

A.6 極座標変数の直交座標微分

極座標変数の直交座標微分 (87) の導出。

微分を定義通り計算すると

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial x} x^2 = \frac{1}{2r} 2x = \frac{x}{r} = \sin \theta \cos \phi \\
\frac{\partial}{\partial x} \cos \theta &= \frac{\partial}{\partial x} \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\
\frac{\partial \theta}{\partial x} \frac{\partial \cos \theta}{\partial \theta} &= z \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-1/2} \\
\frac{\partial \theta}{\partial x} (-\sin \theta) &= z \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} \frac{\partial}{\partial x} x^2 \\
-\sin \theta \frac{\partial \theta}{\partial x} &= -\frac{1}{2} z (x^2 + y^2 + z^2)^{-3/2} 2x \\
-\sin \theta \frac{\partial \theta}{\partial x} &= -\frac{xz}{r^3} \\
\frac{\partial \theta}{\partial x} &= \frac{xz}{r^3 \sin \theta} = \frac{r \sin \theta \cos \phi \cdot r \cos \theta}{r^3 \sin \theta} = \frac{\cos \theta \cos \phi}{r} \\
\frac{\partial}{\partial x} \tan \phi &= \frac{\partial}{\partial x} \frac{y}{x} \\
\frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \tan \phi &= y \frac{\partial}{\partial x} x^{-1} \\
\frac{\partial \phi}{\partial x} \frac{1}{\cos^2 \phi} &= y(-x^{-2}) \\
\frac{\partial \phi}{\partial x} &= -\cos^2 \phi \frac{y}{x^2} = -\cos^2 \phi \frac{r \sin \theta \sin \phi}{(r \sin \theta \cos \phi)^2} = -\frac{\sin \phi}{r \sin \theta} \\
\frac{\partial r}{\partial y} &= \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial y} y^2 = \frac{1}{2r} 2y = \frac{y}{r} = \sin \theta \sin \phi \\
\frac{\partial}{\partial y} \cos \theta &= \frac{\partial}{\partial y} \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\
\frac{\partial \theta}{\partial y} \frac{\partial \cos \theta}{\partial \theta} &= z \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-1/2} \\
\frac{\partial \theta}{\partial y} (-\sin \theta) &= z \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} \frac{\partial}{\partial y} y^2 \\
-\sin \theta \frac{\partial \theta}{\partial y} &= -\frac{1}{2} z (x^2 + y^2 + z^2)^{-3/2} 2y \\
-\sin \theta \frac{\partial \theta}{\partial y} &= -\frac{yz}{r^3} \\
\frac{\partial \theta}{\partial y} &= \frac{yz}{r^3 \sin \theta} = \frac{r \sin \theta \sin \phi \cdot r \cos \theta}{r^3 \sin \theta} = \frac{\cos \theta \sin \phi}{r} \\
\frac{\partial}{\partial y} \tan \phi &= \frac{\partial}{\partial y} \frac{y}{x} \\
\frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \tan \phi &= \frac{1}{x}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial y} \frac{1}{\cos^2 \phi} &= \frac{1}{x} \\
\frac{\partial \phi}{\partial y} &= \frac{\cos^2 \phi}{r \sin \theta \cos \phi} = \frac{\cos \phi}{r \sin \theta} \\
\frac{\partial r}{\partial z} &= \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial z} z^2 = \frac{1}{2r} 2z = \frac{z}{r} = \cos \theta \\
\frac{\partial}{\partial z} \cos \theta &= \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\
\frac{\partial \theta}{\partial z} \frac{\partial \cos \theta}{\partial \theta} &= (x^2 + y^2 + z^2)^{-1/2} + z \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-1/2} \\
\frac{\partial \theta}{\partial z} (-\sin \theta) &= \frac{1}{r} + z \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} \frac{\partial}{\partial z} z^2 \\
-\sin \theta \frac{\partial \theta}{\partial z} &= \frac{r^2}{r^3} - \frac{1}{2} z (x^2 + y^2 + z^2)^{-3/2} 2z \\
-\sin \theta \frac{\partial \theta}{\partial z} &= -\frac{-r^2 + z^2}{r^3} \\
\frac{\partial \theta}{\partial z} &= \frac{z^2 - r^2}{r^3 \sin \theta} = \frac{r^2 \cos^2 \theta - r^2}{r^3 \sin \theta} = \frac{\cos^2 \theta - 1}{r \sin \theta} = \frac{-\sin^2 \theta}{r \sin \theta} = -\frac{\sin \theta}{r} \\
\frac{\partial}{\partial z} \tan \phi &= \frac{\partial}{\partial z} \frac{y}{x} \\
\frac{\partial \phi}{\partial z} \frac{\partial}{\partial \phi} \tan \phi &= 0 \\
\frac{\partial \phi}{\partial z} &= 0
\end{aligned}$$

以上をまとめると (81) を得る。

別解) 直交座標変数 (x, y, z) の極座標微分は簡単に計算でき

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} &= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \\
&= \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{A3}
\end{aligned}$$

である。求めたい変換はこれの逆変換なので

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}$$

と逆行列を計算すれば良い。定義通り余因子行列を計算すると

$$\begin{aligned}
& \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{pmatrix}^{-1} \\
&= \frac{1}{\Delta} \begin{pmatrix} r^2 \sin^2 \theta \cos \phi & -(-r \sin \theta \cos \theta \cos \phi) & -r(\sin^2 \theta + \cos^2 \theta) \sin \phi \\ -(-r^2 \sin^2 \theta \sin \phi) & r \sin \theta \cos \theta \sin \phi & -(-r(\sin^2 \theta + \cos^2 \theta) \cos \phi) \\ r^2 \sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi) & -(r \sin^2 \theta (\cos^2 \phi + \sin^2 \phi)) & 0 \end{pmatrix} \\
&= \frac{1}{\Delta} \begin{pmatrix} r^2 \sin^2 \theta \cos \phi & r \sin \theta \cos \theta \cos \phi & -r \sin \phi \\ r^2 \sin^2 \theta \sin \phi & r \sin \theta \cos \theta \sin \phi & r \cos \phi \\ r^2 \sin \theta \cos \theta & -r \sin^2 \theta & 0 \end{pmatrix}
\end{aligned}$$

で行列式 Δ は変数変換のヤコビアンなので $r^2 \sin \theta$ 、よって

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} &= \frac{1}{r^2 \sin \theta} \begin{pmatrix} r^2 \sin^2 \theta \cos \phi & r \sin \theta \cos \theta \cos \phi & -r \sin \phi \\ r^2 \sin^2 \theta \sin \phi & r \sin \theta \cos \theta \sin \phi & r \cos \phi \\ r^2 \sin \theta \cos \theta & -r \sin^2 \theta & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \\
&= \begin{pmatrix} \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\ \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\ \cos \theta & -\frac{\sin \theta}{r} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}
\end{aligned}$$

この各成分が以下のように対応するので、式 (8.1) を得る。

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \phi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \phi}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \phi}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}$$

となる。

別解の別解) 極座標の ∇ が

$$\nabla = \left(\frac{\partial}{\partial r} \quad \frac{1}{r} \frac{\partial}{\partial \theta} \quad \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right)$$

であることに注意し、極座標に変換した後の微分演算子を ∇ に揃えて表記すると、最初の変換は

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} &= \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \\
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{pmatrix} &= \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}
\end{aligned}$$

と書くことができる。ここでの変換行列は $R^{-1} = R^t$ を満たす直交行列なので、逆行列は転置で与えられ、

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} &= \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{pmatrix} \\
&= \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{pmatrix} \\
&= \begin{pmatrix} \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\ \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\ \cos \theta & -\frac{\sin \theta}{r} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}
\end{aligned}$$

となる。

A.7 極座標のラプラシアン（直接計算）

極座標のラプラシアン

$$\nabla^2 \psi = \frac{\partial^2}{\partial r^2} \psi + \frac{2}{r} \frac{\partial}{\partial r} \psi + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \psi + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \psi + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi \quad (\text{A2})$$

の導出。

まず、関数 $f(r, \theta, \phi)$ の x, y, z による 2 階微分を計算する。 $(\partial f / \partial x)$ を一つの関数とみなして偏微分の座標変換の式を使うと、

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \\ &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial x} \right) \\ &= \sin \theta \cos \phi \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial x} \right) \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2} f &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \\ &= \sin \theta \sin \phi \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial y} \right) + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial y} \right) \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \frac{\partial^2}{\partial z^2} f &= \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) \\ &= \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial z} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial z} \right) \end{aligned} \quad (\text{A6})$$

以下、この各項を評価する。式 (A4) 第 1 項は

$$\begin{aligned} \sin \theta \cos \phi \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) &= \sin \theta \cos \phi \frac{\partial}{\partial r} \left(\sin \theta \cos \phi \frac{\partial f}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial f}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial f}{\partial \phi} \right) \\ &= \sin^2 \theta \cos^2 \phi \frac{\partial^2}{\partial r^2} f \\ &\quad + \sin \theta \cos \theta \cos^2 \phi \frac{-1}{r^2} \frac{\partial}{\partial \theta} f + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\ &\quad - \sin \phi \cos \phi \frac{-1}{r^2} \frac{\partial}{\partial \phi} f - \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\ &= \sin^2 \theta \cos^2 \phi \frac{\partial^2}{\partial r^2} f - \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \frac{\partial}{\partial \theta} f + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\ &\quad + \frac{\sin \phi \cos \phi}{r^2} \frac{\partial}{\partial \phi} f - \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \end{aligned} \quad (\text{A7})$$

式 (A4) 第 2 項は

$$\begin{aligned}
\frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) &= \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \cos \phi \frac{\partial}{\partial r} f + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} f - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} f \right) \\
&= \frac{\cos \theta \cos^2 \phi}{r} \cos \theta \frac{\partial}{\partial r} f + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\
&\quad + \frac{\cos \theta \cos^2 \phi}{r^2} (-\sin \theta) \frac{\partial}{\partial \theta} f + \frac{\cos^2 \theta \cos^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} f \\
&\quad - \frac{\cos \theta \sin \phi \cos \phi}{r^2} \left(\frac{-1}{\sin^2 \theta} \cos \theta \right) \frac{\partial}{\partial \phi} f - \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\
&= \frac{\cos^2 \theta \cos^2 \phi}{r} \frac{\partial}{\partial r} f + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\
&\quad - \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \frac{\partial}{\partial \theta} f + \frac{\cos^2 \theta \cos^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} f \\
&\quad + \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} f - \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f
\end{aligned} \tag{A8}$$

式 (A4) 第 3 項は

$$\begin{aligned}
-\frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial x} \right) &= -\frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\sin \theta \cos \phi \frac{\partial}{\partial r} f + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} f - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} f \right) \\
&= -\frac{\sin \phi}{r} (-\sin \phi) \frac{\partial}{\partial r} f - \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\
&\quad - \frac{\sin \phi \cos \theta}{r^2 \sin \theta} (-\sin \phi) \frac{\partial}{\partial \theta} f - \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\
&\quad + \frac{\sin \phi}{r^2 \sin^2 \theta} \cos \phi \frac{\partial}{\partial \phi} f + \frac{\sin^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f \\
&= \frac{\sin^2 \phi}{r} \frac{\partial}{\partial r} f - \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\
&\quad + \frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} f - \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\
&\quad + \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} f + \frac{\sin^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f
\end{aligned} \tag{A9}$$

となる。次に式 (A5) 第 1 項は

$$\begin{aligned}
\sin \theta \sin \phi \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) &= \sin \theta \sin \phi \frac{\partial}{\partial r} \left(\sin \theta \sin \phi \frac{\partial}{\partial r} f + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} f + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} f \right) \\
&= \sin^2 \theta \sin^2 \phi \frac{\partial^2}{\partial r^2} f \\
&\quad + \sin \theta \cos \theta \sin^2 \phi \frac{-1}{r^2} \frac{\partial}{\partial \theta} f + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\
&\quad + \sin \phi \cos \phi \frac{-1}{r^2} \frac{\partial}{\partial \phi} f + \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\
&= \sin^2 \theta \sin^2 \phi \frac{\partial^2}{\partial r^2} f - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \frac{\partial}{\partial \theta} f + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f
\end{aligned}$$

$$- \frac{\sin \phi \cos \phi}{r^2} \frac{\partial}{\partial \phi} f + \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \quad (\text{A10})$$

式 (A5) 第2項は

$$\begin{aligned} \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial y} \right) &= \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \sin \phi \frac{\partial}{\partial r} f + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} f + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} f \right) \\ &= \frac{\cos \theta \sin^2 \phi}{r} \cos \theta \frac{\partial}{\partial r} f + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\ &\quad + \frac{\cos \theta \sin^2 \phi}{r^2} (-\sin \theta) \frac{\partial}{\partial \theta} f + \frac{\cos^2 \theta \sin^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} f \\ &\quad + \frac{\cos \theta \sin \phi \cos \phi}{r^2} \left(\frac{-\cos \theta}{\sin^2 \theta} \right) \frac{\partial}{\partial \phi} f + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\ &= \frac{\cos^2 \theta \sin^2 \phi}{r} \frac{\partial}{\partial r} f + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} f \\ &\quad - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \frac{\partial}{\partial \theta} f + \frac{\cos^2 \theta \sin^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} f \\ &\quad - \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} f + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \end{aligned} \quad (\text{A11})$$

式 (A5) 第3項は

$$\begin{aligned} \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial y} \right) &= \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\sin \theta \sin \phi \frac{\partial}{\partial r} f + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} f + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} f \right) \\ &= \frac{\cos \phi}{r} \cos \phi \frac{\partial}{\partial r} f + \frac{\cos \phi \sin \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\ &\quad + \frac{\cos \theta \cos \phi}{r^2 \sin \theta} \cos \phi \frac{\partial}{\partial \theta} f + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\ &\quad + \frac{\cos \phi}{r^2 \sin^2 \theta} (-\sin \phi) \frac{\partial}{\partial \phi} f + \frac{\cos^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f \\ &= \frac{\cos^2 \phi}{r} \frac{\partial}{\partial r} f + \frac{\cos \phi \sin \phi}{r} \frac{\partial^2}{\partial r \partial \phi} f \\ &\quad + \frac{\cos \theta \cos^2 \phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} f + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} f \\ &\quad - \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} f + \frac{\cos^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f \end{aligned} \quad (\text{A12})$$

となる。最後に式 (A6) 第1項は

$$\begin{aligned} \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial z} \right) &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial}{\partial r} f - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} f \right) \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} f \\ &\quad - \sin \theta \cos \theta \left(\frac{-1}{r^2} \right) \frac{\partial}{\partial \theta} f - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} f \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} f + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} f - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} f \end{aligned} \quad (\text{A13})$$

式 (A6) 第2項は

$$\begin{aligned}
-\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial z} \right) &= -\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial}{\partial r} f - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} f \right) \\
&= -\frac{\sin \theta}{r} (-\sin \theta) \frac{\partial}{\partial r} f - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} f \\
&\quad + \frac{\sin \theta}{r^2} \cos \theta \frac{\partial}{\partial \theta} f + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} f \\
&= \frac{\sin^2 \theta}{r} \frac{\partial}{\partial r} f - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} f + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} f + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} f \quad (\text{A14})
\end{aligned}$$

となる。以上を利用し

$$\nabla^2 \psi = \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial y^2} \psi + \frac{\partial^2}{\partial z^2} \psi$$

を ψ の極座標微分の各項ごとに調べる。

$$\begin{aligned}
\frac{\partial^2}{\partial r^2} \psi \text{ の項} : & \sin^2 \theta \cos^2 \phi \frac{\partial^2}{\partial r^2} \psi + \sin^2 \theta \sin^2 \phi \frac{\partial^2}{\partial r^2} \psi + \cos^2 \theta \frac{\partial^2}{\partial r^2} \psi \quad \leftarrow (\text{A7}), (\text{A10}), (\text{A13}) \\
&= [\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta] \frac{\partial^2}{\partial r^2} \psi \\
&= (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2}{\partial r^2} \psi \\
&= \frac{\partial^2}{\partial r^2} \psi \\
\frac{\partial^2}{\partial r \partial \theta} \psi \text{ の項} : & \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} \psi + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} \psi + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} \psi \\
&\quad + \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2}{\partial r \partial \theta} \psi - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} \psi - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} \psi \\
&\quad \leftarrow (\text{A7}), (\text{A8}), (\text{A10}), (\text{A11}), (\text{A13}), (\text{A14}) \\
&= \left[2 \frac{\sin \theta \cos \theta \cos^2 \phi}{r} + 2 \frac{\sin \theta \cos \theta \sin^2 \phi}{r} - 2 \frac{\sin \theta \cos \theta}{r} \right] \frac{\partial^2}{\partial r \partial \theta} \psi \\
&= \left[2 \frac{\sin \theta \cos \theta}{r} - 2 \frac{\sin \theta \cos \theta}{r} \right] \frac{\partial^2}{\partial r \partial \theta} \psi \\
&= 0 \\
\frac{\partial^2}{\partial r \partial \phi} \psi \text{ の項} : & -\frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} \psi - \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} \psi + \frac{\sin \phi \cos \phi}{r} \frac{\partial^2}{\partial r \partial \phi} \psi + \frac{\cos \phi \sin \phi}{r} \frac{\partial^2}{\partial r \partial \phi} \psi \\
&\quad \leftarrow (\text{A7}), (\text{A9}), (\text{A10}), (\text{A12}) \\
&= 0 \\
\frac{\partial^2}{\partial \theta^2} \psi \text{ の項} : & \frac{\cos^2 \theta \cos^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} \psi + \frac{\cos^2 \theta \sin^2 \phi}{r^2} \frac{\partial^2}{\partial \theta^2} \psi + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} \psi \quad \leftarrow (\text{A8}), (\text{A11}), (\text{A14}) \\
&= \left[\frac{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi)}{r^2} + \frac{\sin^2 \theta}{r^2} \right] \frac{\partial^2}{\partial \theta^2} \psi \\
&= \left[\frac{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi)}{r^2} + \frac{\sin^2 \theta}{r^2} \right] \frac{\partial^2}{\partial \theta^2} \psi
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\cos^2 \theta + \sin^2 \theta)}{r^2} \frac{\partial^2}{\partial \theta^2} \psi \\
&= \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \psi \\
\frac{\partial^2}{\partial \theta \partial \phi} \psi \text{ の項} &: -\frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \psi - \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \psi + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \psi \\
&\quad + \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \psi \quad \leftarrow (\text{A8}), (\text{A9}), (\text{A11}), (\text{A12}) \\
&= 0 \\
\frac{\partial^2}{\partial \phi^2} \psi \text{ の項} &: \frac{\sin^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi + \frac{\cos^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi \quad \leftarrow (\text{A9}), (\text{A12}) \\
&= \frac{\sin^2 \phi + \cos^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi \\
&= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi \\
\frac{\partial}{\partial r} \psi \text{ の項} &: \frac{\cos^2 \theta \cos^2 \phi}{r} \frac{\partial}{\partial r} \psi + \frac{\sin^2 \phi}{r} \frac{\partial}{\partial r} \psi + \frac{\cos^2 \theta \sin^2 \phi}{r} \frac{\partial}{\partial r} \psi + \frac{\cos^2 \phi}{r} \frac{\partial}{\partial r} \psi + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial r} \psi \\
&\quad \leftarrow (\text{A8}), (\text{A9}), (\text{A11}), (\text{A12}), (\text{A14}) \\
&= \left[\frac{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi)}{r} + \frac{\sin^2 \phi + \cos^2 \phi}{r} + \frac{\sin^2 \theta}{r} \right] \frac{\partial}{\partial r} \psi \\
&= \left[\frac{(\cos^2 \theta + \sin^2 \theta)}{r} + \frac{1}{r} \right] \frac{\partial}{\partial r} \psi \\
&= \frac{2}{r} \frac{\partial}{\partial r} \psi \\
\frac{\partial}{\partial \theta} \psi \text{ の項} &: -\frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \frac{\partial}{\partial \theta} \psi - \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \frac{\partial}{\partial \theta} \psi + \frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \psi \\
&\quad - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \frac{\partial}{\partial \theta} \psi - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \frac{\partial}{\partial \theta} \psi + \frac{\cos \theta \cos^2 \phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \psi \\
&\quad + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} \psi + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} \psi \\
&\quad \leftarrow (\text{A7}), (\text{A8}), (\text{A9}), (\text{A10}), (\text{A11}), (\text{A12}), (\text{A13}), (\text{A14}) \\
&= \left[-2 \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} + \frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \right. \\
&\quad \left. - 2 \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} + \frac{\cos \theta \cos^2 \phi}{r^2 \sin \theta} \right. \\
&\quad \left. + 2 \frac{\sin \theta \cos \theta}{r^2} \right] \frac{\partial}{\partial \theta} \psi \\
&= \left[-2 \frac{\sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi)}{r^2} + 2 \frac{\sin \theta \cos \theta}{r^2} + \frac{\cos \theta (\sin^2 \phi + \cos^2 \phi)}{r^2 \sin \theta} \right] \frac{\partial}{\partial \theta} \psi \\
&= \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \psi
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \phi} \psi \text{ の項} : & \frac{\sin \phi \cos \phi}{r^2} \frac{\partial}{\partial \phi} \psi + \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \psi + \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \psi - \frac{\sin \phi \cos \phi}{r^2} \frac{\partial}{\partial \phi} \psi \\
& - \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \psi - \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \psi \quad \leftarrow (\text{A7}), (\text{A8}), (\text{A9}), (\text{A10}), (\text{A11}), (\text{A12}) \\
& = 0
\end{aligned}$$

以上まとめると

$$= \frac{\partial^2}{\partial r^2} \psi + \frac{2}{r} \frac{\partial}{\partial r} \psi + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \psi + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \psi + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi$$

を得る。

A.8 極座標のラプラシアン (計量テンソル)

極座標のラプラシアン

$$\nabla^2\psi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \psi \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \psi \right] \quad (83)$$

の計量テンソルを使った導出。

3次元の曲線座標でのラプラシアンは一般に

$$\nabla^2\psi = \frac{1}{\sqrt{g}} \sum_{i,j=1}^3 \frac{\partial}{\partial q^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial q^j} \right) \psi$$

と与えられる (導出は国広悌二「量子力学」(東京図書)p.78、猪木慶治・川合光「量子力学I」(講談社サイエンティフィク)p.286などを参照)。計量テンソルなどの定義は

$$g_{ij} = \sum_{k=1}^3 \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j}, \quad g = \det g_{ij}, \quad \sum_{k=1}^3 g^{ik} g_{kj} = \delta_j^i \quad (g^{ij} \text{は } g_{ij} \text{の逆行列})$$

であり、極座標の場合の計算には

$$(q_1, q_2, q_3) = (r, \theta, \phi)$$

とすれば良い。計量テンソル g_{ij} を計算すると⁵

$$\begin{aligned} g_{11} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial r} \\ &= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta \\ &= \sin^2\theta + \cos^2\theta \\ &= 1 \\ g_{12} = g_{21} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\ &= \sin\theta \cos\phi r \cos\theta \cos\phi + \sin\theta \sin\phi r \cos\theta \sin\phi + \cos\theta(-r \sin\theta) \\ &= r \sin\theta \cos\theta \cos^2\phi + r \cos\theta \sin\theta \sin^2\phi - r \sin\theta \cos\theta \\ &= r \sin\theta \cos\theta - r \sin\theta \cos\theta \\ &= 0 \\ g_{13} = g_{31} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\ &= \sin\theta \cos\phi(-r \sin\theta \sin\phi) + \sin\theta \sin\phi r \sin\theta \cos\phi + \cos\theta \times 0 \\ &= -r \sin^2\theta \sin\phi \cos\phi + r \sin^2\theta \sin\phi \cos\phi \end{aligned}$$

⁵極座標での微小線素が $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ であることを使えば係数から計量テンソルがすぐ得られる。

$$\begin{aligned}
&= 0 \\
g_{22} &= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta} \\
&= r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \\
&= r^2 (\cos^2 \theta + \sin^2 \theta) \\
&= r^2 \\
g_{23} = g_{32} &= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} \\
&= r \cos \theta \cos \phi (-r \sin \theta \sin \phi) + r \cos \theta \sin \phi r \sin \theta \cos \phi + \cos \theta \times 0 \\
&= -r^2 \sin \theta \cos \theta \sin \phi \cos \phi + r^2 \sin \theta \cos \theta \sin \phi \cos \phi \\
&= 0 \\
g_{33} &= \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi} \frac{\partial z}{\partial \phi} \\
&= r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi \\
&= r^2 \sin^2 \theta
\end{aligned}$$

つまり

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

を得る。これを用いて

$$\begin{aligned}
\sqrt{g} &= \sqrt{r^4 \sin^2 \theta} = r^2 \sin \theta \\
g^{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}
\end{aligned}$$

ここで対角行列の逆行列は各成分の逆数をとったもので与えられることを用いた。以上より

$$\begin{aligned}
\nabla^2 \psi &= \frac{1}{r^2 \sin \theta} \sum_{ij} \frac{\partial}{\partial q^i} \left(r^2 \sin \theta g^{ij} \frac{\partial}{\partial q^j} \right) \psi \\
&= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta g^{11} \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(r^2 \sin \theta g^{22} \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(r^2 \sin \theta g^{33} \frac{\partial}{\partial \phi} \right) \right] \psi \\
&= \left[\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial}{\partial \phi} \right) \right] \psi \\
&= \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi
\end{aligned}$$

を得る。第1項は

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) \psi = \frac{1}{r^2} \left(2r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right) \psi = \frac{\partial^2}{\partial r^2} \psi + \frac{2}{r} \frac{\partial}{\partial r} \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi)$$

となるので

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\psi\right]$$

を得る。