4 Resonances in scattering theory

4.1 Resonances as poles of scattering amplitude

- Discrete eigenstates of Hamiltonian ← outgoing boundary condition (QM, §2)
- Outgoing boundary condition ← zeros of Jost function $\not{f}_\ell(p) = 0$ (scattering theory, §3)
- S-matrix from wave function: amplitude of outgoing wave normalized by the incoming one

$$u_\ell(r) \to \hat{h}_\ell^-(pr) - s_\ell(p)\hat{h}_\ell^+(pr) \quad (r \to \infty) \tag{50}$$

  Comparing with Eq. (45), we find

$$s_\ell(p) = \frac{\not{f}_\ell(-p)}{\not{f}_\ell(p)} \tag{51}$$

  ⇒ discrete eigenstates are represented by **poles of S-matrix**

- Scattering amplitude: from Eq. (29)

$$f_\ell(p) = \frac{s_\ell(p) - 1}{2ip} = \frac{\not{f}_\ell(-p) - \not{f}_\ell(p)}{2ip\not{f}_\ell(p)} \tag{52}$$

  ⇒ discrete eigenstates are represented by **poles of scattering amplitude**

- $s_\ell(p)$ and $f_\ell(p)$ are meromorphic functions of p ($\sim$ no singularities except for poles)

4.2 Eigenenergies and Riemann sheets

- Analytic continuation of $\not{f}_\ell(p)$, $s_\ell(p)$, $f_\ell(p)$ defined in physical region $p > 0$ to complex plane
- Complex momentum p and complex energy E

$$p = |p|e^{i\theta_p}, \quad E = |E|e^{i\theta_E} \tag{53}$$

- Relation between p and E

$$E = \frac{p^2}{2\mu} = \frac{|p|^2}{2\mu}e^{2i\theta_p} \quad \Rightarrow \quad |E| = \frac{|p|^2}{2\mu}, \quad \theta_E = 2\theta_p \tag{54}$$

  - When θ_p varies $0 \to 2\pi$, θ_E varies $0 \to 4\pi$
  - p and $-p$ (θ_p and $\theta_p + \pi$) are mapped onto the same value of E

- Meromorphic functions of p [$s_\ell(p)$, $f_\ell(p)$] are defined on two-sheeted Riemann surface of E
  $0 \le \theta_E < 2\pi$: first Riemann sheet of E (upper half-plane of p, $0 \le \theta_p < \pi$)
  $2\pi \le \theta_E < 4\pi$: **second Riemann sheet** of E (lower half-plane of p, $\pi \le \theta_p < 2\pi$)

- Fig. 8: Complex p and E planes
  Branch cut along real axis of E plane (branch point at $E = 0$)

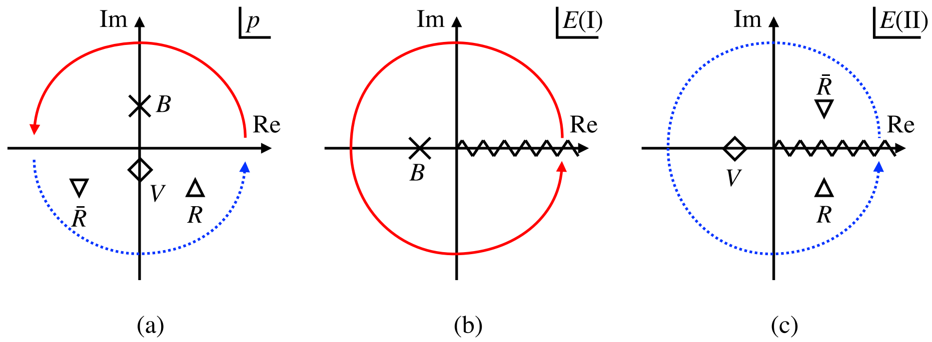

Figure 8: Poles in the complex plane. (a): p plane, (b): E plane (first Riemann sheet), (c): E plane (second Riemann sheet). B, V, R, and $\bar{R}$ represent a bound state, a virtual state, a resonance, and an anti-resonance, respectively.

4.3 Classification of eigenstates

- Eigenstate of Hamiltonian: zero of Jost function $\mathscr{f}_\ell(p) = 0$
- From Eq. (44), if $\mathscr{f}_\ell(p) = 0$,

$$\mathscr{f}_\ell(-p^*) = [\mathscr{f}_\ell(p)]^* = 0 \tag{55}$$

  $\Rightarrow$ If p is a solution, then $-p^*$ is also a solution

- Solutions with $p = -p^*$ (on the imaginary axis)
  - Bound state (B): $\times$ in Fig. 8

$$\mathrm{Re}\ p_B = 0, \quad \mathrm{Im}\ p_B > 0 \tag{56}$$

    Energy $E_B = p_B^2/(2\mu)$ is real and negative (first Riemann sheet)
  - Virtual state (anti-bound state, V): $\diamondsuit$

$$\mathrm{Re}\ p_V = 0, \quad \mathrm{Im}\ p_V < 0 \tag{57}$$

    The energy $E_V = p_V^2/(2\mu)$ is real and negative (second Riemann sheet)
    The residue of the pole ($\sim$ norm) is negative [28, 29], non-physical degree of freedom?

- Solutions with $p \neq -p^*$ (always appearing in pairs)
  - Exist only in lower half-plane of p
    $\leftarrow$ Complex E is allowed only when wave function is not square-integrable

- **Resonance** (R): △

$$\text{Re } p_R > 0, \quad \text{Im } p_R < 0 \tag{58}$$

Re $E_R > 0$, Im $E_R < 0$ (second Riemann sheet)

- Anti-resonance ($\bar{R}$): ▽

$$\text{Re } p_{\bar{R}} < 0, \quad \text{Im } p_{\bar{R}} < 0 \tag{59}$$

Appears together with resonance
A solution growing in time [30] ("conjugate" of the resonance)

4.4 Resonances and observables

- Only real energies are experimentally accessible
Effect of a resonance pole at $E = E_R = M_R - i\Gamma_R/2$ on observables in partial wave ℓ

- $f_\ell(E)$ having a pole at $E = E_R$:

$$f_\ell(E) = f_{\ell,\text{BW}}(E) + f_{\ell,\text{BG}}(E), \tag{60}$$

Breit-Wigner term $f_{\ell,\text{BW}}(E)$: resonance contribution

$$f_{\ell,\text{BW}}(E) = \frac{-\Gamma_R}{2p}\frac{1}{E - E_R} = \frac{-\Gamma_R}{2p}\frac{E - M_R - i\Gamma_R/2}{(E - M_R)^2 + \Gamma_R^2/4} \tag{61}$$

Nonresonant background $f_{\ell,\text{BG}}(E)$: analytic at $E = E_R$ and slowly varying

- At real $E \sim M_R$, contribution from $f_{\ell,\text{BW}}(E)$ increases (in particular, for narrow Γ_R)

- When $f_{\ell,\text{BG}}(E)$ is **assumed** to be small and negligible,

$$f_\ell(E) \approx f_{\ell,\text{BW}}(E) \quad (f_{\ell,\text{BG}}(E) \approx 0). \tag{62}$$

we observe a "resonance" at **real energy** $E = M_R$:

  (i) Re $f_\ell(E) = 0$ and Im $f_\ell(E)$ becomes maximal ← Eq. (61)

  (ii) Cross section $\sigma(E)$ exhibits a peak ← (i) and the optical theorem (see Exercise 2)

  (iii) Phase shift $\delta_\ell(E)$ increases rapidly and crosses $\pi/2$
  ← From Eq. (29), Im $s_\ell(M_R) = 0$ when Re $f_\ell(M_R) = 0$
  ← Except for $\delta_\ell = 0$ (non-interacting case), we have $\delta_\ell = \pi/2$ (modulo π)

- When $f_{\ell,\text{BG}}(E)$ is non-negligible, interference term contributes:

$$|f_\ell(E)|^2 = |f_{\ell,\text{BW}}(E)|^2 + |f_{\ell,\text{BG}}(E)|^2 + 2\text{Re}\,[f_{\ell,\text{BW}}(E) f^*_{\ell,\text{BG}}(E)], \tag{63}$$

Peaks can be generated kinematically without a pole (e.g. threshold cusps) [31]
⇒ Importance of a careful analysis to determine resonance poles

4.5 Effective range expansion

- Low-energy (small p) behavior of scattering amplitude $f_\ell(p)$

$$f_\ell(p) = \frac{s_\ell(p) - 1}{2ip} = \frac{1}{p\cot\delta_\ell(p) - ip} = \frac{p^{2\ell}}{p^{2\ell+1}\cot\delta_\ell(p) - ip^{2\ell+1}} \tag{64}$$

- From Eq. (43), Jost function can be written as

$$\not{f}_\ell(p) = F_\ell(p^2) + ip\,G_\ell(p^2) \quad \Leftrightarrow \quad \not{f}_\ell(-p) = F_\ell(p^2) - ip\,G_\ell(p^2) \tag{65}$$

F_ℓ and G_ℓ are functions of p^2 and behave as $(p \to 0)$

$$F_\ell(p^2) = \mathcal{O}(p^0), \quad G_\ell(p^2) = \mathcal{O}(p^{2\ell}) \tag{66}$$

- From Eq. (52)

$$f_\ell(p) = \frac{\not{f}_\ell(-p) - \not{f}_\ell(p)}{2ip\not{f}_\ell(p)} = \frac{p^{2\ell}}{-p^{2\ell}F_\ell(p^2)/G_\ell(p^2) - ip^{2\ell+1}} \tag{67}$$

- Comparing Eqs. (64) and (67), we obtain

$$p^{2\ell+1}\cot\delta_\ell(p) = -p^{2\ell}\frac{F_\ell(p^2)}{G_\ell(p^2)} \tag{68}$$

Right-hand side is a function of p^2 with $\mathcal{O}(p^0)$ as $p \to 0$, and its Taylor expansion reads

$$\Rightarrow \quad p^{2\ell+1}\cot\delta_\ell(p) = -\frac{1}{a_\ell} + \frac{r_\ell}{2}p^2 + \mathcal{O}(p^4) \tag{69}$$

which is called the **effective range expansion**.

- s-wave case ($\ell = 0$)

$$f_0(p) = \frac{1}{-\frac{1}{a_0} + \frac{r_0}{2}p^2 + \mathcal{O}(p^4) - ip}$$

  - a_0: scattering length (opposite sign convention is also used in hadron physics)
  - r_0: effective range, roughly corresponds to the interaction range, but can be negative
  - Eq. (68) can have poles (CDD poles [32])
    When a CDD pole exists at low energy, Padé approximant is useful [33, 34]

- Low-energy scattering: assuming higher-order terms in p are negligible

$$f_0(p) \approx \frac{1}{-\frac{1}{a_0} - ip}$$

Pole at $p = \frac{i}{a_0}$

  - $a_0 > 0$: pole in the upper half-plane $\Rightarrow$ bound state ($E = -1/(2\mu a_0^2) < 0$)
  - $a_0 < 0$: pole in the lower half-plane $\Rightarrow$ virtual state ($E = -1/(2\mu a_0^2) < 0$)
  - $a_0 \to \pm\infty$: pole at $p = 0$ ($E = 0$, **unitary limit**)

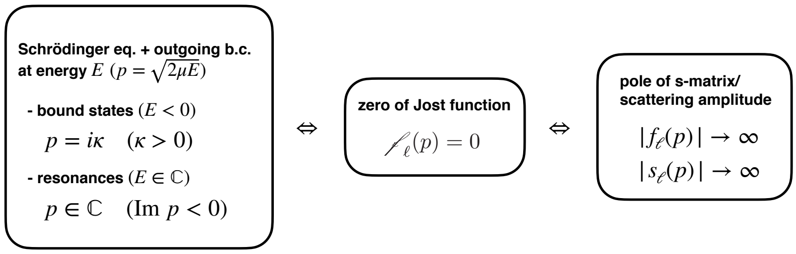

Figure 9: Various conditions for resonances. The outgoing boundary condition of the wave function is related to the poles of the scattering amplitude through the zeros of the Jost function.

4.6 Summary of §3 and §4

- Definitions of S-matrix, phase shift, scattering amplitude, etc.
- Correspondence between poles of scattering amplitude and resonance states (Fig. 9)
- Effective range expansion: description of low-energy scattering