13 Nuclear force and entanglement suppression

13.1 Two-nucleon wave function

- Nucleon follows $\mathrm{SU}(2)_S \times \mathrm{SU}(2)_I$ **symmetry**
  - Spin $\mathrm{SU}(2)_S \leftarrow$ Lorentz (Poincaré) symmetry of QCD
  - Isospin $\mathrm{SU}(2)_I \leftarrow$ Isospin symmetry of QCD (interchange of the u and d quarks)
- One-nucleon state:

$$|N\rangle = \psi(\boldsymbol{r}) \otimes |\mathrm{spin}\rangle \otimes |\mathrm{isospin}\rangle\,, \tag{311}$$

$$\psi(\boldsymbol{r}) : \text{Spatial wave function}$$

$$|\mathrm{spin}\rangle : \text{Spin wave function (spin-1/2), } \{|\uparrow\rangle\,, |\downarrow\rangle\}$$

$$|\mathrm{isospin}\rangle : \text{Isospin wave function (isospin-1/2), } \{|p\rangle\,, |n\rangle\}$$

- Two-nucleon system:

$$\text{Spatial part}: \;\; \psi(\boldsymbol{r}_A)\psi(\boldsymbol{r}_B) = \Psi(\boldsymbol{R})\phi(\boldsymbol{r}), \quad \boldsymbol{R} = \frac{\boldsymbol{r}_A + \boldsymbol{r}_B}{2}, \quad \boldsymbol{r} = \boldsymbol{r}_A - \boldsymbol{r}_B, \tag{312}$$

$$\text{Spin and isospin}: \;\; \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \tag{313}$$

- Two-nucleon state (center-of-mass frame):

$$|NN\rangle = \phi_\ell(\boldsymbol{r}) \otimes |\mathrm{Spin}\rangle \otimes |\mathrm{Isospin}\rangle\,, \tag{314}$$

$$\phi_\ell(\boldsymbol{r}) : \text{Relative wave function (relative coordinate } \boldsymbol{r}\text{, orbital angular momentum } \ell)$$

$$|\mathrm{Spin}\rangle : \text{Spin wave function: } |S, S_z\rangle = |0,0\rangle\,, |1,+1\rangle\,, |1,0\rangle\,, |1,-1\rangle$$

$$|\mathrm{Isospin}\rangle : \text{Isospin wave function: } |I, I_z\rangle = |0,0\rangle\,, |1,+1\rangle\,, |1,0\rangle\,, |1,-1\rangle$$

- In the computational basis and the Bell basis:

$$|0,0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = |\Psi^-\rangle\,, \quad |1,0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} = |\Psi^+\rangle\,, \tag{315}$$

$$|1,+1\rangle = |\uparrow\uparrow\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}, \quad |1,-1\rangle = |\downarrow\downarrow\rangle = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}. \tag{316}$$

13.2 Symmetries of the nuclear force

- Spatial exchange operator P_r: Exchange of the two nucleons ($\boldsymbol{r}_A \leftrightarrow \boldsymbol{r}_B$): $\boldsymbol{r} \to -\boldsymbol{r}$.
  Using the property of the spherical harmonics,

$$P_r\phi_\ell(\boldsymbol{r}) = \phi_\ell(-\boldsymbol{r}) = (-1)^\ell\phi_\ell(\boldsymbol{r}). \tag{317}$$

$\phi_\ell(\boldsymbol{r})$ is **symmetric** (eigenvalue $+1$) for even ℓ, **antisymmetric** (eigenvalue -1) for odd ℓ.

Table 3: Spin and isospin of the s-wave two-nucleon states.

Orbital angular momentum	Spin	Isospin
$\ell = 0$ (symmetric)	$S = 0$ (antisymmetric)	$I = 1$ (symmetric)
	$S = 1$ (symmetric)	$I = 0$ (antisymmetric)

- Eigenvalues of $\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B$

$$\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B \left|0,0\right\rangle = -3\left|0,0\right\rangle, \quad \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B \left|1,S_z\right\rangle = +1\left|1,S_z\right\rangle. \tag{318}$$

- Spin exchange operator (SWAP gate):

$$P_\sigma = \frac{1+\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{2}, \tag{319}$$

$$P_\sigma \left|0,0\right\rangle = -\left|0,0\right\rangle, \quad P_\sigma \left|1,S_z\right\rangle = +\left|1,S_z\right\rangle. \tag{320}$$

This operator exchanges the spins of nucleons A and B.
$S = 0$ is **antisymmetric**, $S = 1$ is **symmetric** under the exchange.

- Isospin exchange operator:

$$P_\tau \left|0,0\right\rangle = -\left|0,0\right\rangle, \quad P_\tau \left|1,I_z\right\rangle = +\left|1,I_z\right\rangle. \tag{321}$$

$I = 0$ is **antisymmetric**, $I = 1$ is **symmetric** under the exchange.

- Fermi statistics: $\left|NN\right\rangle$ must be **antisymmetric** under

$$P_r P_\sigma P_\tau \left|NN\right\rangle = -\left|NN\right\rangle. \tag{322}$$

Allowed states for low-energy (Table 3): I is uniquely determined for a fixed S

- Low-energy s-wave scattering is described by **only two channels**:
  - Spin $S = 0$, isospin $I = 1$: ${}^1S_0$
  - Spin $S = 1$, isospin $I = 0$: ${}^3S_1$

Note: under the $\mathrm{SU}(2)_S \times \mathrm{SU}(2)_I$ symmetry, **${}^1S_0$ and ${}^3S_1$ are independent**.

- Scattering lengths of the physical nuclear force [58]:

$$a_0 = a_{S=0} \approx -23.76 \text{ fm}, \quad a_1 = a_{S=1} \approx 5.42 \text{ fm}. \tag{323}$$

- Typical length scale of the nuclear force:

$$R_{\text{typ}} \sim \frac{1}{m_\pi} \approx 1.43 \text{ fm}. \tag{324}$$

Range of the one-pion-exchange force, the longest-range interaction between hadrons.

- Scattering lengths $\gg R_{\text{typ}}$
$\Rightarrow$ close to the unitary limit, **nonrelativistic conformal symmetry** [10]:

$$|a_0| \gg \frac{1}{m_\pi}, \quad |a_1| \gg \frac{1}{m_\pi}. \tag{325}$$

- **Both** the $S = 0$ and $S = 1$ channels are close to the unitary limit:

$$\frac{1}{|a_0|} \sim \frac{1}{|a_1|} \ll m_\pi. \tag{326}$$

Similarity between $S = 0$ and $S = 1 \Rightarrow$ **spin-flavor SU(4) symmetry** [8, 9].

- One-nucleon states form the fundamental representation of SU(4):

$$\{|p\uparrow\rangle, |p\downarrow\rangle, |n\uparrow\rangle, |n\downarrow\rangle\}. \tag{327}$$

The irreducible decomposition of the two-nucleon states is

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10}. \tag{328}$$

${}^1S_0$ (3 isospin states) and ${}^3S_1$ (3 spin states) together form **6**.

- **Emergent symmetries**: cannot be derived directly from QCD

13.3 Entanglement suppression in the two-nucleon system

- Projection operators onto states with total spin S:

$$\mathcal{J}_0 = \frac{1 - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4}, \quad \mathcal{J}_1 = \frac{3 + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4}. \tag{329}$$

- pn scattering (isospin is automatically determined from Table 3)

$$S = e^{2i\delta_0}\mathcal{J}_0 + e^{2i\delta_1}\mathcal{J}_1 \propto \frac{3 + e^{-2i(\delta_1 - \delta_0)}}{4} + \frac{1 - e^{-2i(\delta_1 - \delta_0)}}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B. \tag{330}$$

δ_S: phase shift with total spin S.

- Entanglement power (EP) of a general operator U:

$$E(U) = \frac{1}{6} - \frac{1}{18}\sum_{a<b}\cos(4\beta_a)\cos(4\beta_b), \quad U = \exp\left\{i\sum_a \beta_a \sigma_a \otimes \sigma_a\right\}. \tag{331}$$

For $\beta_1 = \beta_2 = \beta_3 = \beta$,

$$E(U) = \frac{1}{6}\sin^2(4\beta), \quad U \propto \frac{3+e^{-4i\beta}}{4} + \frac{1-e^{-4i\beta}}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B. \tag{332}$$

Comparing Eq. (330) with Eq. (332), we obtain $2\beta = \delta_1 - \delta_0$

- EP of pn scattering [11, 61]:

$$E(S) = \frac{1}{6}\sin^2[2(\delta_1 - \delta_0)] \geq 0. \tag{333}$$

- The minimum value $E(S) = 0$ is obtained when $\sin[2(\delta_1 - \delta_0)] = 0$:

  - Solution 1: $|\delta_1 - \delta_0| = 0$
  - Solution 2: $|\delta_1 - \delta_0| = \pi/2$

- Solution 1: $S = 0$ and $S = 1$ have the same phase shift
$\Rightarrow$ **spin-flavor SU(4) symmetry**.

- If Solution 2 is realized in the low-energy limit ($p \to 0$):

$$-\frac{1}{a_0} = \lim_{p\to 0}[p\cot\delta_0], \quad -\frac{1}{a_1} = \lim_{p\to 0}[p\cot\delta_1]. \tag{334}$$

Using $\delta_1 = \delta_0 + \pi/2$ and multiplying the two equations,

$$\frac{1}{a_0}\frac{1}{a_1} = \lim_{p\to 0}\Big[p\cot\delta_0 \cdot p\cot(\delta_0 + \pi/2)\Big] = \lim_{p\to 0}\Big[p^2\cot\delta_0(-\tan\delta_0)\Big] = 0. \tag{335}$$

Either $1/a_0$ or $1/a_1$ must vanish, namely, the **scattering length diverges**.
$\Rightarrow$ **nonrelativistic conformal symmetry**.

13.4 Interpretation in terms of quantum gates

- Interpretation of the solutions [61]

- Solution 1: setting $\delta_1 = \delta_0$,

$$S = e^{2i\delta_0}\mathcal{J}_0 + e^{2i\delta_0}\mathcal{J}_1 = e^{2i\delta_0}(\mathcal{J}_0 + \mathcal{J}_1). \tag{336}$$

proportional to the **sum** of the projection operators

- Solution 2: setting $\delta_1 = \delta_0 + \pi/2$,

$$S = e^{2i\delta_0}\mathcal{J}_0 + e^{2i(\delta_0+\pi/2)}\mathcal{J}_1 = e^{2i\delta_0}\mathcal{J}_0 + e^{2i\delta_0}e^{i\pi}\mathcal{J}_1 = -e^{2i\delta_0}(\mathcal{J}_1 - \mathcal{J}_0). \tag{337}$$

  proportional to the **difference** of the projection operators.

- Using the projection operators in Eq. (329),

$$\mathcal{J}_0 + \mathcal{J}_1 = \frac{1 - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4} + \frac{3 + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4} = 1, \tag{338}$$

$$\mathcal{J}_1 - \mathcal{J}_0 = \frac{3 + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4} - \frac{1 - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{4} = \frac{1 + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B}{2} = \mathrm{SWAP}. \tag{339}$$

  Solution 1: S matrix acts as the **identity gate** (identity operator).
  Solution 2: S matrix acts as the **SWAP gate** (spin exchange operator).

- Identity and SWAP **map a product state to another product state**:

$$1 \; |\psi_A\rangle \otimes |\psi_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle\,, \tag{340}$$

$$\mathrm{SWAP} \; |\psi_A\rangle \otimes |\psi_B\rangle = |\psi_B\rangle \otimes |\psi_A\rangle\,. \tag{341}$$

  The output state remains a product state (EE is zero) $\to$ EP vanishes

- **Only the identity and the SWAP gate** give $E(S) = 0$.
  Indeed, the only solutions of Eq. (331) with vanishing EP are [61]

$$\begin{cases} \beta_1 = \beta_2 = \beta_3 = 0 & \Rightarrow U = 1, \\ \beta_1 = \beta_2 = \beta_3 = \dfrac{\pi}{4} & \Rightarrow U = e^{i\pi/4} \times \mathrm{SWAP}. \end{cases} \tag{342}$$