

14 Extension to general spin and flavor

14.1 Spin and flavor of baryons

- Single-quark state

$$|q\rangle = \psi(\mathbf{r}) \otimes |\text{color}\rangle \otimes |\text{spin}\rangle \otimes |\text{flavor}\rangle, \quad (343)$$

$\psi(\mathbf{r})$: Spatial wave function

$|\text{color}\rangle$: Color wave function: **3** of SU(3), $\{|r\rangle, |g\rangle, |b\rangle\}$

$|\text{spin}\rangle$: Spin wave function: spin-1/2 of SU(2), $\{|\uparrow\rangle, |\downarrow\rangle\}$

- **Flavor** wave function $|\text{flavor}\rangle$

$\{|u\rangle, |d\rangle\}$: Flavor (isospin) SU(2)

$\{|u\rangle, |d\rangle, |s\rangle\}$: Flavor SU(3)

- Flavor symmetry is an approximate symmetry ($p \sim uud$, $n \sim udd$, $\Lambda \sim uds$)

$$m_p = 938 \text{ MeV}, \quad m_n = 940 \text{ MeV}, \quad m_\Lambda = 1116 \text{ MeV} \quad (344)$$

SU(2): valid at the level of a few percent;

SU(3): valid at roughly the 20% level.

- Wave function of a baryon (three-quark state)

$$|qqq\rangle = \phi_\ell(\boldsymbol{\rho}, \boldsymbol{\lambda}) \otimes |\text{Color}\rangle \otimes |\text{Spin}\rangle \otimes |\text{Flavor}\rangle \quad (345)$$

Completely antisymmetric under the exchange of any two quarks.

- $\phi_\ell(\boldsymbol{\rho}, \boldsymbol{\lambda})$: Jacobi coordinates $\{\boldsymbol{\rho}, \boldsymbol{\lambda}\}$ and total orbital angular momentum ℓ .
- Color wave function $|\text{Color}\rangle$:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \quad (346)$$

The completely antisymmetric **1** (color confinement):

$$|\text{Color}\rangle = \frac{|rgb\rangle + |brg\rangle + |gbr\rangle - |grb\rangle - |bgr\rangle - |rbg\rangle}{\sqrt{6}} \quad (347)$$

- Spin wave function $|\text{Spin}\rangle$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \quad (348)$$

- Flavor wave function $|\text{Flavor}\rangle$: the same decomposition as Eq. (346).

For the ground state ($\ell = 0$), $|qqq\rangle$ is completely antisymmetric if (Fig. 25)

$$|\text{Spin}\rangle \otimes |\text{Flavor}\rangle = \begin{cases} |S = 1/2\rangle \otimes |\mathbf{8}\rangle & \text{Spin-1/2 baryon octet} \\ |S = 3/2\rangle \otimes |\mathbf{10}\rangle & \text{Spin-3/2 baryon decuplet} \end{cases} \quad (349)$$

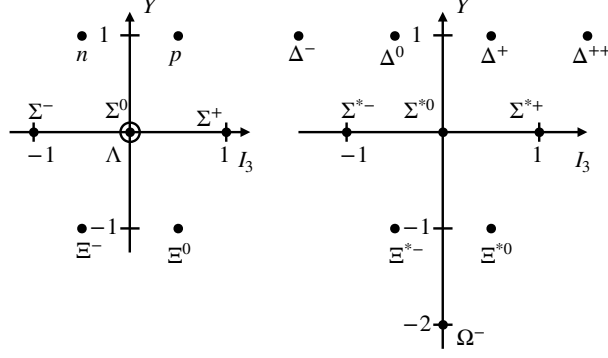


Figure 25: Weight diagrams of the ground-state octet (left) and decuplet (right) baryons.

14.2 Scattering of octet baryons

- Spin and flavor of the two-octet-baryon system

$$\frac{1}{2} \otimes \frac{1}{2} = \underbrace{0}_{\text{A}} \oplus \underbrace{1}_{\text{S}} \quad (350)$$

$$8 \otimes 8 = \underbrace{1 \oplus 8_S \oplus 27}_{\text{S}} \oplus \underbrace{8_A \oplus 10 \oplus \overline{10}}_{\text{A}} \quad (351)$$

For the s wave: spin-A with flavor-S or spin-S with flavor-A

- S matrix: **six independent** scattering channels

$$S = \mathcal{J}_0(\mathcal{F}_1 e^{2i\delta_{0,1}} + \mathcal{F}_{8_S} e^{2i\delta_{0,8_S}} + \mathcal{F}_{27} e^{2i\delta_{0,27}}) + \mathcal{J}_1(\mathcal{F}_{8_A} e^{2i\delta_{1,8_A}} + \mathcal{F}_{10} e^{2i\delta_{1,10}} + \mathcal{F}_{\overline{10}} e^{2i\delta_{1,\overline{10}}}) \quad (352)$$

$\mathcal{J}_i$: projector onto the spin- i state in Eq. (B29), $\mathcal{F}_{\mathbf{R}}$: flavor projector onto $\mathbf{R}$.

- Charge Q and strangeness S conservations: independent scattering in each (Q, S) sector. Example: $(Q, S) = (+1, 0)$: nuclear force (pn scattering), $\delta_{0,27}$ (1S_0) and $\delta_{1,\overline{10}}$ (3S_1)
- Sector with the largest number of coupled channels: $(Q, S) = (0, -2)$

$$\Sigma^+ \Sigma^-, \Sigma^0 \Sigma^0, \Lambda \Sigma^0, \Xi^- p, \Xi^0 n, \Lambda \Lambda \quad (353)$$

Consequences of entanglement suppression [62]:

$$S \propto 1 \quad \text{Spin-flavor SU(16) symmetry} \quad (354)$$

$$S \propto \text{SWAP} \quad \text{Flavor SU(8) and NR conformal symmetries} \quad (355)$$

Enlarged flavor space $\rightarrow$ enlarged symmetry

Symmetries of other sectors (e.g. nuclear force) are contained as subgroup symmetries:

$$\text{SU(16)} \supset \text{SU(4)} \quad (356)$$

- Spin-flavor SU(16) symmetry: symmetry of baryons ($16 = 8 \times 2$)

$$\{|p \uparrow\rangle, |p \downarrow\rangle, |n \uparrow\rangle, |n \downarrow\rangle, |\Lambda \uparrow\rangle, |\Lambda \downarrow\rangle, \dots\} \quad (357)$$

c.f.) In the large- N_c limit, spin-flavor SU(6) is the symmetry of quarks ($6 = 3 \times 2$):

$$\{|u \uparrow\rangle, |u \downarrow\rangle, |d \uparrow\rangle, |d \downarrow\rangle, |s \uparrow\rangle, |s \downarrow\rangle\} \quad (358)$$

$\Rightarrow$ Entanglement suppression imposes a **stronger constraint than the large- N_c limit** [11].

Lattice QCD indicates evidence for approximate SU(16) symmetry [63].

- In the two-flavor case, both quarks $\{u, d\}$ and baryons $\{p, n\}$ belong to spin-1/2.
 $\Rightarrow$ The spin-flavor symmetry is SU(4) for both quarks and baryons.

14.3 Spin-3/2 states

- Spin-3/2 state (4-dimensional qudit): a complex four-component vector

$$|\psi_4\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C} \quad (359)$$

Without any constraints, there are **eight degrees of freedom** ($\mathcal{H} = \mathbb{C}^4$).

Normalization and phase d.o.f. $\rightarrow |\psi_4\rangle$ is described by **six degrees of freedom**.

- Fubini–Study measure on the projective space $\mathbb{C}^4 / \sim = \mathbb{CP}^3$

$$|\psi_4\rangle = \begin{pmatrix} \cos \theta_1 \sin \theta_2 \sin \theta_3 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 e^{i\nu_1} \\ \cos \theta_2 \sin \theta_3 e^{i\nu_2} \\ \cos \theta_3 e^{i\nu_3} \end{pmatrix}, \quad d\omega_4 = \frac{3!}{\pi^3} \prod_{i=1}^3 d\theta_i d\nu_i \cos \theta_i \sin^{2i-1} \theta_i, \quad \int d\omega_4 = 1 \quad (360)$$

The six degrees of freedom are θ_i, ν_i ($i = 1, 2, 3$).

- Spin operators $\mathbf{s}$: 4×4 complex matrices

$$s_x = \begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix}, \quad s_y = \begin{pmatrix} 0 & -i\sqrt{3}/2 & 0 & 0 \\ i\sqrt{3}/2 & 0 & -i & 0 \\ 0 & i & 0 & -i\sqrt{3}/2 \\ 0 & 0 & i\sqrt{3}/2 & 0 \end{pmatrix}, \quad (361)$$

$$s_z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \quad (362)$$

- Eigenstates of s_z

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (363)$$

$$s_z |\uparrow\rangle = \frac{3}{2} |\uparrow\rangle, \quad s_z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle, \quad s_z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle, \quad s_z |\downarrow\rangle = -\frac{3}{2} |\downarrow\rangle \quad (364)$$

14.4 Scattering of decuplet baryons

- Spin and flavor of the two-decuplet-baryon system

$$\frac{3}{2} \otimes \frac{3}{2} = \underbrace{0 \oplus 2}_A \oplus \underbrace{1 \oplus 3}_S \quad (365)$$

$$10 \otimes 10 = \underbrace{27 \oplus 28}_S \oplus \underbrace{10 \oplus 35}_A \quad (366)$$

For the s wave: spin-A with flavor-S or spin-S with flavor-A

- S matrix: **eight independent** scattering channels

$$S = \mathcal{J}_0(\mathcal{F}_{27}e^{2i\delta_{0,27}} + \mathcal{F}_{28}e^{2i\delta_{0,28}}) + \mathcal{J}_1(\mathcal{F}_{10}e^{2i\delta_{1,10}} + \mathcal{F}_{35}e^{2i\delta_{1,35}}) \\ + \mathcal{J}_2(\mathcal{F}_{27}e^{2i\delta_{2,27}} + \mathcal{F}_{28}e^{2i\delta_{2,28}}) + \mathcal{J}_3(\mathcal{F}_{10}e^{2i\delta_{3,10}} + \mathcal{F}_{35}e^{2i\delta_{3,35}}) \quad (367)$$

- Projection operators for spin-3/2: linear combinations of the quadratic Casimir operator

$$\mathcal{J}_0 = \frac{33}{128} + \frac{31}{96} (\mathbf{s}_A \cdot \mathbf{s}_B) - \frac{5}{72} (\mathbf{s}_A \cdot \mathbf{s}_B)^2 - \frac{1}{18} (\mathbf{s}_A \cdot \mathbf{s}_B)^3, \quad \dots \quad (368)$$

- (Q, S) conservation: largest number of coupled channels in $(Q, S) = (0, -2)$

$$\Delta^+\Xi^{*-}, \Delta^0\Xi^{*0}, \Sigma^{*0}\Sigma^{*0}, \Sigma^{*+}\Sigma^{*-} \quad (369)$$

Consequences of entanglement suppression [64]:

$$S \propto 1 \quad \text{Spin-flavor SU(40) symmetry} \quad (370)$$

$$S \propto \text{SWAP} \quad \text{Spin SU(4), flavor SU(10), and NR conformal symmetry} \quad (371)$$

- Phenomenological application: $\Omega\Omega$ scattering in the $(Q, S) = (-2, -6)$ sector.
Identical particles with no flavor d.o.f.: only antisymmetric $S = 0$ and $S = 2$ states
- Lattice QCD indicates that the $S = 0$ channel is close to the unitary limit [65].
- Entanglement suppression in $\Omega\Omega$ scattering [64] (with proper normalization):

$$S \propto 1_A \quad \text{Spin SU(4) symmetry} \quad (372)$$

$$S \propto \text{SWAP}_A \quad \text{NR conformal symmetry} \quad (373)$$

$\Rightarrow$ spin-2 channel is either at the unitary limit ($S \propto 1_A$) or weakly interacting ($S \propto \text{SWAP}_A$).