Unique forbidden beta decays and neutrino mass

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Outlook

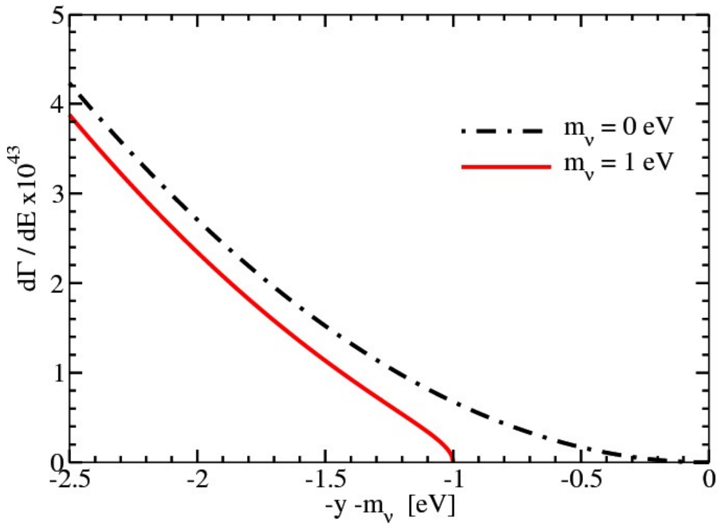
- Kurie function for allowed beta decays
- Unique forbidden beta decays
- Effect of Fermi functions
- Non unique first forbidden beta decay
- Comparison of Kurie plots for allowed and forbidden beta decays

Standard approach

$$^{3}H \rightarrow {}^{3}He + e^{-} + \tilde{\nu}_{e}$$

1934 – Fermi pointed out that shape of electron spectrum in beta decay near the endpoint is sensitive to neutrino mass

E. Fermi, Z. Phys. **88** (1934) 161



Endpoint beta spectrum

First measurement by G. Hanna, B. Pontecorvo:
Phys. Rev. **75,** 983 (1949) with estimation $m_{\nu} \sim 1$ keV

Standard approach

For the $\Delta J^{\pi} = 1^{+}$ transition the electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{G_\beta^2}{2\pi^3} g_A^2 |M|^2 p_e E_e F_0(Z_f, E_e)(E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$|M|^2 = \frac{1}{2J_i + 1}\left|\left\langle J_f \left\| \sum_n \tau_n^+ \sigma_1(n) \right\| J_i \right\rangle\right|^2$$

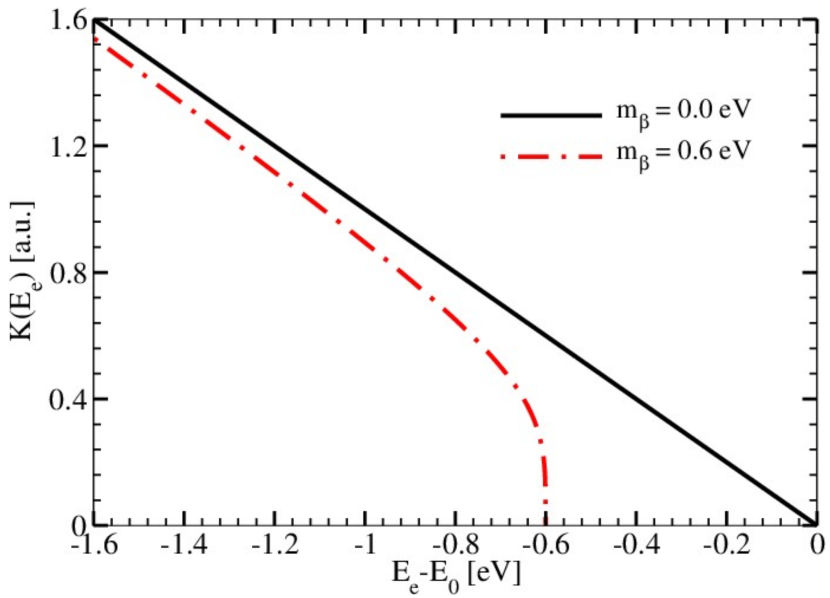
$$m_\beta^2 = \sum_{k=1}^{3} |U_{ek}|^2 m_k^2$$

Standard approach

Franz Newell Devereux Kurie
credits to LBNL

$$K(E_e) = \sqrt{\frac{d\Gamma/dE_e}{p_e E_e F_0(Z_f, E_e)}} = \frac{G_\beta g_A |M|}{\sqrt{2\pi^3}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}}$$

The advantage of Kurie plot is that non-linearity implies non-zero neutrino mass.



Standard approach

For the reasons that will be clear later we can rewrite the expression as:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{1^+} p_e E_e F_0(Z_f, E_e)(E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$B_{1^+} = \frac{1}{2J_i + 1} \left| \left\langle J_f \left\| \sum_n \tau_n^+ \{\sigma_1(n) \otimes Y_0(n)\}_1 \right\| J_i \right\rangle \right|^2$$

A constant factor : $Y_0 = \frac{1}{\sqrt{4\pi}}$

$$\frac{1}{2\pi^3} G_\beta^2 \rightarrow \frac{2}{\pi^2} G_\beta^2$$

Unique forbidden beta decays

First unique forbidden beta decay

$$\Delta J^{\pi} = 2^{-}$$

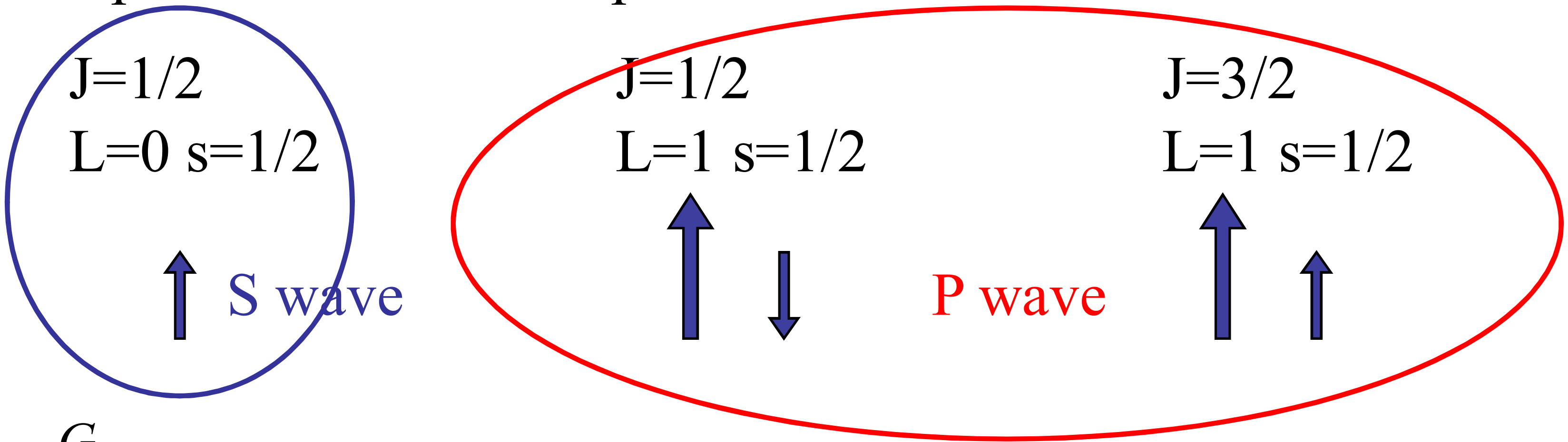
Nucleus:	^{79}Se	$^{93}Zr^{*}$	^{107}Pd	$^{135}Cs^{*}$	^{182}Hf	^{187}Re
Q[keV]:	151	60	34.1	0.5	104.6	2.469

* - decay to the excited nuclear state

First unique forbidden beta decay

The change of the angular momentum and parity between mother and daughter nuclei : $\Delta J^{\pi} = 2^{-}$

The electron is emitted in the presence of the Coulomb field of the daughter nucleus, therefore the wave function is expressed in terms of spherical waves



$$H_{\beta} = \frac{G_{\beta}}{\sqrt{2}} \overline{\psi}_e(x)\gamma^{\mu}(1-\gamma_5)\psi_{\nu}(x) j_{\mu}(x) + h.c.,$$

$$\psi_e = \psi_{S_{1/2}} + \psi_{P_{1/2}} + \psi_{P_{3/2}} + \ldots, \quad \psi_{\nu}(\vec{r}) \cong (1 + i\vec{k}.\vec{r})\mathrm{v}(k)$$

First unique forbidden beta decay

Electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{2^-} p_e E_e (E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$
$$\times\left(\left(\frac{1}{3} p_e R\right)^2 F_{P_{3/2}}(Z, E_e) + \left(\frac{1}{3} p_\nu R\right)^2 F_{S_{1/2}}(Z, E_e)\right)$$

$$B_{2^-} = \frac{1}{2J_i + 1}\left|\left\langle J_f \left\| \sum_n \frac{r_n}{R} \tau_n^+ \{\sigma_1(n) \otimes Y_1(n)\}_2 \right\| J_i \right\rangle\right|^2$$

$$p_\nu = \sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

First unique forbidden beta decay

Electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{2^-} p_e E_e (E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$
$$\times\left(\left(\frac{1}{3}p_e R\right)^2 F_{P_{3/2}}(Z,E_e) + \left(\frac{1}{3}p_\nu R\right)^2 F_{S_{1/2}}(Z,E_e)\right)$$

Electron in the $p_{3/2}$ state

Electron in the $s_{1/2}$ state

$$\frac{g_{-1}^2 + f_{+1}^2}{j_0^2(p_e R)} \approx \frac{g_{-1}^2 + f_{+1}^2}{1} = F_{S_{1/2}}(Z,E_e)$$

$$\frac{g_{-2}^2 + f_{+2}^2}{j_1^2(p_e R)} \approx \frac{g_{-2}^2 + f_{+2}^2}{\left(p_e R/3\right)^2} = F_{P_{3/2}}(Z,E_e)$$

$$\alpha Z \to 0$$

$$F_S, F_P \to 1$$

Fermi functions: A) analytical approximation

Fermi functions are important in calculation of the phase space.
Common approach: approximation of the solution for the
point-like nucleus

$$F_{k-1}(Z,E)=\left(\frac{\Gamma(1+2k)}{\Gamma(k)\Gamma(1+2\gamma_k)}\right)^2 (2pR)^{2(\gamma_k-k)}\left|\Gamma(\gamma_k+iy)\right|^2 e^{\pi y}$$

Notation: S-wave: k=1

P-wave: k=2

$$k = 1,2,3,...$$

$$\kappa = -1,+1,-2,+2,...$$

$$S_{1/2}, P_{1/2}, P_{3/2}, D_{3/2}$$

$$\gamma_k = \sqrt{k^2-(\alpha Z)^2}$$

$$y = \alpha ZE/p$$

Fermi functions: B) exact analytical solution

Exact solution of the Dirac equation for the point-like nucleus

$$F_0(Z,E) \rightarrow \left(g_{-1}^2 + f_{+1}^2\right)/1 \qquad F_1(Z,E) \rightarrow \frac{\left(g_{-2}^2 + f_{+2}^2\right)}{\left(pr/3\right)^2}$$

with

$$rg_\kappa = \sqrt{\frac{E+m}{2E}}\frac{\left|\Gamma(1+\gamma_k+iy)\right|}{\Gamma(1+2\gamma_k)}(2pr)^{\gamma_k}\,\Im\left\{e^{i(pr+\xi)}F(\gamma_k - iy, 1+2\gamma_k, -2ipr)\right\}$$

$$rf_\kappa = \sqrt{\frac{E-m}{2E}}\frac{\left|\Gamma(1+\gamma_k+iy)\right|}{\Gamma(1+2\gamma_k)}(2pr)^{\gamma_k}\,\Re\left\{e^{i(pr+\xi)}F(\gamma_k - iy, 1+2\gamma_k, -2ipr)\right\}$$

$$e^{-2i\xi} = \frac{\gamma_k - iy}{\kappa - iym/E}$$

and a remark

$$j_0(pr) \approx 1$$

$$j_1(pr) \approx pr/3$$

Fermi functions: C) finite nuclear size

Numerical solution - g,f - of the Dirac equation for the finite nuclear size by use of the RADIAL package:
Comput.Phys.Commun. **90**(1995) 151

$$V_o(r) = \begin{cases} -\dfrac{\alpha Z}{2R}\left(3-\left(\dfrac{r}{R}\right)^2\right) & \text{for} \quad r \le R \\ -\dfrac{\alpha Z}{r} & \text{for} \quad r > R \end{cases}$$

Fermi functions: D) screening effect

Numerical solution of the Dirac equation for the finite nuclear size with screening effect taken into account by Thomas-Fermi function via Majorana solution: Am.J.Phys. **70**, 852 (2002)

$$\varphi(x) = Z_{eff} / Z$$

$$\varphi'' = \frac{\varphi^{3/2}}{\sqrt{x}}$$

Thomas-Fermi eq.

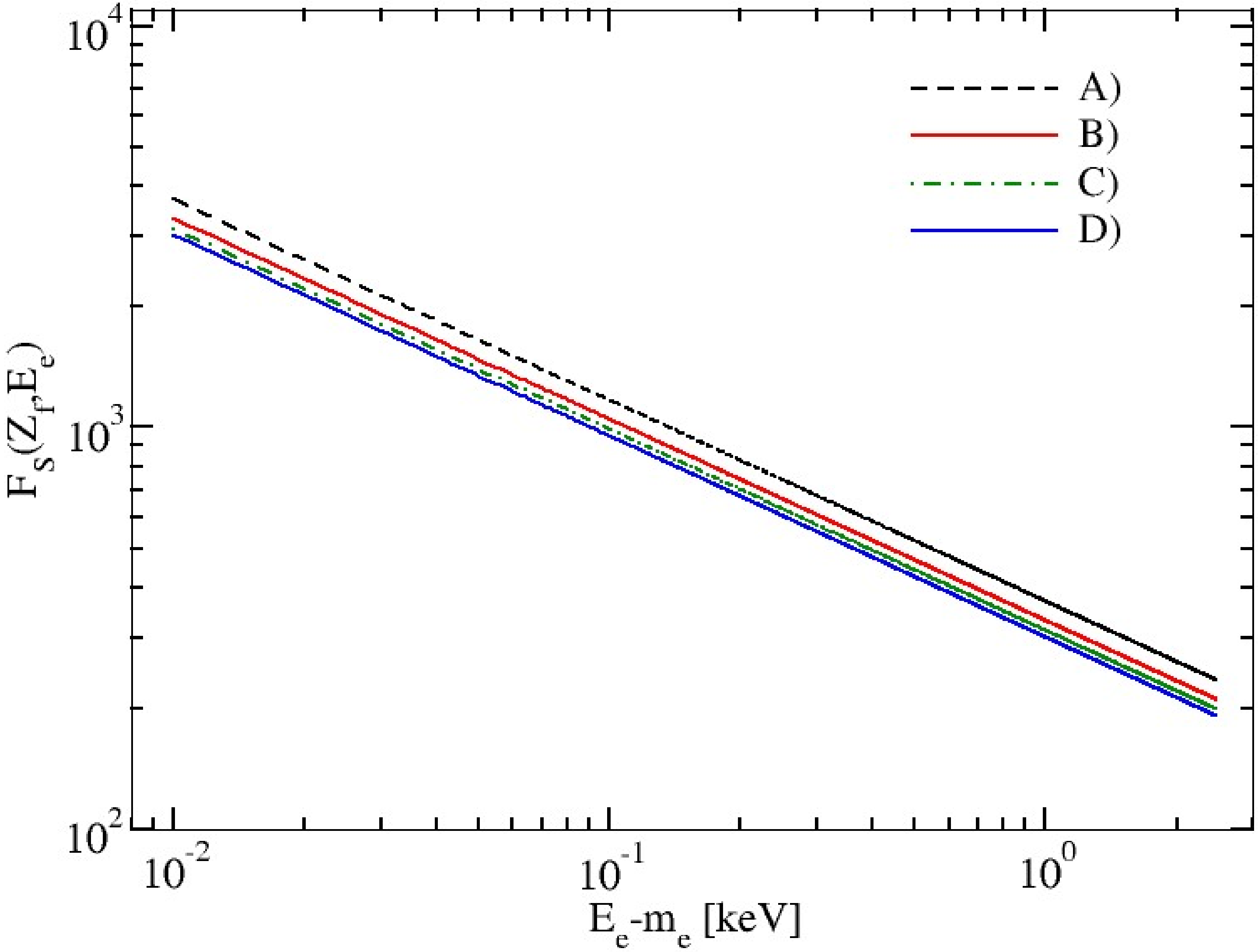
$$V(r) = \varphi(r) V_o(r)$$

$$\varphi(0) = 1$$

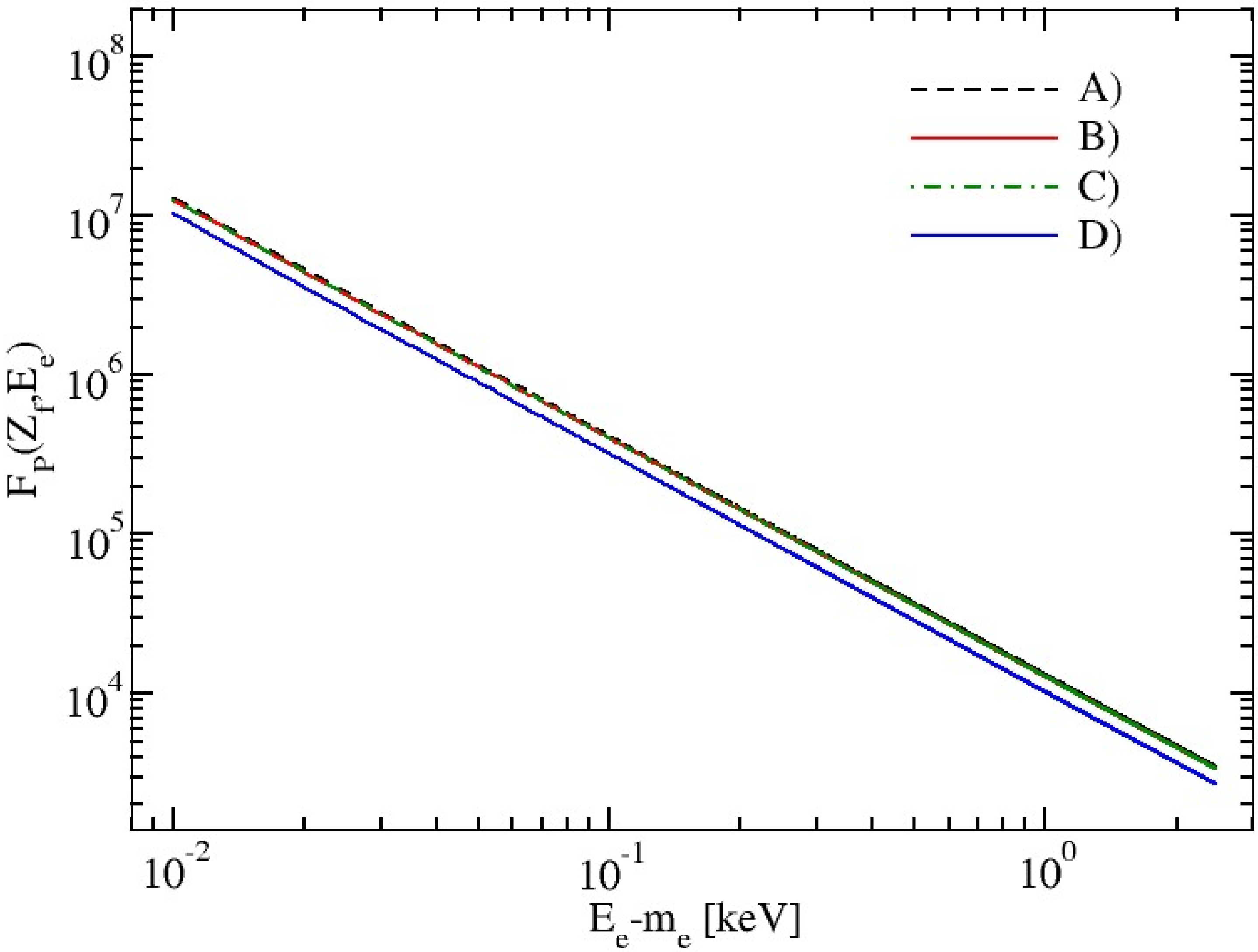
$$\varphi(\infty) = 1/Z$$

See for details: Štefánik, Dvornický, Šimkovic, and Vogel, PRC **92** 055502 (2015)

Fermi functions in rhenium beta decay



Fermi functions in rhenium beta decay



First unique forbidden beta decay

Kurie function:

$$K_{2^-}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{P_{3/2}} \left(p_e R / 3\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{2^-}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{2^-}(E_e)}$$

$$S_{2^-}(E_e) = \left(1 + \frac{p_\nu^2 F_{S_{1/2}}(Z, E_e)}{p_e^2 F_{P_{3/2}}(Z, E_e)} \right)$$

First unique forbidden beta decay

Kurie function:

$$K_{2^-}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{P_{3/2}} \left(p_e R/3\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{2^-}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{2^-}(E_e)}$$

The same formula as in case of allowed transitions.

Term originating from the forbiddenness of the decay

First unique forbidden beta decay

Kurie function:

$$K_{2^-}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{P_{3/2}} \left(p_e R/3\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{2^-}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{2^-}(E_e)}$$

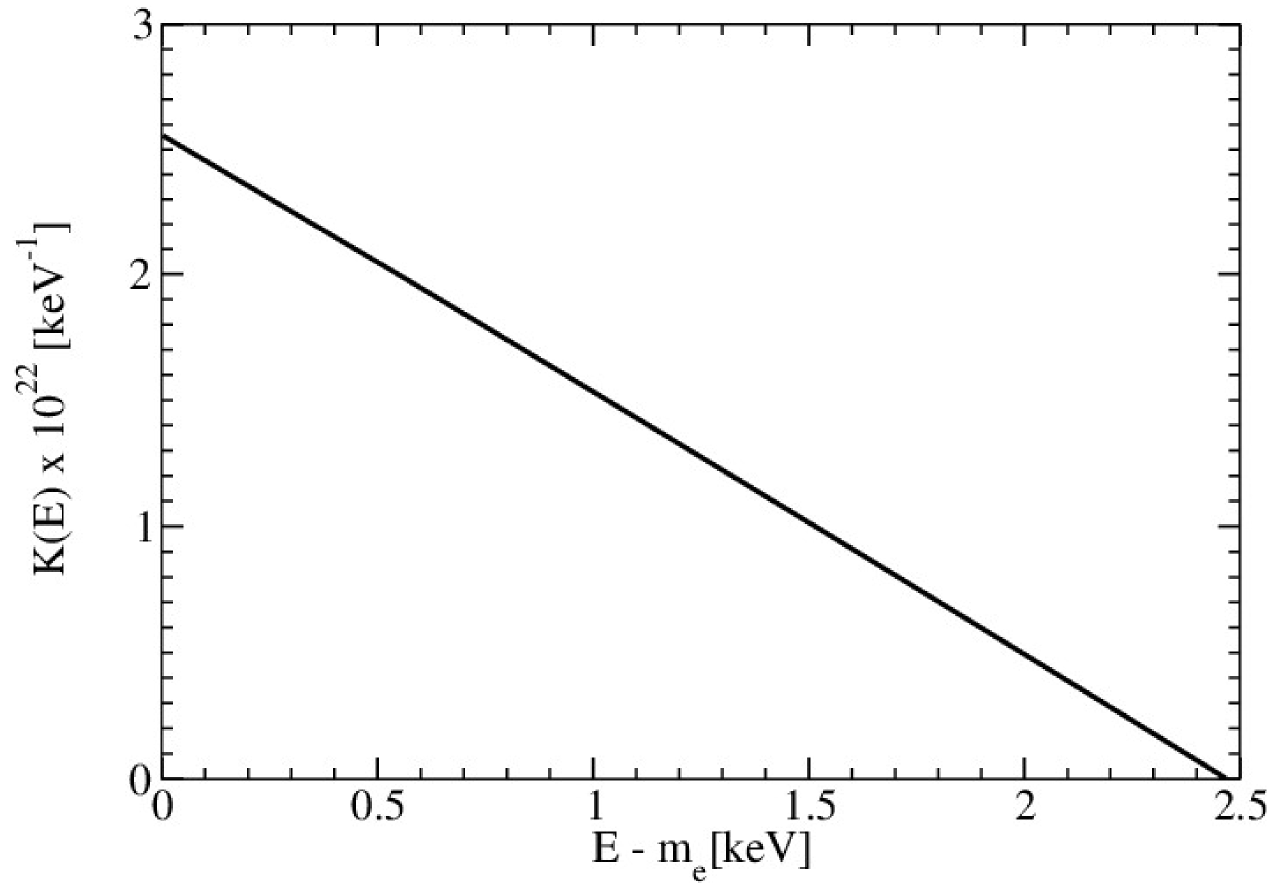
$$S_{2^-}(E_e) = \left(1 + \frac{p_\nu^2 F_{S_{1/2}}(Z, E_e)}{p_e^2 F_{P_{3/2}}(Z, E_e)}\right)$$ ⇐ Practically constant ~ 1

Close to the endpoint: $E_e \to E_0 - m_\beta$

$p_\nu \to 0$

First unique forbidden beta decay

Example: ^{187}Re with Q=2.469 keV



Second unique forbidden beta decay

$$\Delta J^{\pi} = 3^{+}$$

Nucleus:	$^{99}Tc^{*}$	$^{115}In^{*}$	^{129}I	$^{135}Cs^{*}$	$^{138}La^{*}$
Q[keV]:	204	0.189	194	47.76	255.3

Second unique forbidden beta decay

Electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{3^+} p_e E_e (E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$\times\left(\left(\frac{1}{15}(p_e R)^2\right)^2 F_{D_{5/2}}(Z,E_e) + \frac{6}{5}\left(\frac{1}{3}p_e R \frac{1}{3}p_\nu R\right)^2 F_{P_{3/2}}(Z,E_e) + \left(\frac{1}{15}(p_\nu R)^2\right)^2 F_{S_{1/2}}(Z,E_e)\right)$$

$$B_{3^+} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n^2}{R^2}\tau_n^+ \left\{\sigma_1(n)\otimes Y_2(n)\right\}_3 \right\| J_i \right\rangle\right|^2$$

$$\frac{g_{-3}^2 + f_{+3}^2}{j_2^2(p_e R)} \approx \frac{g_{-3}^2 + f_{+3}^2}{\left((p_e R)^2/15\right)^2} = F_{D_{5/2}}(Z,E_e)$$

Second unique forbidden beta decay

Kurie function:

$$K_{3^+}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{D_{5/2}} \left((p_e R)^2 / 15\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{3^+}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{3^+}(E_e)}$$

$$S_{3^+}(E_e) = \left(1 + \frac{10 p_\nu^2 F_{P_{3/2}}(Z, E_e)}{3 p_e^2 F_{D_{5/2}}(Z, E_e)} + \frac{p_\nu^4 F_{S_{1/2}}(Z, E_e)}{p_e^4 F_{D_{5/2}}(Z, E_e)} \right)$$

Third unique forbidden beta decay

$\Delta J^{\pi} = 4^{-}$ Electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{4^-} p_e E_e (E_0 - E_e) \sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$\times \frac{1}{105^2} \Big(F_{F_{7/2}}(E_e, R)(p_e R)^6 + 7F_{D_{5/2}}(E_e, R) p_e^4 p_\nu^2 R^6$$

$$+ 7F_{P_{3/2}}(E_e, R) p_e^2 p_\nu^4 R^6 + F_{S_{1/2}}(E_e, R)(p_\nu R)^6 \Big)$$

$$B_{4^-} = \frac{1}{2J_i + 1} \left| \left\langle J_f \left\| \sum_n \frac{r_n^2}{R^2} \tau_n^+ \{\sigma_1(n) \otimes Y_3(n)\}_4 \right\| J_i \right\rangle \right|^2$$

$$\frac{g_{-4}^2 + f_{+4}^2}{j_3^2(p_e R)} \approx \frac{g_{-4}^2 + f_{+4}^2}{\left((p_e R)^3 / 105\right)^2} = F_{F_{7/2}}(Z, E_e)$$

Third unique forbidden beta decay

Kurie function:

$$K_{4^-}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{F_{7/2}} \left((p_e R)^3 / 105\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{4^-}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{4^-}(E_e)}$$

$$S_{4^-}(E_e) = \left(1 + \frac{7 p_\nu^2 F_{D_{5/2}}(Z, E_e)}{p_e^2 F_{F_{7/2}}(Z, E_e)} + \frac{7 p_\nu^4 F_{P_{3/2}}(Z, E_e)}{p_e^4 F_{F_{7/2}}(Z, E_e)} + \frac{p_\nu^6 F_{S_{1/2}}(Z, E_e)}{p_e^6 F_{F_{7/2}}(Z, E_e)} \right)$$

Fourth unique forbidden beta decay

$$\Delta J^{\pi} = 5^{+}$$

Nucleus:	$^{48}Ca^{*}$	^{60}Fe
Q[keV]:	151.1	237

Fourth unique forbidden beta decay

Electron energy spectrum:

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{5^+} p_e E_e (E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$\times\left(\left(\frac{1}{945}(p_e R)^4\right)^2 F_{G_{9/2}}(Z,E_e) + \frac{4}{3}\left(\frac{1}{105}(p_e R)^3 \frac{1}{3} p_\nu R\right)^2 F_{F_{7/2}}(Z,E_e)\right.$$

$$+\frac{10}{7}\left(\frac{1}{15}(p_e R)^2 \frac{1}{15}(p_\nu R)^2\right)^2 F_{D_{5/2}}(Z,E_e) + \frac{4}{3}\left(\frac{1}{3} p_e R \frac{1}{105}(p_\nu R)^3\right)^2 F_{P_{3/2}}(Z,E_e)$$

$$\left.+\left(\frac{1}{945}(p_\nu R)^4\right)^2 F_{S_{1/2}}(Z,E_e)\right)$$

Fourth unique forbidden beta decay

$$B_{5^+} = \frac{1}{2J_i + 1} \left| \left\langle J_f \left\| \sum_n \frac{r_n^4}{R^4} \tau_n^+ \left\{ \sigma_1(n) \otimes Y_4(n) \right\}_5 \right\| J_i \right\rangle \right|^2$$

$$\frac{g_{-4}^2 + f_{+4}^2}{j_3^2(p_e R)} \approx \frac{g_{-4}^2 + f_{+4}^2}{\left((p_e R)^3 / 105\right)^2} = F_{F_{7/2}}(Z, E_e)$$

$$\frac{g_{-5}^2 + f_{+5}^2}{j_4^2(p_e R)} \approx \frac{g_{-5}^2 + f_{+5}^2}{\left((p_e R)^4 / 945\right)^2} = F_{G_{9/2}}(Z, E_e)$$

Fourth unique forbidden beta decay

Kurie function:

$$K_{5^+}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{G_{9/2}} \left((p_e R)^4 / 945 \right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{5^+}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{5^+}(E_e)}$$

$$S_{5^+}(E_e) = \left(1 + \frac{12 p_\nu^2 F_{F_{7/2}}(Z, E_e)}{p_e^2 F_{G_{9/2}}(Z, E_e)} + \frac{126 p_\nu^4 F_{D_{5/2}}(Z, E_e)}{5 p_e^4 F_{G_{9/2}}(Z, E_e)} \right.$$

$$\left. + \frac{12 p_\nu^6 F_{P_{3/2}}(Z, E_e)}{p_e^6 F_{G_{9/2}}(Z, E_e)} + \frac{p_\nu^8 F_{S_{1/2}}(Z, E_e)}{p_e^8 F_{G_{9/2}}(Z, E_e)} \right)$$

First non-unique forbidden beta decay

$$\Delta J^{\pi} = 0^{-}$$

Nucleus:	^{144}Ce	^{151}Sm
Q[keV]:	318.7	76.6

First forbidden non-unique beta decay

Electron energy spectrum for a particular transition: $0^{\pm} \rightarrow 0^{\mp}$

$$\frac{d\Gamma}{dE_e} = \frac{2}{\pi^2} G_\beta^2 g_A^2 B_{0^-} p_e E_e (E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\beta^2}$$

$$\times\left(\left(\frac{1}{3}p_e R\right)^2 F_{P_{1/2}}(Z,E_e) + \left(\frac{1}{3}R\right)^2 F_{SP}(E_e,R)\left(2\frac{p_\nu^2}{E_\nu}p_e + 6\lambda E_0 p_e\right)\right.$$

$$\left. + \left(\frac{1}{3}R\right)^2 F_{S_{1/2}}(Z,E_e)\left(p_\nu^2 + 9\lambda^2 E_0^2 + 6\lambda E_0 \frac{p_\nu^2}{E_\nu}\right)\right)$$

$$B_{0^-} = \frac{1}{2J_i + 1}\left|\left\langle J_f \left\| \sum_n \frac{r_n}{R}\tau_n^+ \{\sigma_1(n) \otimes Y_1(n)\}_0 \right\| J_i \right\rangle\right|^2$$

$$\frac{g_{+1}^2 + f_{-1}^2}{j_1^2(p_e R)} \approx \frac{g_{+1}^2 + f_{-1}^2}{(p_e R/3)^2} = F_{P_{1/2}}(Z,E_e)$$

$$\frac{g_{+1}f_{+1} - g_{-1}f_{-1}}{j_1(p_e R)j_0(p_e R)} \approx \frac{g_{+1}f_{+1} - g_{-1}f_{-1}}{p_e R/3} = F_{SP}(Z,E_e)$$

First forbidden non-unique beta decay

Kurie function:

$$K_{0^-}(E_e, m_\beta) \equiv \sqrt{\frac{d\Gamma / dE_e}{p_e E_e F_{P_{1/2}} \left(p_e R / 3\right)^2}}$$

$$= \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_{0^-}} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_{0^-}(E_e)}$$

$$S_{0^-}(E_e) = \left(1 + \frac{\left(2 p_\nu^2 / E_\nu + 6\lambda E_0\right) F_{SP}(E_e, R)}{p_e F_{P_{1/2}}(Z, E_e)} + \frac{\left(p_\nu^2 + 9\lambda^2 E_0^2 + 6\lambda E_0 p_\nu^2 / E_\nu\right) F_{S_{1/2}}(Z, E_e)}{p_e^2 F_{P_{1/2}}(Z, E_e)} \right)$$

Parameter λ originates from the velocity dependent nuclear matrix element. Its value is a constant for a given nucleus.

First non-unique forbidden beta decay

$$\Delta J^{\pi} = 1^{-}$$

For example: ^{210}Bi - to be submitted soon

Kurie functions

For allowed, first non-unique forbidden, first-, second-, third-, and fourth- unique forbidden beta decays, respectively, label I takes the values 1^+, 0^-, 2^-, 3^+, 4^-, and 5^+.

$$K_I(E_e, m_\beta) = \frac{\sqrt{2}}{\pi} G_\beta g_A \sqrt{B_I} (E_0 - E_e) \sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}} \sqrt{S_I(E_e)}$$

Beta strengths

Allowed transitions :

$$B_{1^+} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \tau_n^+ \{\sigma_1(n)\otimes Y_0(n)\}_1 \right\| J_i \right\rangle\right|^2$$

First non-unique forbidden:

$$B_{0^-} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n}{R} \tau_n^+ \{\sigma_1(n)\otimes Y_1(n)\}_0 \right\| J_i \right\rangle\right|^2$$

Unique forbidden:

$$B_{2^-} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n}{R} \tau_n^+ \{\sigma_1(n)\otimes Y_1(n)\}_2 \right\| J_i \right\rangle\right|^2$$

$$B_{3^+} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n^2}{R^2} \tau_n^+ \{\sigma_1(n)\otimes Y_2(n)\}_3 \right\| J_i \right\rangle\right|^2$$

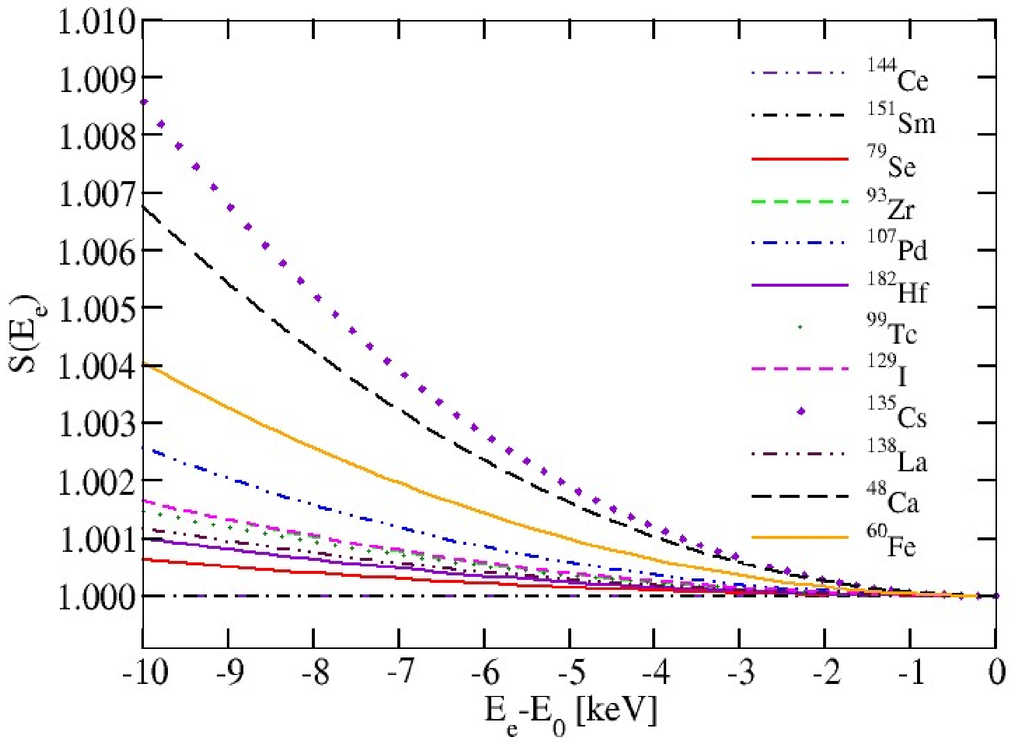
$$B_{4^-} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n^3}{R^3} \tau_n^+ \{\sigma_1(n)\otimes Y_1(n)\}_4 \right\| J_i \right\rangle\right|^2$$

$$B_{5^+} = \frac{1}{2J_i+1}\left|\left\langle J_f \left\| \sum_n \frac{r_n^4}{R^4} \tau_n^+ \{\sigma_1(n)\otimes Y_4(n)\}_5 \right\| J_i \right\rangle\right|^2$$

Kurie functions

TABLE 1. List of nuclei which β decays are classified as unique forbidden. Selection was performed by fulfilling two conditions simultaneously: $T_{1/2} > 10$ yrs and $Q < m_e$. The Q values were taken from [7]. The first nuclear excited state with spin-parity J^π of final nucleus is labeled with subscript as J^π_1.

Parent($J_i^{\pi_i}$)	Daughter($J_f^{\pi_f}$)	ΔJ^π	Q-value [keV]	$T_{1/2}$ [yrs]
^{48}Ca(0^+)	^{48}Sc(5^+_1)	5^+	151.1	-
^{60}Fe(0^+)	^{40}Co(5^+)	5^+	237	-
^{79}Se($7/2^+$)	^{79}Br($3/2^-$)	2^-	151	-
^{93}Zr($5/2^+$)	^{93}Nb($1/2^-_1$)	2^-	60	1.61×10^6
^{99}Tc($9/2^+$)	^{99}Ru($3/2^+_1$)	3^+	204	2.111×10^5
^{107}Pd($5/2^+$)	^{107}Ag($1/2^-$)	2^-	34.1	6.5×10^6
^{115}In($9/2^+$)	^{115}Sn($3/2^+_1$)	3^+	0.189	4.41×10^{14}
^{129}I($7/2^+$)	^{129}Xe($1/2^+$)	3^+	194	-
^{135}Cs($7/2^+$)	^{135}Ba($1/2^+_1$)	3^+	47.76	-
^{135}Cs($7/2^+$)	^{135}Ba($11/2^-_1$)	2^-	0.5	-
^{138}La(5^+)	^{138}Ce(2^+_1)	3^+	255.3	1.02×10^{11}
^{182}Hf(0^+)	^{182}Ta(2^-)	2^-	104.6	8.9×10^6
^{187}Re($5/2^+$)	^{187}Os($1/2^-$)	2^-	2.469	4.33×10^{10}

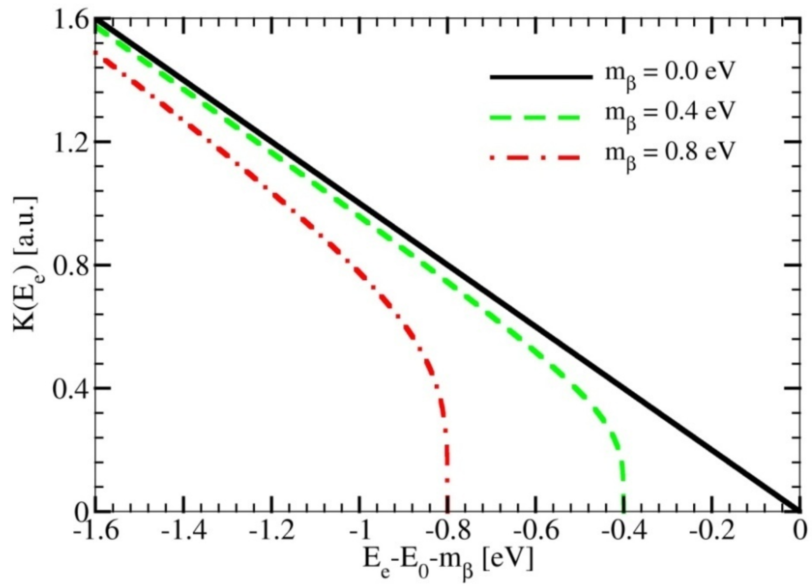


Conclusion

Kurie functions for forbidden beta decays

Properly defined Kurie functions for the unique forbidden and first non-unique forbidden beta decays reflect the behavior of a Kurie function for the allowed transitions, i.e. they are linear close to the endpoint only in the case of massless neutrinos

$$K(E_e, m_\beta) \cong (E_0 - E_e)\sqrt[4]{1 - \frac{m_\beta^2}{(E_0 - E_e)^2}}$$



Thank you
for your attention