6 Compositeness and weak-binding relation

6.1 Introduction

- Study of hadron structure : problem with unknown potential
  - Inter-quark potential is not accessible by confinement
  - Inter-hadron potential is not observable in a strict sense
    (Change of potential can be absorbed by change of w.f. keeping observables unchanged)
- Internal structure of hadron resonances : superposition of all possible configurations

$$|\,\Lambda(1405)\,\rangle = C_{3q}|\,uds\,\rangle + C_{5q}|\,uds\bar{q}q\,\rangle + C_{MB}|\,MB\,\rangle + \cdots \tag{44}$$

  C_i : weight of each component

- Traditional method : Comparison of model calculation with experimental data
  Calculation with $|\,uds\,\rangle$ model, calculation of $|\,\bar{K}N\,\rangle$ model, etc.
  $\rightarrow$ Dominant C_i is given by the model which well reproduces the experimental data
- Problems :
  - By refinements (with many parameters introduced), any model can describe data
  - Is Eq. (44) well-defined? (orthogonality of $5q$ and MB? structure of unstable resonance?)
  - Wave function is not an observable? energy scale?
- Weak-binding relation : Model-independent method
  $\rightarrow$ relate observables to internal structure
- History :
  - 1960's : Discussion to identify elementary particles out of composite particles
  - 1965 : Weak-binding relation by Weinberg (bound state) [18]
  - 2003 : Application to hadron physics (integration of spectral function) [36]
  - 2011 : "Compositeness" in hadron physics [37]
  - 2015 : Generalization of weak-binding relation to resonances [19, 20]

6.2 Weak-binding relation

- Compositeness X of stable bound state (deuteron case) [18]

$$|\,d\,\rangle = \sqrt{X}|\,NN\,\rangle + \sqrt{Z}|\,\text{others}\,\rangle, \quad X + Z = 1$$

  $|\,d\,\rangle$: wave function of deuteron
  $|\,NN\,\rangle$: two-nucleon (s-wave) component
  $|\,\text{others}\,\rangle$: all other components
  Z : elementarity/elementariness

- $(X, Z) = (1, 0)$: composite like state, $(X, Z) = (0, 1)$: elementary like state
- Cautions
  - $|\,NN\,\rangle$ is labeled by continuous variable such as momentum, and X is sum of all the components
  - $|\,\text{others}\,\rangle$ contains $6q$ states, $N\Delta$ states, NN(d-wave) states, and so on
  - Because $|\,\text{others}\,\rangle$ is historically introduced as a single-particle state, Z is called "elementarity"
- **Weak-binding relation**

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}, \quad R = \frac{1}{\sqrt{2\mu B}}$$

  a_0 : NN scattering length ($^3S_1$ channel)
  B : deuteron binding energy, $\mu = M_N/2$: NN reduced mass
  R : deuteron radius (length scale associated with the binding energy)
  R_{typ} : typical length scale of interaction

- When B is small (R_{typ}/R is negligible), compositeness X is determined by observables (a_0, B)
- Condition
  - X is the compositeness in s wave scattering channel (not applicable to $\ell \neq 0$)
  - Stable bound state (no decay)
- Using $B = 2.22$ MeV and $a_0 = 5.42$ fm for deuteron, [38]

$$X = 1.68^{+2.15}_{-0.83}$$

  Uncertainty estimated by $\mathcal{O}(R_{\text{typ}}/R)$ term with $R_{\text{typ}} \sim 1/m_\pi = 1.43$ fm
  By definition (see below) $X \leq 1 \Rightarrow 0.85 \leq X \leq 1$: more than 80 % of deuteron is NN composite

- Without using the nuclear force and deuteron w.f., structure is determined by observables

6.3 Compositeness in effective field theory

- Hamiltonian of EFT [19, 20]

$$\begin{aligned} H &= H_{\text{free}} + H_{\text{int}} \\ H_{\text{free}} &= \int d\boldsymbol{r}\left[\frac{1}{2M}\nabla\psi^\dagger\cdot\nabla\psi + \frac{1}{2m}\nabla\phi^\dagger\cdot\nabla\phi + \frac{1}{2M_0}\nabla B_0^\dagger\cdot\nabla B_0 + \omega_0 B_0^\dagger B_0\right] \\ H_{\text{int}} &= \int d\boldsymbol{r}\left[g_0\left(B_0^\dagger\phi\psi + \psi^\dagger\phi^\dagger B_0\right) + v_0\psi^\dagger\phi^\dagger\phi\psi\right] \end{aligned} \tag{45}$$

  $\phi(\boldsymbol{r})$, $\psi(\boldsymbol{r})$, $B_0(\boldsymbol{r})$: field operators, M, m, M_0 : masses, ω_0 : bare energy
  Each term of free Hamiltonian H_{free} is equivalent to the kinetic term of $\mathcal{L}_{\text{eff}}$ in §5
  g_0, v_0 : coupling constants
  H_{int} : contact interactions (Fig. 19), $\psi\phi \leftrightarrow \psi\phi$ and $\psi\phi \leftrightarrow B_0$

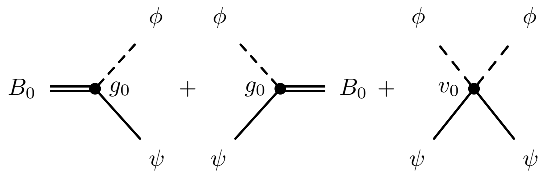

Figure 19: Feynman rules for the interaction Lagrangian (45).

- Eigenstates of H_{free}

$$H_{\text{free}}|\, B_0\,\rangle = \omega_0|\, B_0\,\rangle \quad \text{(discrete state } B_0\text{, elementary component)}$$

$$H_{\text{free}}|\, \boldsymbol{p}\,\rangle = \frac{\boldsymbol{p}^2}{2\mu}|\, \boldsymbol{p}\,\rangle \quad (\psi\phi \text{ scattering states with relative momentum } \boldsymbol{p}\text{, composite component})$$

Explicit forms and orthonormality

$$|\, B_0\,\rangle = \frac{1}{\sqrt{V_p}}\tilde{B}_0^\dagger(\boldsymbol{0})|\, 0\,\rangle, \quad |\, \boldsymbol{p}\,\rangle = \frac{1}{\sqrt{V_p}}\tilde{\psi}^\dagger(\boldsymbol{p})\tilde{\phi}^\dagger(-\boldsymbol{p})|\, 0\,\rangle$$

$$\langle\, B_0\,|\, B_0\,\rangle = 1, \quad \langle\, B_0\,|\, \boldsymbol{p}\,\rangle = 0, \quad \langle\, \boldsymbol{p}'\,|\, \boldsymbol{p}\,\rangle = (2\pi)^3\delta^3(\boldsymbol{p}' - \boldsymbol{p})$$

$|\, 0\,\rangle$: vacuum, $V_p = (2\pi)^3\delta^3(\boldsymbol{0})$: phase space, $\tilde{\alpha}(\boldsymbol{p}) =$ F.T. $\alpha(\boldsymbol{r})$: momentum space field operators

- Complete set : from particle number conservation,

$$1 = |\, B_0\,\rangle\langle\, B_0\,| + \int \frac{d\boldsymbol{p}}{(2\pi)^3}|\, \boldsymbol{p}\,\rangle\langle\, \boldsymbol{p}\,| \tag{46}$$

Relation with Feshbach projections [39] ($P + Q = 1$)

$$P = \int \frac{d\boldsymbol{p}}{(2\pi)^3}|\, \boldsymbol{p}\,\rangle\langle\, \boldsymbol{p}\,| \quad (\text{projection to } \psi\phi \text{ scattering states})$$

$$Q = |\, B_0\,\rangle\langle\, B_0\,| \quad (\text{projection to discrete state } B_0)$$

Field theory equivalent to the single-resonance approximation of Q channel in §4

- Bound state $|\, B\,\rangle$ with binding energy B as eigenstate of H (physical state)

$$H|\, B\,\rangle = -B|\, B\,\rangle$$

- Compositeness X (overlap with scattering states), elementarity Z (overlap with discrete state)

$$X \equiv \langle\, B\,|P|\, B\,\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3}|\langle\, \boldsymbol{p}\,|\, B\,\rangle|^2 \geq 0$$

$$Z \equiv \langle\, B\,|Q|\, B\,\rangle = |\langle\, B_0\,|\, B\,\rangle|^2 \geq 0$$

Regarding B_0 as a bare state, Z is field renormalization constant

- From the normalization of the bound state $\langle\, B\,|\, B\,\rangle = 1$ and the completeness relation (46),

$$Z + X = 1$$

By definition, X and Z are nonnegative, and sum is normalized to 1 : interpreted as probabilities

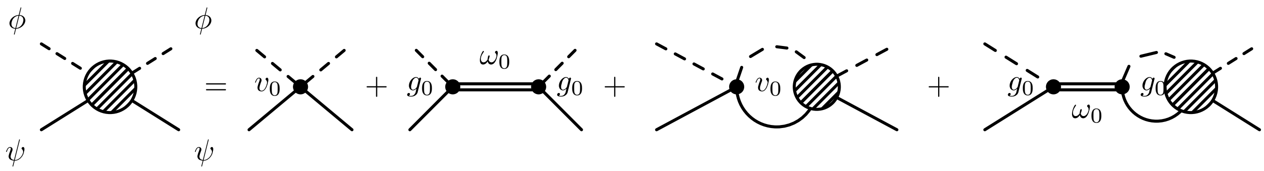

Figure 20: Feynman diagrams of the $\psi\phi$ scattering amplitude.

6.4 Weak-binding relation in effective field theory

- $\psi\phi$ scattering amplitude (derived as in §5, see Fig. 20)

$$f(E) = -\frac{\mu}{2\pi}\frac{1}{1/v(E) - G(E)} \tag{47}$$

$$v(E) = v_0 + \frac{g_0^2}{E - \omega_0}, \tag{48}$$

$$G(E) = \frac{1}{2\pi^2}\int_0^\Lambda dp \frac{p^2}{E - p^2/(2\mu) + i0^+} \tag{49}$$

- Cutoff Λ : momentum scale at which hadron interaction can be regarded as pointlike

$$\Rightarrow R_{\text{typ}} = \frac{1}{\Lambda}$$

- Expression of compositeness by scattering amplitude

$$X = \left.\frac{G'(E)}{G'(E) - [1/v(E)]'}\right|_{E=-B}, \quad A'(E) = \frac{dA(E)}{dE}$$

- Expanding the scattering length in terms of (R_{typ}/R)

$$a_0 = -f(E=0) = \frac{\mu}{2\pi}\left[1/v(0) - G(0)\right]^{-1} \tag{50}$$

$$= \cdots = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}, \quad R = \frac{1}{\sqrt{2\mu B}} \tag{51}$$

- If $\mathcal{O}(R_{\text{typ}}/R)$ is negligible, **compositeness X is determined by observables** (a_0, B)
- Cautions
  - X is the compositeness in P channel
  - When multiple discrete/continuum states are included in the Q channel, the leading order term in Eq. (51) does not change and the same result holds
  - With different UV regularizations (except for dimensional regularization) in Eq. (49), the leading order term in Eq. (51) does not change and the same result holds
  - It is assumed that the coefficients of the effective range expansion does not give larger length scale than R_{typ} (assumption of the order of $\Delta v_{\text{inv}}(0)$ in Exercise 6)

Exercise 6

1) Denoting $1/v(E) = v_{\rm inv}(E)$ and the expansion of a function $F(E)$ around $E = -B$ as

$$F(E) = F(-B) + (E+B)F'(-B) + \Delta F(E), \quad \Delta F(E) = \sum_{n=2}^{\infty} \frac{1}{n!}(E+B)^n F^{(n)}(-B),$$

show that the scattering length is written as $a_0 = \mu/(2\pi)[-BG'(-B)/X + \Delta v_{\rm inv}(0) - \Delta G(0)]^{-1}$.

2) Evaluating the integral in Eq. (49), we obtain

$$G(E) = \frac{\mu}{\pi^2}\left[-\Lambda + \sqrt{-2\mu E - i0^+}\arctan\frac{\Lambda}{\sqrt{-2\mu E - i0^+}}\right].$$

Expanding $\arctan x$ around $x = +\infty$ in this expression, show the following relations with $\Lambda = 1/R_{\rm typ}$ and $R = 1/\sqrt{2\mu B}$ (Note : Eq. (41) in Ref. [20] contains typos.):

$$-BG'(-B) = \frac{\mu}{4\pi R}\left[1 + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right], \quad -\Delta G(0) = \frac{\mu}{4\pi R}\left[1 + \mathcal{O}\left(\left(\frac{R_{\rm typ}}{R}\right)^3\right)\right].$$

3) Assuming $R\Delta v_{\rm inv}(0)/\mu \sim \mathcal{O}((R_{\rm typ}/R)^3)$, derive the weak-binding relation (51).

6.5 Generalization to unstable states

- For the application to hadron resonances, generalization to unstable states is necessary
- Add decay channel to the EFT of previous sections (Fig. 21)
  Fields : ϕ_1, ψ_1, ϕ_2, ψ_2, B_0
  four-point contact interactions : $\psi_1\phi_1 \leftrightarrow \psi_1\phi_1$, $\psi_1\phi_1 \leftrightarrow \psi_2\phi_2$, $\psi_2\phi_2 \leftrightarrow \psi_2\phi_2$
  three-point contact interactions : $\psi_1\phi_1 \leftrightarrow B_0$, $\psi_2\phi_2 \leftrightarrow B_0$
- When threshold energy $-\nu$ of the added channel 2 ($\psi_2\phi_2$) is lower than B, eigenenergy becomes complex, describing unstable resonance state (discussion in §4)

$$H|R\rangle = E_h|R\rangle, \quad E_h \in \mathbb{C}$$

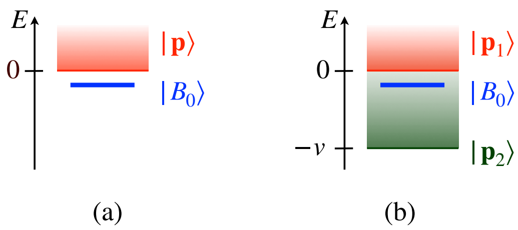

Figure 21: Spectrum of free Hamiltonian. (a) : EFT in Sec. 6.3 and 6.4, (b) : EFT in Sec. 6.5.

- Normalization condition of resonances ($\langle R | R \rangle$ diverges)

$$\langle \tilde{R} | R \rangle = 1$$

$\langle \tilde{R} |$ is the left-eigenvector with the same complex eigenvalue

$$\langle \tilde{R} | H = E_h \langle \tilde{R} |$$

(Because of the non-hermiticity, left-eigenvector is not conjugate of right-eigenvector)

- Complete set

$$\begin{aligned}
1 &= P + Q \\
P &= \int \frac{d\boldsymbol{p}}{(2\pi)^3} | \boldsymbol{p}_1 \rangle \langle \boldsymbol{p}_1 | \quad \text{(projection to } \psi_1\phi_1 \text{ scattering states)} \\
Q &= | B_0 \rangle \langle B_0 | + \int \frac{d\boldsymbol{p}}{(2\pi)^3} | \boldsymbol{p}_2 \rangle \langle \boldsymbol{p}_2 | \quad \text{(projection to discrete state } B_0 \text{ and } \psi_2\phi_2 \text{ scattering states)}
\end{aligned}$$

- Compositeness X, elementarity Z

$$\begin{aligned}
X &= \langle \tilde{R} | P | R \rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \tilde{R} | \boldsymbol{p}_1 \rangle \langle \boldsymbol{p}_1 | R \rangle \in \mathbb{C} \\
Z &= \langle \tilde{R} | Q | R \rangle = \langle \tilde{R} | B_0 \rangle \langle B_0 | R \rangle + \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \tilde{R} | \boldsymbol{p}_2 \rangle \langle \boldsymbol{p}_2 | R \rangle \in \mathbb{C}
\end{aligned}$$

- While sum of X and Z is normalized to 1, each term is **complex** and not interpreted as probabilities
- Expand the scattering length of channel 1 by (R_{typ}/R) (R, a_0 are complex)

$$a_0 = R \left\{ \frac{2X}{1+X} + \mathcal{O}\left(\left| \frac{R_{\text{typ}}}{R} \right| \right) + \mathcal{O}\left(\left| \frac{l}{R} \right|^3 \right) \right\}, \quad R = \frac{1}{\sqrt{-2\mu E_h}}, \quad l = \frac{1}{\sqrt{2\mu\nu}} \tag{52}$$

ν : threshold energy difference, l : corresponding length scale

- If $\mathcal{O}(R_{\text{typ}}/R)$ and $\mathcal{O}(|l/R|^3)$ are negligible, compositeness X is determined by observables (a_0, B)
- Interpretation of complex X : several proposals [19, 20, 25, 40, 41], not yet established
- $\Lambda(1405)$ case :
  - channel 1 is $\bar{K}N$, channel 2 is $\pi\Sigma$
  - Eigenenergy and scattering length : $E_h = -10 - 26i$ MeV, $a_0 = 1.39 - 0.85i$ fm [42, 43]
  - For $|R| \sim 2$ fm, $R_{\text{typ}} \sim 0.25$ fm (ρ meson exchange), $l \sim 1.08$ fm ($\pi\Sigma$ channel)

  Neglecting the correction terms, we obtain

$$X = 1.2 + i0.1$$

  X is close to 1 $\Rightarrow$ $\bar{K}N$ molecular component is dominant