacuum of QCD. We have followed closely the work of Diakonov and Petrov (DP) in the formulation of the light quark propagator in terms of the quark zero-mode in the presence of multi instantons. We have newly added the size distribution of instantons into the DP formalism.

In the first place, we have studied the dynamical chiral symmetry breaking of light quarks with the bosonization method. The introduction of the instanton size distribution affects the behavior of the dynamical quark mass in the infrared region. As the size distribution approaches $n = 3$ from above in the expression of $f(\rho) = \rho^n$, the mass function increases at the zero momentum, $M(q^2 = 0) \rightarrow \infty$. This case was conjectured by Diakonov as the indication of the color confinement [27]. The difference between the dynamical quark masses calculated with various instanton size distributions is particularly large in the infrared region. We expect that the difference in the infrared region affects for instance the chiral condensate increases slightly. On the other hand, the pion decay constant is insensitive to the size distribution.

In conclusion, we have found that the use of the size distribution of instantons influences largely the behavior of the dynamical quark mass in the infrared region. The simple extrapolation of the mass function into the timelike region seems to indicate a confining feature of quarks as the size distribution parameter $n$ being reduced toward $n = 3$ and the packing fraction is increased. It is, therefore, very interesting to calculate the meson properties with the size distribution of instantons to see if a confining feature is seen as a discrete spectrum for mesons. This is an important next step to be worked out.

[28] D. Diakonov and V. Petrov, Non-perturbative Approaches to Quantum Chromodynamics (Petersburg Nuclear Physics Institute, Gatchina, 1995), p. 239.