12 Entanglement power of the S matrix

12.1 S matrix and s-wave scattering

- **S matrix**: **operator** that transforms initial state into final state in scattering

$$|\text{out}\rangle = S\,|\text{in}\rangle\,. \tag{283}$$

- Elastic scattering: characterized by momentum p and scattering angle θ.
  Partial wave decomposition: scattering is described as a function of p for each ℓ (§3).

- In the following, consider only the s-wave ($\ell = 0$), which is dominant at low energies

$$s(p) = e^{2i\delta(p)}. \tag{284}$$

  The **scattering phase shift** δ is a function of p.
  Same S matrix for δ and $\delta + n\pi$ ($n \in \mathbb{Z}$): physically equivalent

- Differential cross section and scattering amplitude:

$$\frac{d\sigma}{d\Omega} = |f(p)|^2, \quad f(p) = \frac{1}{p\cot\delta - ip}. \tag{285}$$

  The phase shift contains all the information about the scattering.

- **Scattering length** a_0: characterizes the low-energy scattering (§4.5):

$$-\frac{1}{a_0} = \lim_{p\to 0}[p\cot\delta]. \tag{286}$$

  Note that $\delta \to 0$ as $p \to 0$, and a_0 can be either positive or negative.

- Low-energy limit of the total cross section:

$$\lim_{p\to 0}\sigma(p) = 4\pi a_0^2, \quad \sigma(p) = \int d\Omega\,|f(p)|^2. \tag{287}$$

12.2 Scattering length and symmetry

- The unitarity bound (36) $\leftarrow$ unitarity of the S matrix:

$$\sigma(p) \le \frac{4\pi}{p^2}. \tag{288}$$

  As $p \to 0$, the upper bound of the cross section diverges.
  $\Rightarrow$ the magnitude of the scattering length $|a_0|$ can take any value from 0 to ∞.

- $|a_0| \to \infty$: the unitarity bound is saturated in the limit $p \to 0$ (**unitary limit**).

- Feshbach resonance in cold atoms (§5): a_0 is controlled by external magnetic field
- Near the unitary limit, low-energy universality emerges [58, 59].
  Ex) when a_0 is large and positive, a shallow bound state exists with
  $$B = \frac{1}{2\mu a_0^2} \tag{289}$$
  Binding energy B is determined solely by a_0.
- Much larger $|a_0|$ than the typical length scale $\to$ **scale invariance**:
  $$\begin{aligned} a_0 = 0 \quad &: \text{ noninteracting for } p \to 0, \\ |a_0| \to \infty \quad &: \text{ infinitely strong interaction for } p \to 0 \text{ (unitary limit)}. \end{aligned}$$
- In nonrelativistic systems, scale invariance $\to$ **nonrelativistic conformal symmetry** [10].
- The largest symmetry of the Schrödinger equation (spacetime translations, rotations, Galilean boosts, scale transformations, and special conformal transformations)

12.3 Quantum gates in two-spin system

- State is transformed by an operator (**quantum gate**) U:
  $$|\phi\rangle = U\,|\psi\rangle\,. \tag{290}$$
- Controlled-NOT (**CNOT**) gate:
  $$\begin{aligned} \text{CNOT} &= P_\uparrow \otimes 1 + P_\downarrow \otimes \sigma_x \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{291}$$
  Action: spin B (target qubit) is flipped when spin A (control qubit) is $|\downarrow\rangle$
  $$\text{CNOT}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle\,, \quad \text{CNOT}\,|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle\,, \tag{292}$$
  $$\text{CNOT}\,|\downarrow\uparrow\rangle = |\downarrow\downarrow\rangle\,, \quad \text{CNOT}\,|\downarrow\downarrow\rangle = |\downarrow\uparrow\rangle\,, \tag{293}$$
  $$\text{CNOT}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{pmatrix}. \tag{294}$$

- **SWAP** gate:

$$\mathrm{SWAP} = \frac{1 + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{2} = \frac{1}{2}\left(1 + \sum_i \sigma_i \otimes \sigma_i\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{295}$$

  Action: exchanges spins A and B

$$\mathrm{SWAP}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle\,, \quad \mathrm{SWAP}\,|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle\,, \tag{296}$$

$$\mathrm{SWAP}\,|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle\,, \quad \mathrm{SWAP}\,|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle\,, \tag{297}$$

$$\mathrm{SWAP}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \\ \beta \\ \delta \end{pmatrix}. \tag{298}$$

- Representation in terms of CNOT (Fig. 24):
  CNOT gate with spin B as the control qubit is

$$1 \otimes P_\uparrow + \sigma_x \otimes P_\downarrow = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{299}$$

  and therefore

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{SWAP}. \end{aligned} \tag{300}$$

  Any two-qubit gate can be constructed from arbitrary single-qubit gates and CNOT gates

- Change of **entanglement**:

$$\begin{cases} \mathrm{CNOT}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle\,, & \text{entanglement remains zero,} \\ \mathrm{CNOT}\,|+\uparrow\rangle = |\Phi^+\rangle\,, & \text{entanglement changes from zero to maximal,} \\ \mathrm{CNOT}\,|\Phi^+\rangle = |+\uparrow\rangle\,, & \text{entanglement changes from maximal to zero.} \end{cases} \tag{301}$$

  Quantum gates can either increase or decrease entanglement.
  The change depends on the input state.

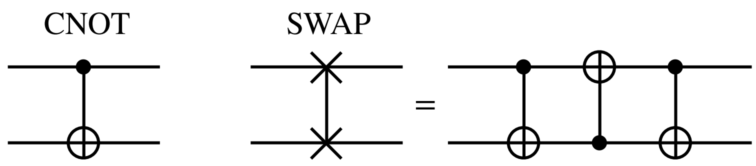

Figure 24: Quantum circuit representation of two-qubit gates. Left: controlled-NOT (CNOT) gate. Right: SWAP gate.

12.4 Entanglement power

- **Entanglement power** (EP): ability of an operator U to **generate** entanglement [60]

$$E(U) = \overline{\mathcal{E}\big(U\,|\text{product state}\rangle\big)}. \tag{302}$$

  Average over initial product states

- Initial product state:

$$\begin{aligned} &\mathcal{E}\big(|\text{product state}\rangle\big) = 0, && (303)\\ \Rightarrow\quad &\mathcal{E}\big(U\,|\text{product state}\rangle\big) = \text{the entanglement generated by } U. && (304)\end{aligned}$$

- Average is taken over **all possible** initial product states
  $\leftarrow$ **state independent** definition of EP

- Representation of product states on the Bloch sphere [61]:

$$\begin{aligned}|\theta_A,\phi_A\rangle\otimes|\theta_B,\phi_B\rangle &= \begin{pmatrix}\cos\frac{\theta_A}{2}\\ \sin\frac{\theta_A}{2}e^{i\phi_A}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta_B}{2}\\ \sin\frac{\theta_B}{2}e^{i\phi_B}\end{pmatrix}\\ &= \begin{pmatrix}\cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\\ \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}e^{i\phi_B}\\ \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}e^{i\phi_A}\\ \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}e^{i(\phi_A+\phi_B)}\end{pmatrix}.\end{aligned} \tag{305}$$

  The directions of spins A and B are specified by (θ_A,ϕ_A) and (θ_B,ϕ_B), respectively.

- The angles (θ,ϕ) cover the entire solid angle:

$$\int d\Omega = \int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi = \int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi = 4\pi. \tag{306}$$

- Explicit expression for the EP:

$$\begin{aligned}E(U) &= \int\frac{d\Omega_A}{4\pi}\int\frac{d\Omega_B}{4\pi}\mathcal{E}\big(U\,|\theta_A,\phi_A\rangle\otimes|\theta_B,\phi_B\rangle\big) && (307)\\ &= 1-\int\frac{d\Omega_A}{4\pi}\int\frac{d\Omega_B}{4\pi}\mathrm{Tr}_A\left[\rho_A^2\right], && (308)\end{aligned}$$

where

$$\rho_A = \mathrm{Tr}_B \left[U\big(|\theta_A, \phi_A\rangle \otimes |\theta_B, \phi_B\rangle\big)\big(\langle\theta_A, \phi_A| \otimes \langle\theta_B, \phi_B|\big)U^\dagger \right]. \tag{309}$$

For a given U, EP is calculated by this equation.

- If U is the S matrix, the EP depends on the scattering phase shift δ.
- Any nonlocal two-spin operator U: characterized by three parameters β_a $(a = 1, 2, 3)$ $\leftarrow$ Cartan decomposition of SU(4)/(SU(2) $\times$ SU(2)) [61].
- EP can be given by β_a as

$$E(U) = \frac{1}{6} - \frac{1}{18} \sum_{a<b} \cos(4\beta_a) \cos(4\beta_b). \tag{310}$$