11 Bell's inequality

11.1 EPR pair and spin correlation

- EPR pair (Bell state with total spin zero):

$$|\Psi^-\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{263}$$

The probabilities of obtaining $|\uparrow\rangle$ and $|\downarrow\rangle$ by **measuring only spin A** are equal:

$$\Pr(\uparrow_A) = \mathrm{Tr}\left[\, |\uparrow\rangle\langle\uparrow|\, \rho_A\right] = \frac{1}{2}, \quad \Pr(\downarrow_A) = \mathrm{Tr}\left[\, |\downarrow\rangle\langle\downarrow|\, \rho_A\right] = \frac{1}{2}. \tag{264}$$

- If $|\uparrow\rangle$ is observed, the wave function collapses to

$$|\Psi^-\rangle \to |\uparrow\downarrow\rangle\,, \tag{265}$$

so the **unmeasured spin B is determined** to be $|\downarrow\rangle$.

- Measurement fixes the state (a superposition before the measurement as Schrödinger's cat).

- Quantum correlation is preserved even when spins A and B are spatially separated.

- Einstein–Podolsky–Rosen [53]: counterintuitive?
Locality: an operation at one location cannot influence a distant system.
Physical reality: physical quantities have definite values before measurement.

- **Spin correlation function**: for unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$,

$$C(\boldsymbol{n},\boldsymbol{m}) = \langle\Psi^-|\boldsymbol{n}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{m}\cdot\boldsymbol{\sigma}|\Psi^-\rangle = n_i m_j \langle\Psi^-|\sigma_i\otimes\sigma_j|\Psi^-\rangle \tag{266}$$

Expectation value by measuring spin A along $\boldsymbol{n}$ and spin B along $\boldsymbol{m}$ (Fig. 21, left).

- Since $|\Psi^-\rangle$ has total spin zero,

$$\begin{aligned}(\sigma_i\otimes 1 + 1\otimes\sigma_i)\,|\Psi^-\rangle &= 0\\ \sigma_i\otimes 1\,|\Psi^-\rangle &= -1\otimes\sigma_i\,|\Psi^-\rangle\,.\end{aligned} \tag{267}$$

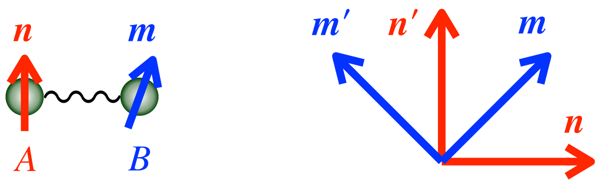

Figure 21: Left: schematic illustration of the spin correlation function. Right: measurement axes that maximize the CHSH correlation in Eq. (281).

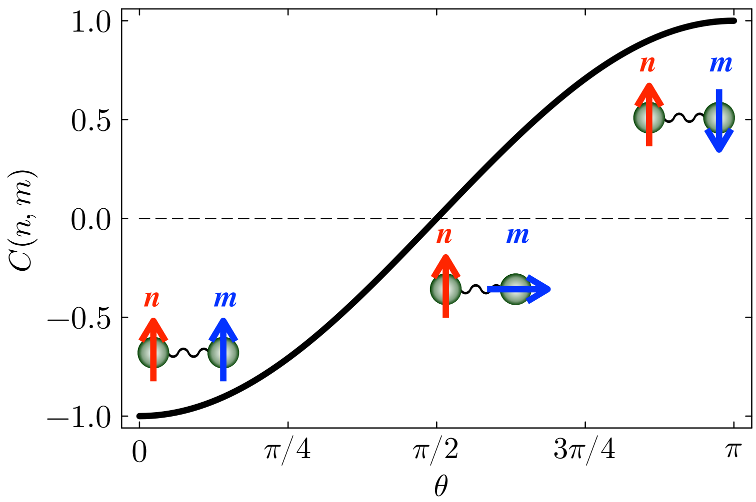

Figure 22: Spin correlation function $C(\boldsymbol{n}, \boldsymbol{m})$ in Eq. (268).

The correlation function is determined by the angle θ between $\boldsymbol{n}$ and $\boldsymbol{m}$:

$$\begin{aligned} C(\boldsymbol{n}, \boldsymbol{m}) &= -n_i m_j \langle \Psi^- | \sigma_i \sigma_j \otimes 1 | \Psi^- \rangle \\ &= -n_i m_j \langle \Psi^- | (\delta_{ij} + i\epsilon_{ijk}\sigma_k) \otimes 1 | \Psi^- \rangle \\ &= -\boldsymbol{n} \cdot \boldsymbol{m} \\ &= -\cos\theta. \end{aligned} \tag{268}$$

- $C(\boldsymbol{n}, \boldsymbol{m}) = 1$ for antiparallel axes, 0 for perpendicular axes, and -1 for parallel axes (Fig. 22). Since $|\Psi^-\rangle$ always consists of antiparallel spins, the correlation is maximal at $\theta = \pi$.

11.2 Hidden variable theory

- Locally realistic theory with an unknown parameter λ and a probability distribution $\rho(\lambda)$

- Expectation value of an operator O is given by

$$\langle O \rangle_{\mathrm{HV}} = \int d\lambda\, \rho(\lambda) O(\lambda), \quad \int d\lambda\, \rho(\lambda) = 1. \tag{269}$$

$\rightarrow \rho(\lambda)$ plays a role of the density matrix ρ.
The distribution is chosen so that the **expectation value agrees** with quantum mechanics:

$$\langle O \rangle_{\mathrm{HV}} = \mathrm{Tr}\,[O\rho]\,. \tag{270}$$

- The measurement outcome is predetermined.
The probabilistic nature $\leftarrow$ unknown variable λ.

- Spin correlation,

$$C(\boldsymbol{n},\boldsymbol{m})_{\mathrm{HV}} = \int d\lambda\, \rho(\lambda) N(\boldsymbol{n},\lambda) M(\boldsymbol{m},\lambda) \tag{271}$$

  $N(\boldsymbol{n},\lambda) \in \{1,-1\}$: outcome of a single measurement (fixed λ) of spin A along the $\boldsymbol{n}$ axis

- With suitable choice of distributions, $C(\boldsymbol{n},\boldsymbol{m})_{\mathrm{HV}} = C(\boldsymbol{n},\boldsymbol{m})$ for **two axes** $\{\boldsymbol{n},\boldsymbol{m}\}$.

- When $\boldsymbol{n} = \boldsymbol{m}$, Eq. (268) gives $C(\boldsymbol{n},\boldsymbol{n}) = -1$. Therefore,

$$C(\boldsymbol{n},\boldsymbol{n})_{\mathrm{HV}} = \int d\lambda\, \rho(\lambda) N(\boldsymbol{n},\lambda) M(\boldsymbol{n},\lambda) = -1. \tag{272}$$

  Since $N(\boldsymbol{n},\lambda)M(\boldsymbol{n},\lambda) = \pm 1$ for every λ, this is possible only if

$$N(\boldsymbol{n},\lambda) = -M(\boldsymbol{n},\lambda) \qquad \text{(perfectly anticorrelated, c.f., Eq. (267))} \tag{273}$$

- Bell's inequality [55]: for correlations involving **three or more axes**, the predictions of quantum mechanics and hidden variable theory differ.
  (Equivalently, quantum mechanics cannot be described by a single $\rho(\lambda)$.)

11.3 CHSH inequality

- Extension of Bell's inequality by Clauser, Horne, Shimony, and Holt (CHSH) [54].

- Choose two measurement axes for each spin:

$$\text{spin } A : \boldsymbol{n}, \boldsymbol{n}', \tag{274}$$
$$\text{spin } B : \boldsymbol{m}, \boldsymbol{m}'. \tag{275}$$

- CHSH correlation: a combination of four spin correlations,

$$S = C(\boldsymbol{n},\boldsymbol{m}) - C(\boldsymbol{n},\boldsymbol{m}') + C(\boldsymbol{n}',\boldsymbol{m}) + C(\boldsymbol{n}',\boldsymbol{m}'). \tag{276}$$

- **CHSH inequality**: in a hidden variable theory (local realism),

$$0 \le |S| \le 2. \tag{277}$$

- Let $s(\lambda)$ be the contribution from a single measurement:

$$\begin{aligned} s(\lambda) &= NM - NM' + N'M + N'M' \\ &= N(M - M') + N'(M + M'). \end{aligned} \tag{278}$$

  Since each measurement outcome is ± 1, there are four possible combinations for M and M':

M	M'	$M-M'$	$M+M'$	s
1	1	0	2	$2N'$
1	-1	2	0	$2N$
-1	1	-2	0	$-2N$
-1	-1	0	-2	$-2N'$

Since N and N' also take only the values ± 1,

$$s(\lambda) = \pm 2. \tag{279}$$

The maximum and minimum values of a single measurement are $+2$ and -2.

- Averaging over all measurements:

$$S = \int d\lambda\, \rho(\lambda)\, s(\lambda), \tag{280}$$

the average must lie between the maximum and minimum values $\to$ Eq. (277).

- In quantum mechanics, choosing the measurement axes as shown in Fig. 21 (right),

$$\begin{aligned} S &= -(\cos\theta_{nm} - \cos\theta_{nm'} + \cos\theta_{n'm} + \cos\theta_{n'm'}) \\ &= -\left(\cos\frac{\pi}{4} - \cos\frac{3\pi}{4} + \cos\frac{\pi}{4} + \cos\frac{\pi}{4}\right) \\ &= -\left(\frac{1}{\sqrt{2}} - (-\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= -2\sqrt{2}, \end{aligned} \tag{281}$$

which violates the CHSH inequality (277).

11.4 Experimental tests

- Tests using **photon** pairs [56]
  - Photon pairs produced in a Ca cascade decay.
  - Measure the polarization correlation between the emitted photons.
  - Bell's inequality is violated, in agreement with the prediction of quantum mechanics.
  - Several experimental **loopholes** were identified, leading to continual improvements.
- Tests using **proton** pairs [57]
  - A deuteron (d) beam is incident on a liquid hydrogen (^{1}H) target:

$$^1\mathrm{H}(d, {}^2\mathrm{He})n. \tag{282}$$

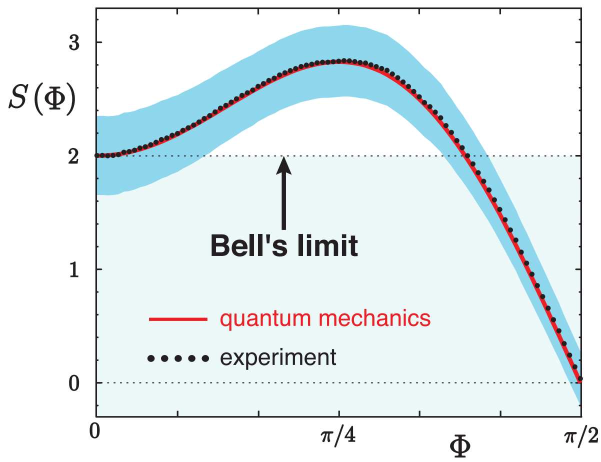

Figure 23: Spin correlation of a proton pair. Adapted from Ref. [57].

- "^{2}He": a low-energy pp pair in the spin-singlet state (see Sec. 13).
  A smaller relative momentum gives a higher spin-correlation purity.
- The proton spins in the final state are measured through rescattering.
  A larger proton momentum improves the polarization analyzing power.
- The reaction (282) produces pp pairs with small relative momentum but large total momentum (i.e., each proton has a large momentum).
- Result: the experimental data (black dots in Fig. 23) agree well with the quantum-mechanical prediction (red solid curve in Fig. 23).