2 Resonances in quantum mechanics

2.1 Overview of resonance states

- **Resonance**: a quantum-mechanically formed metastable "**state**" that decays over time
- Decay products are scattering states (continuum) $\rightarrow$ **scattering theory**
  Example: $T_{cc} \rightarrow D^0 D^{*+} \Leftrightarrow T_{cc}$ is a resonance in the $D^0 D^{*+}$ scattering
- Inelastic scattering and channels
  - Elastic scattering: initial state = final state ($D^0 D^{*+} \rightarrow D^0 D^{*+}$, ...)
  - Inelastic scattering: transitions to different final states ($D^0 D^{*+} \rightarrow D^{*0} D^{*+}$, ...)
  - Channels: multi-body states specified by their particle content ($D^0 D^{*+}$, $D^{*0} D^{*+}$, ...)
- Various characterizations of resonances: how are they related?
  - Peak in cross sections: Fig. 3(a)
  - $\pi/2$ crossing of the phase shift $\delta(E)$: Fig. 3(b)
  - **Pole** of the scattering amplitude in the complex energy plane: Fig. 3(c)
  - **Eigenstate** of the Hamiltonian with **complex** energy: Fig. 3(d)

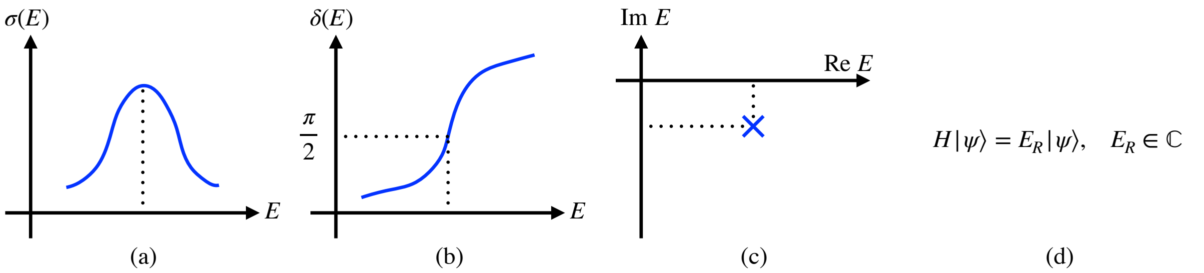

Figure 3: Schematic characterization of resonances. (a) Peak in the total cross section $\sigma(E)$, (b) $\pi/2$ crossing of the phase shift $\delta(E)$, (c) pole of the scattering amplitude, (d) complex eigenenergy.

- **Shape resonance** (potential resonance): Fig. 4(b)
  - Single-channel scattering
    Typical potential: short-range attraction plus a repulsive barrier
  - Energy $E > 0$
  - Unstable due to tunneling

- **Feshbach resonance**: Fig. 4(c)
  - Coupled-channel scattering
    P: entrance channel, Q: closed channel
  - Threshold of Q at $E = \Delta > 0$, threshold of P at $E = 0$
  - A bound state in channel Q at $0 < E < \Delta$
  - Unstable via the $Q \to P$ transition

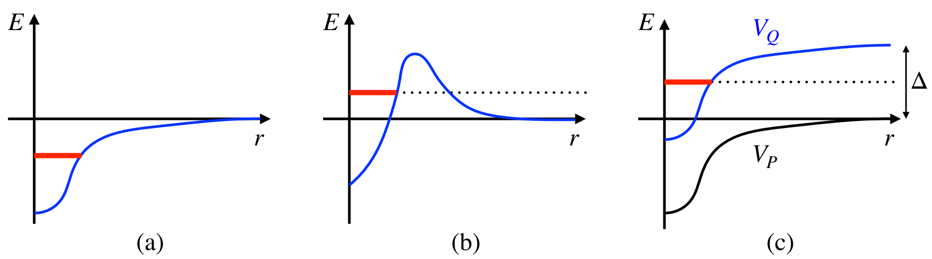

Figure 4: Illustration of a bound state (a), shape resonance (b), and Feshbach resonance (c).

2.2 Resonances as eigenstates of Hamiltonian

- α decay of atomic nuclei: Gamow introduced a **complex eigenenergy** [26]

$$E = M_R - \frac{i\Gamma_R}{2} \tag{6}$$

- Time evolution of the wavefunction ($\hbar = 1$)

$$\Psi(t) \propto \exp\{-iEt\} = \exp\{-iM_R t\}\exp\{-\Gamma_R t/2\}$$

  Probability decreases exponentially: $|\Psi(t)|^2 \propto \exp\{-\Gamma_R t\}$

- Inconsistent with "expectation value of Hamiltonian H (hermitian operator) is real"?
  $\leftarrow$ Space on which the operator acts [domain $D(H)$] needs to be specified
  - Eigenvalues are real if $D(H)$ is Hilbert space (square-integrable functions $L^2(\mathbb{R}^d)$)

$$\int |\Psi(x)|^2 dx < \infty \tag{7}$$

  - Example of a non-square-integrable wavefunction: plane wave $\Psi(x) \sim e^{\pm ipx}$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} 1\ dx \to \infty \tag{8}$$

- Can we find an eigenstate with complex E?

2.3 Square-well potential

- Schrödinger equation

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2}+V(r)\right)u_0(r)=Eu_0(r), \quad 0\le r\le\infty \tag{9}$$

$u_0(r)\sim$ radial wavefunction of spherical 3D potential with $\ell=0$: $\psi_{\ell,m}(\boldsymbol{r})=\frac{u_\ell(r)}{r}Y_\ell^m(\hat{\boldsymbol{r}})$

- Attractive square-well potential ($V_0>0$)

$$V(r)=\begin{cases}-V_0 & 0\le r\le b\\ 0 & b<r\end{cases} \tag{10}$$

- General solutions (without boundary conditions)

$$u_0(r)\propto\begin{cases}e^{\pm ikr} & 0\le r\le b, \quad k=\sqrt{2\mu(E+V_0)}\\ e^{\pm ipr} & b<r, \quad p=\sqrt{2\mu E}\end{cases} \tag{11}$$

- Scattering solutions (with boundary condition $u_0(r\to 0)=0$)

$$u_0(r)=\begin{cases}C\sin(kr) & 0\le r\le b\\ A^-(p)e^{-ipr}+A^+(p)e^{+ipr} & b<r\end{cases} \tag{12}$$

$$A^\pm(p)=\frac{C}{2}\left[\sin(kb)\mp i\frac{k}{p}\cos(kb)\right]e^{\mp ipb} \tag{13}$$

  - Solutions exist for any $E>0$: **continuous** spectrum
  - Scattering solutions are not normalizable (they do not vanish as $r\to\infty$)
    $\to$ Overall normalization C is arbitrary
  - $A^\pm(p)$ are determined by the continuity of u_0 and du_0/dr at $r=b$
  - The waves $e^{\pm ipr}$ propagate in the $\pm r$ directions
    A^+ (A^-) is the amplitude of the outgoing (incoming) wave

2.4 Discrete eigenstates

- Discrete eigenstates: obtained by imposing boundary conditions both at $r\to 0$ and $r\to\infty$
- Bound state solution : eigenenrgy $E<0\Leftrightarrow$ purely imaginary eigenmomentum $p=\sqrt{2\mu E}$

$$p=i\kappa, \quad \kappa>0 \tag{14}$$

Wavefunction at $r\to\infty$ behaves as

$$u_0(r)=A^-(i\kappa)e^{+\kappa r}+A^+(i\kappa)e^{-\kappa r} \quad (r\to\infty) \tag{15}$$

Table 1: Numerical solutions for Eq. (17) with $V_0 = 10\mu^{-1}b^{-2}$.

	p $[b^{-1}]$	$E = p^2/(2\mu)$ $[\mu^{-1}b^{-2}]$
Bound state B	$+3.68i$	$-$ 6.78
1st resonance R_1	$1.06 - 1.02i$	$0.05 - 1.08i$
2nd resonance R_2	$6.29 - 1.41i$	$18.8 - 8.86i$
3rd resonance R_3	$9.90 - 1.69i$	$47.6 - 16.8i$
⋮		

- Boundary condition: $u_0(r)$ is square integrable $\rightarrow$ eliminate the diverging component $e^{+\kappa r}$

$$A^-(i\kappa) = 0 \tag{16}$$

For $p = i\kappa$, the incoming wave (e^{-ipr}) vanishes, leaving only the outgoing wave (e^{+ipr})

- $A^-(p) = 0$: **Outgoing (Siegert) boundary condition**

$$\tan(\sqrt{p^2 + 2\mu V_0}\ b) = -i\frac{\sqrt{p^2 + 2\mu V_0}}{p} \tag{17}$$

Substituting $p = i\kappa$, the well-known bound-state condition $\kappa = -k\cot(kb)$ is recovered

- Bound states: solution of Eq. (17) with pure imaginary p
$\leftrightarrow$ physical scattering occurs for real and positive p
$\Rightarrow$ bound state solution is obtained by **analytic continuation** of Eq. (17)

- Resonance states : solutions of Eq. (17) with **complex** p

- Attractive square well potential has infinitely many resonance solutions [27, 14]
Table 1: numerical solutions of Eq. (17) with $V_0 = 10\mu^{-1}b^{-2}$
Poles of $1/A^-(p)$ in complex p plane (Fig. 5, left)

- Imaginary part of eigenmomentum is negative

$$p = p_R - ip_I, \quad p_R, p_I > 0 \tag{18}$$

behavior of wavefunction

$$u_0(r) \rightarrow A^+(p)e^{ipr} \propto \underbrace{e^{ip_R r}}_{\text{oscillation}}\ \underbrace{e^{+p_I r}}_{\text{increasing}}$$

$u_0(r)$ diverges with oscillation for $r \rightarrow \infty$ and is **not square integrable** (Fig. 5, right)

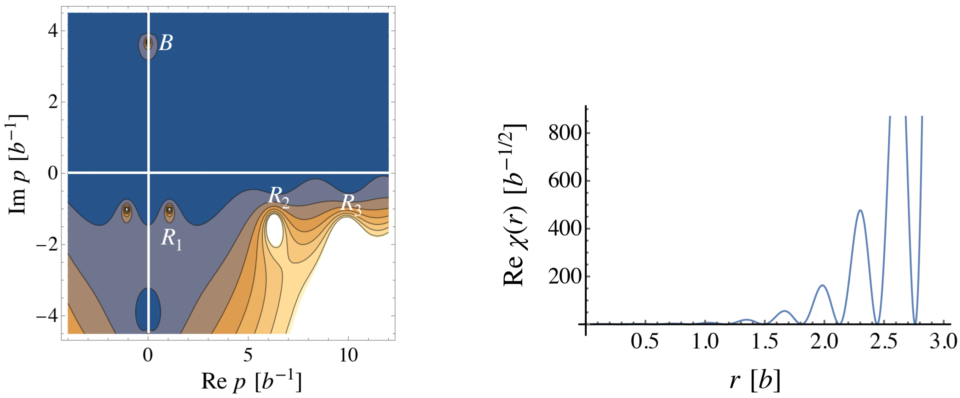

Figure 5: Left: contour plot of $1/|A^-(p)|$ of square well potential (10) with $V_0 = 10\mu^{-1}b^{-2}$. Right: real part of wavefunction of the third resonance R_3.

2.5 Summary of §2

- Discrete eigenstates ← outgoing boundary condition
- Resonances: eigenstates of the Hamiltonian with complex eigenenergy
  Analogous to bound states (analytic continuation of eigenmomentum)
- Resonance wavefunctions diverge as $r \to \infty$ (complex p)