# $Z_3$-QCD and persistent homology

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**Abstract**

The $Z_3$-symmetric QCD ($Z_3$-QCD) is investigated. The pure gluon theory has $Z_3$-symmetry, but the effects of the dynamical quarks break the symmetry. However, in three flavor QCD with symmetric quark masses, the $Z_3$-symmetry can be restored by introducing of flavor-dependent imaginary chemical potential. In this report, the lattice calculation of $Z_3$-QCD at finite isospin chemical potential is shown. The persistent homology analyses are also done to analyze the transition.

# 1 Introduction

The determination of QCD phase diagram is a very important subject not only particle and nuclear physics but also in astrophysics and cosmology. The first principle calculation, namely, the lattice QCD (LQCD) calculation has shown that the QCD transition is crossover at vanishing chemical potential [1]. However, due to the famous sign problem, it is very difficult to do LQCD calculation at intermediate and high density. Another problem in QCD transition is a lack of symmetry. The chiral symmetry is regarded as a good approximate symmetry, but it is not directly related to the confinement-deconfinement transition. It is well known that the pure gluon theory has $Z_3$-symmetry, which is related to the confinement-deconfinement transition, but the introduction of the dynamical quarks breaks the symmetry.

However, in three flavor QCD with symmetric quark masses, the $Z_3$-symmetry can be restored by introducing the flavor-dependent chemical potential. We call this $Z_3$-symmetric QCD "$Z_3$-QCD" [2, 3, 4, 5, 6, 7, 8, 9] in this report. The lattice calculation of $Z_3$-QCD at vanishing chemical potential was done and showed that the finite temperature confinement-deconfinement transition is very sharp [10]. The result is consistent with the earlier results of phenomenological model. What about the transition at finite density? In $Z_3$-QCD at real quark chemical potential, the sign problem is weaker than

the ordinary QCD but still remains [11, 12, 13]. Hence, here we show the lattice calculation of $Z_3$-QCD at real isospin chemical potential, where the sign problem is absent.

This report is organized as follows. In Sec. 2 we explain the $Z_3$-QCD briefly. The results of lattice calculation of $Z_3$-QCD at intermediate temperature and real isospin chemical potential are shown. In Sec. 3 the persistent homology analyses for the transition is shown. Section 4 is devoted to summary.

## 2 $Z_3$-QCD

It is well-known that the pure gluon theory has $Z_3$-symmetry under the gauge transformation with SU(3) elements $U(x,\tau)$ which satisfy

$$U(\mathbf{x}, 1/T) = \exp(2\pi ik/3)U(\mathbf{x}, 0), \tag{1}$$

where $\mathbf{x}$ is the spatial coordinate, $\tau$ is the Euclidean time, $T$ is the temperature, and $k$ is any integer. Note that $\exp(2\pi ik/3)$ is an element of $Z_3$ which is the center of SU(3). Polyakov-loop, which is not invariant under the $Z_3$-transformation, is the order parameter of the confinement-deconfinement transition.

In full QCD with dynamical quarks the $Z_3$-symmetry is explicitly broken since the temporal anti-periodic boundary condition of quark filed $q$ changes into

$$q(\mathbf{x}, 1/T) = -\exp(2\pi ik/3)q(\mathbf{x}, 0). \tag{2}$$

However, in the three flavor QCD with symmetric quark masses, the $Z_3$-symmetry is restored by introducing the flavor ($f$)-dependent boundary condition

$$q_f(\mathbf{x}, 1/T) = -\exp(2\pi i(f-2)/3)q_f(\mathbf{x}, 0) \quad (f = 1, 2, 3) \tag{3}$$

instead of the ordinary anti-periodic boundary condition. By redefining quark field $q_f$, the flavor dependent boundary condition (2) can be transformed into the ordinary anti-periodic boundary condition but the flavor dependent imaginary chemical potential

$$\mu_1 = -i2\pi T/3, \qquad \mu_2 = 0, \qquad \mu_3 = i2\pi T/3 \tag{4}$$

appears. This is $Z_3$-QCD. It is expected that $Z_3$-QCD approaches to the ordinal QCD in the low temperature limit $T \to 0$ [11, 14].

It is also expected that the sign problem is weaker in $Z_3$-QCD than in the ordinary QCD but still remains [11, 12, 13]. Hence, we consider (real) isospin chemical potential instead of (real) quark chemical potential. Consequently, we should treat the flavor-dependent complex chemical potential. Consider symmetric six flavor QCD with flavor dependent chemical potential

$$\begin{aligned} \mu_1 = \mu - i2\pi T/3, \quad \mu_2 = \mu, \quad \mu_3 = \mu + i2\pi T/3, \\ \mu_4 = -\mu - i2\pi T/3, \quad \mu_5 = -\mu, \quad \mu_6 = -\mu + i2\pi T/3 \end{aligned} \tag{5}$$

with a real parameter $\mu$. This theory is $Z_3$-symmetric and has no sign problem. Since the symmetric six flavor QCD does not correspond to the physical world, we multiply the quark action by the additional factor $\frac{1}{3}$ and use the parameter of symmetric two flavor QCD. In the limit $T \to 0$, where the temporal boundary condition is irrelevant, our model coincides with the ordinary two-flavor QCD. The parameter we use is the same as presented in the right panel of TABLE I in Ref. [15] but the simple Wilson fermion is used to save the computational time. The LQCD program we use is based on the lattice took kit [16]. The spacial (temporal) lattice size is 8 (4). We generate about 8,000∼12,000 trajectories and remove the first 2,000 trajectories for the thermalization of all the parameters and measured the physical quantities at every 50 trajectories.

Figure 1 shows the $\mu$-dependence of the expectation value of the absolute value $\langle |P| \rangle$ of the spatial averaged Polyakov-loop and the isospin density $n$. (Note that the expectation value $\langle P \rangle$ of Polyakov-loop itself vanishes due to the $Z_3$-symmetry even if the spontaneous breaking happens. ) We see that, as $\mu$ increases, $n$ increases rapidly in the region of $\mu/T > 3.5$ but $\langle |P| \rangle$ increases only slowly and is saturated at $\mu/T \sim 4.0$. It can be expected that the system is still in confined phase when $\beta = 1.5$ and $\mu/T \leq 4.0$. (Similar tendency is seen in the case of $\beta = 1.80$ which corresponds to $T \sim 0.45 T_c$. )

# 3 Persistent homology

To examine the result presented in the previous section, we have performed the persistent homology (PH) analyses. PH is one of the useful methods in topological data analyses and analyze $multi-scale$ topological structures in discrete data set. For the introductory review, see Refs. [17] and [18], and

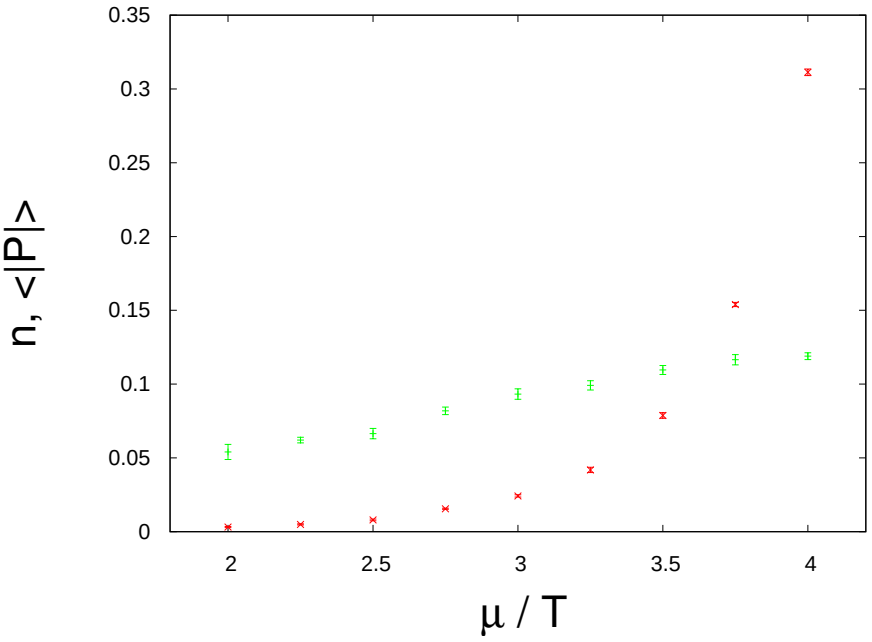


Figure 1: $\mu$-dependence of $\langle|P|\rangle$ (green) and $n$ (red). $n$ is normalized by the lattice spacing. We use the inverse coupling $\beta = 1.50$ which corresponds to $T/T_c \sim 0.37$ where $T_c$ ($\sim 260$ MeV) is the critical temperature of confinement-deconfinement transition at zero-chemical potential in $Z_3$-QCD.

the references therein. Here we only give an intuitive explanation. Suppose the discrete data which are presented as points in some data space. For simplicity, consider two dimensional data space. We replace the points by the filled circles with finite radius $r$ and call $t = r^2$ *time*. As $t$ increases, circles merge each other and, at time $t_B$, may form a hole surrounded by a close curved line (with finite width). When $t$ becomes bigger, the hole disappears at $t_D$ ($> t_B$). This process is called a *filtration*. Hence we regards $t_B$ ($t_D$) as the birth time (the death time) of a topological structure, namely, a hole, in the data space. The set of $(t_B, t_D)$ for all holes, which appear and disappear in the filtration, forms the persistent diagram (PD). The generalization for the higher-dimensional data space is straightforward.

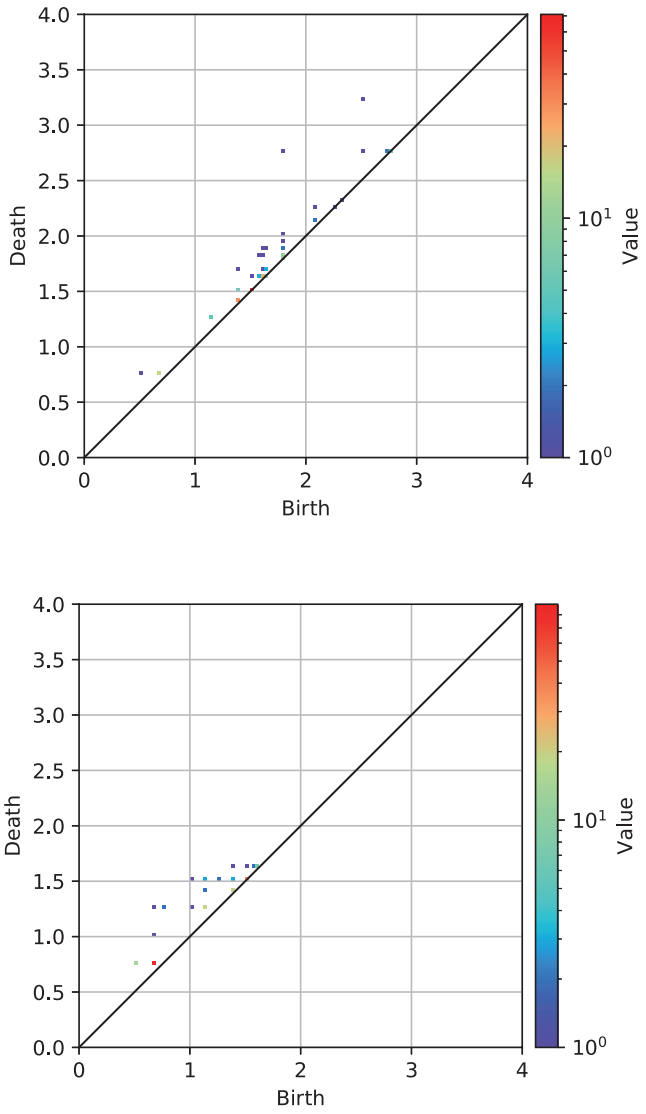


Figure 2: PD for the spatial distribution of the phase of Polyakov-loop at $\mu = 0$. The horizontal (vertical ) axis is the birth (death) time. We set $\beta = 1.50$ (2.40), which corresponds to $T = 0.37$ (1.45) $T_c$, in the upper (lower) panel. The data spaces of the first $Z_3$-domain were used for analyses.

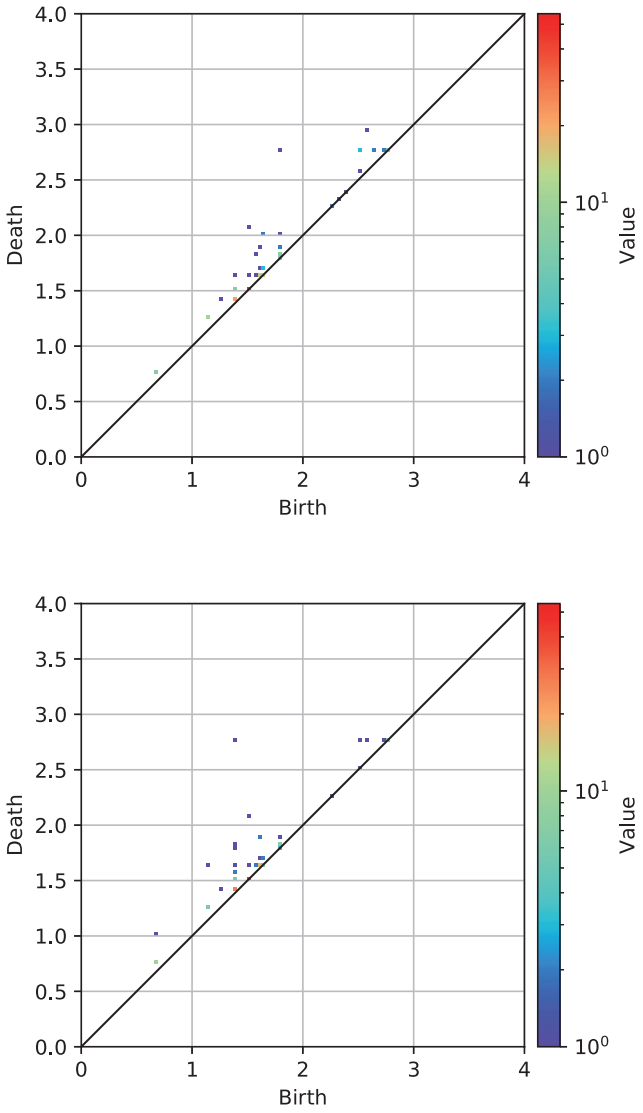


Figure 3: PD for the spatial distribution of the phase of Polyakov-loop at $\beta = 1.50$. The horizontal (vertical ) axis is the birth (death) time. We set $\mu/T = 3.25$ (4.0) in the upper (lower) panel. The data spaces of the first $Z_3$-domain were used for analyses.

Figure 2 shows examples of PD for the phase of Polyakov-loop. Using the same procedure in Ref. [17], we classified the phase $\phi_{\mathbf{x}}$ of the Polyakov-loop $P_{\mathbf{x}}$ at the spatial point $\mathbf{x}$ into the three $Z_3$-domains. When $\phi_{\mathbf{x}}$ is in the first (second, third) $Z_3$-domain, we plot a point at $\mathbf{x}$ in the first (second, third) three-dimensional data space. For each data space, PH analyses are done. In the numerical estimation of the persistent homology, we employ the homcloud [19]. The result in the upper panel corresponds to $T \sim 0.37T_c$ and in confined phase. It is seen that there are holes with large $t_B$ and $t_D$. The result in the lower panel corresponds to $T \sim 1.45T_c$ and in deconfined phase. In this case, the large structure disappears in PD.

Figure 3 is the same as Fig. 2 but for the cases of $\beta = 1.50$ and $\mu/T = 3.25,\ 4.0$. Although, as is seen in Fig. 1, the corresponding isospin densities are much different, PD in the lower panel is similar as the one in the upper panel. These PD also resemble the PD with the one in the upper panel of Fig. 2. This result confirms the observation in the previous section. At $\beta = 1.5$, the system is in confined phase for $\mu/T \leq 4.0$.

# 4 Summary

The lack of the symmetry which governs the confinement-deconfinement transition yields a difficulty to interpret the transition. To investigate this problem, we have developed the $Z_3$-symmetric QCD, namely, $Z_3$-QCD. We have done the lattice $Z_3$-QCD calculation for the intermediate temperature, when the isospin chemical potential $\mu$ is finite. It is found that the system is in confined phase when $T \sim 0.37T_c$ and $\mu/T \leq 4.0$. The persistent homology analyses confirmed the conclusion. It seems that the exact $Z_3$-symmetry strongly prevents quarks from the deconfinement. If the pion condensation does not occur, the property of $Z_3$-QCD at intermediate temperature and finite isospin chemical potential may resemble the one of the ordinary QCD at low temperature and finite quark chemical potential. We hope that our finding may give an insight into the property of the confinement-deconfinement transition at low temperature and finite density. Detail analyses will appear elsewhere.

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