10 Classical and Quantum Correlations

10.1 Pure and mixed states

- **Pure state**: density matrix by a single state

$$\rho = |\psi\rangle\langle\psi| \tag{229}$$

Property (idempotency): ρ is a projection operator

$$\rho^2 = |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{=1} \langle\psi| = |\psi\rangle\langle\psi| = \rho, \tag{230}$$

$$\mathrm{Tr}\left[\rho^2\right] = \mathrm{Tr}\left[\rho\right] = 1 \quad \text{(pure state)} \tag{231}$$

- **Mixed state**: density matrix by a weighted sum of states with **real** coefficients $0 \le p_i \le 1$

$$\rho = \sum_i p_i |i\rangle\langle i|, \quad \sum_i p_i = 1. \tag{232}$$

An ensemble of states prepared with probability p_i ("mixed ensemble" in J.J. Sakurai).
Classical probabilistic mixture of states, **not** a quantum superposition
The decomposition of a mixed state is not unique (Schrödinger's mixture theorem).

- A mixed state is normalized but does not satisfy $\rho^2 = \rho$:

$$\mathrm{Tr}\left[\rho\right] = \sum_i p_i \mathrm{Tr}\left[|i\rangle\langle i|\right] = \sum_i p_i = 1, \tag{233}$$

$$\rho^2 = \sum_{i,j} p_i p_j |i\rangle\langle i|j\rangle\langle j| = \sum_i p_i^2 |i\rangle\langle i| \neq \rho. \tag{234}$$

For $0 \le p_i \le 1$, $\sum_i p_i^2 \le \sum_i p_i$, therefore

$$\mathrm{Tr}\left[\rho^2\right] \le 1 \quad \text{(mixed state)}. \tag{235}$$

Equality holds for a pure state

- **Purity** $\mathrm{Tr}\left[\rho^2\right]$: distance from a pure state

$$\underbrace{1}_{\text{pure state}} \ge \mathrm{Tr}\left[\rho^2\right] \ge \underbrace{\frac{1}{d}}_{\text{maximally mixed state}}, \quad d = \dim \mathcal{H} \tag{236}$$

The equalities hold for

$$\{p_i\} = \begin{cases} \{1, 0, \ldots, 0\} & \text{(pure state, rank}(\rho) = 1), \\ \{1/d, 1/d, \ldots, 1/d\} & \text{(maximally mixed state, } \rho \propto 1). \end{cases} \tag{237}$$

- Example: pure states for a single-spin system

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{238}$$

$$|+\rangle\langle+| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \tag{239}$$

After diagonalization, Eq. (239) takes the same form as Eq. (238).

- Example: mixed state for a single-spin system

$$\frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{240}$$

Formally, this is $|+\rangle\langle+|$ without the off-diagonal elements

10.2 Correlations between two spins

- **Measurement** of s_z of both spins A and B for a two-spin state $|\psi\rangle$
  The probability of obtaining both spins up is

$$\Pr(\uparrow\uparrow) = |\langle\uparrow\uparrow|\psi\rangle|^2 = \mathrm{Tr}\left[P_{\uparrow\uparrow}\rho\right], \quad P_{\uparrow\uparrow} = P_\uparrow \otimes P_\uparrow. \tag{241}$$

- Consider the following density matrices:

$$\rho_p = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| \quad \text{(pure \textbf{product state})}, \tag{242}$$

$$\rho_m = \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| \quad \text{(\textbf{mixed state})}, \tag{243}$$

$$\rho_e = |\Phi^+\rangle\langle\Phi^+| = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} \frac{\langle\uparrow\uparrow| + \langle\downarrow\downarrow|}{\sqrt{2}} \quad \text{(pure \textbf{Bell state})}. \tag{244}$$

Same "correlation" that "if spin A is up, then spin B is also up"?

- Measurement of the spin (z axis)
  - For ρ_p: since $|\psi\rangle = |\uparrow\uparrow\rangle$,

$$\Pr(\uparrow\uparrow) = |\langle\uparrow\uparrow|\uparrow\uparrow\rangle|^2 = 1, \quad \Pr(\uparrow\downarrow) = |\langle\uparrow\downarrow|\uparrow\uparrow\rangle|^2 = 0, \quad \cdots \tag{245}$$

  - For ρ_m:

$$\Pr(\uparrow\uparrow) = \mathrm{Tr}\left[P_{\uparrow\uparrow}\rho_m\right] = \langle\uparrow\uparrow|\left(\frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)|\uparrow\uparrow\rangle = \frac{1}{2}, \tag{246}$$

$$\Pr(\uparrow\downarrow) = \mathrm{Tr}\left[P_{\uparrow\downarrow}\rho_m\right] = \langle\uparrow\downarrow|\left(\frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)|\uparrow\downarrow\rangle = 0, \quad \cdots \tag{247}$$

– For ρ_e: since $|\psi\rangle = |\Phi^+\rangle$,

$$\Pr(\uparrow\uparrow) = |\langle\uparrow\uparrow|\Phi^+\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \quad \Pr(\uparrow\downarrow) = |\langle\uparrow\downarrow|\Phi^+\rangle|^2 = 0, \quad \cdots \tag{248}$$

- Measurement of the spin (**x axis**)

  – For ρ_p: from Eq. (174),

$$|{+}{+}\rangle = \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \tag{249}$$

which gives

$$\Pr({+}{+}) = |\langle{+}{+}|\uparrow\uparrow\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}, \tag{250}$$

$$\Pr({+}{-}) = |\langle{+}{-}|\uparrow\uparrow\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}, \quad \cdots \tag{251}$$

  – For ρ_m:

$$\begin{aligned}\Pr({+}{+}) &= \mathrm{Tr}\left[P_{++}\rho_m\right] = \langle{+}{+}|\left(\frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)|{+}{+}\rangle \\ &= \frac{1}{2}\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{4}, \quad \cdots\end{aligned} \tag{252}$$

  – For ρ_e:

$$|\Phi^+\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \tag{253}$$

$$\begin{aligned}\frac{|{+}{+}\rangle + |{-}{-}\rangle}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\\1/\sqrt{2}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\\-1/\sqrt{2}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1/2+1/2\\1/2-1/2\\1/2-1/2\\1/2+1/2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = |\Phi^+\rangle\end{aligned} \tag{254}$$

Namely, **$|\Phi^+\rangle$ is also a Bell state in the x basis**. Therefore,

$$\rho_e = \left(\frac{|{+}{+}\rangle + |{-}{-}\rangle}{\sqrt{2}}\right)\left(\frac{\langle{+}{+}| + \langle{-}{-}|}{\sqrt{2}}\right). \tag{255}$$

Replacing $(\uparrow,\downarrow) \leftrightarrow (+,-)$, the calculation is identical to that in the z basis:

$$\Pr({+}{+}) = \Pr(\uparrow\uparrow) = \frac{1}{2}, \quad \Pr({+}{-}) = \Pr(\uparrow\downarrow) = 0, \quad \cdots \tag{256}$$

Table 2: Probabilities of spin-measurement outcomes for the density matrices ρ_p, ρ_m, and ρ_e.

density matrix	$\uparrow\uparrow$	$\uparrow\downarrow$	$\downarrow\uparrow$	$\downarrow\downarrow$	$++$	$+-$	$-+$	$--$
ρ_p	1	0	0	0	1/4	1/4	1/4	1/4
ρ_m	1/2	0	0	1/2	1/4	1/4	1/4	1/4
ρ_e	1/2	0	0	1/2	1/2	0	0	1/2

- Summary (Table 2)
  - ρ_p: spins aligned only when both are **up along the z axis**, s_x are completely random
  - ρ_m: spins aligned for both **up and down along the z axis**, s_x are completely random
    Observing $|\uparrow_A\rangle$ determines $|\uparrow_B\rangle$, while observing $|+_A\rangle$ leaves spin B completely random.
  - ρ_e: spins aligned in **all cases**, both x and z directions
    Observing $|\uparrow_A\rangle$ determines $|\uparrow_B\rangle$, and observing $|+_A\rangle$ determines $|+_B\rangle$.

- Difference between ρ_m and ρ_e:

$$\begin{aligned}\rho_e &= \frac{1}{2}\left|\uparrow\uparrow\right\rangle\left\langle\uparrow\uparrow\right| + \frac{1}{2}\left|\uparrow\uparrow\right\rangle\left\langle\downarrow\downarrow\right| + \frac{1}{2}\left|\downarrow\downarrow\right\rangle\left\langle\uparrow\uparrow\right| + \frac{1}{2}\left|\downarrow\downarrow\right\rangle\left\langle\downarrow\downarrow\right| \\ &= \rho_m \underbrace{+\frac{1}{2}\left|\uparrow\uparrow\right\rangle\left\langle\downarrow\downarrow\right| + \frac{1}{2}\left|\downarrow\downarrow\right\rangle\left\langle\uparrow\uparrow\right|}_{\text{interference terms}}\end{aligned} \tag{257}$$

  ρ_e: a **coherent** superposition with complex coefficients: **quantum correlations**.
  ρ_m: a probabilistic mixture of density matrices with real weights: classical correlations

10.3 Entanglement and reduced density matrices

- If ρ is a pure **product state**, then ρ_A is also a pure state:

$$\rho = \Big(|\psi\rangle \otimes |\phi\rangle\Big)\Big(\langle\psi| \otimes \langle\phi|\Big) = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|, \tag{258}$$

$$\rho_A = \mathrm{Tr}_B\left[\rho\right] = |\psi\rangle\langle\psi|\,\mathrm{Tr}\left[|\phi\rangle\langle\phi|\right] = |\psi\rangle\langle\psi|. \tag{259}$$

  For $\rho_p = \left|\uparrow\uparrow\right\rangle\left\langle\uparrow\uparrow\right| = |\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow|$, substituting $\alpha = 1$, $\beta = \gamma = \delta = 0$ into Eq. (210) gives

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\uparrow\rangle\langle\uparrow|. \tag{260}$$

- If ρ is an **entangled pure state**, then ρ_A **is a mixed state**.
  For $\rho_e = |\Phi^+\rangle\langle\Phi^+|$, substituting $\alpha = \delta = 1/\sqrt{2}$, $\beta = \gamma = 0$ into Eq. (210) gives

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| \tag{261}$$

- Even when ρ is in a pure state, ρ_A can be a mixed state.
  Conversely, every mixed state can be regarded as ρ_A **of a pure state** in a larger Hilbert space (**purification**).

- The linear entropy measures the deviation of the **purity of** ρ_A from unity (a pure state):

$$\mathcal{E} = 1 - \mathrm{Tr}\left[\rho_A^2\right]. \tag{262}$$