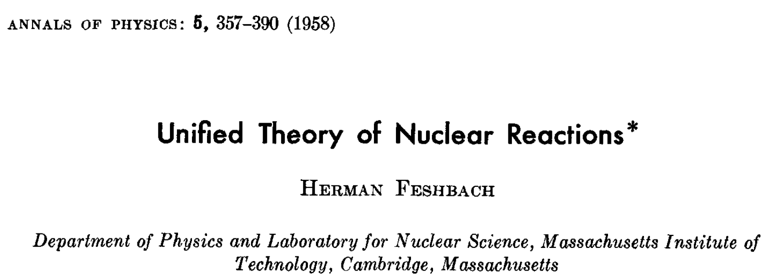
5 Theory of Feshbach resonances

5.1 Introduction

- Feshbach resonance: a resonance in **coupled-channel** scattering
- Threshold energy E_{th} and channels at energy E
  - open channels ($E > E_{\text{th}}$): scattering occurs
  - closed channels ($E < E_{\text{th}}$): no scattering occurs
- Original work: theory of compound nuclear reactions [35, 36] (Fig. 10, left)
- Cold atoms experiments: control of scattering length by a magnetic field [37] (Fig. 10, right)



ANNALS OF PHYSICS: **5**, 357–390 (1958)

Unified Theory of Nuclear Reactions*

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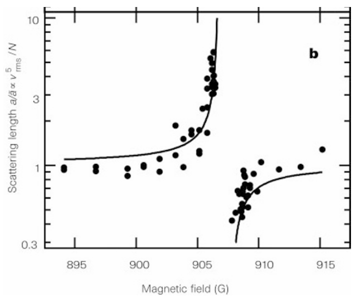

Figure 10: Left: H. Feshbach, Ann. Phys. **5**, 357 (1958). Right: Control of the scattering length in cold atoms by a magnetic field, adopted from S. Inouye, Nature (London) **392**, 151 (1998).

5.2 Two-channel Hamiltonian

- Two channels P and Q, with the threshold of P set at $E_{\text{th}}(P) = 0$ [38]
- Schrödinger equation in matrix form (a set of equations)

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \tag{70}$$

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} = \begin{pmatrix} \dfrac{\hat{\boldsymbol{p}}^2}{2\mu_P} + \hat{V}_P & \hat{V}_t \\ \hat{V}_t & \dfrac{\hat{\boldsymbol{p}}^2}{2\mu_Q} + \Delta + \hat{V}_Q \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix} \tag{71}$$

  - $\hat{V}_P, \hat{V}_Q$: potentials in each channel (Fig. 4), vanishing at $r \to \infty$
  - $\hat{V}_t$: channel-transition potential

- $\Delta = E_{\rm th}(Q) - E_{\rm th}(P) > 0$ (originating from the Zeeman splitting $\propto$ magnetic field)
- For $0 < E < \Delta$, P is open and Q is closed

- Projection operators

$$\hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{72}$$

$$\hat{P}^2 = \hat{P}, \quad \hat{Q}^2 = \hat{Q}, \quad \hat{P}\hat{Q} = \hat{Q}\hat{P} = 0, \quad \hat{P} + \hat{Q} = \hat{I} \tag{73}$$

Each component of $|\psi\rangle$ or $\hat{H}$ can be written as $|X\rangle = \hat{X}|\psi\rangle$ and $\hat{H}_{XY} = \hat{X}\hat{H}\hat{Y}$.

- **Effective Hamiltonian for channel P** $\leftarrow$ eliminating $|Q\rangle$

Lower component of Eq. (70)

$$\begin{aligned} \hat{H}_{QP}|P\rangle + \hat{H}_{QQ}|Q\rangle &= E|Q\rangle \\ \hat{H}_{QP}|P\rangle &= (E - \hat{H}_{QQ})|Q\rangle \\ |Q\rangle &= (E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}|P\rangle \end{aligned} \tag{74}$$

Upper component of Eq. (70)

$$\begin{aligned} \hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle &= E|P\rangle \\ \hat{H}_{PP}|P\rangle + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}|P\rangle &= E|P\rangle \\ \Rightarrow \quad \hat{H}^{\rm eff}(E)|P\rangle &= E|P\rangle, \end{aligned} \tag{75}$$

$$\hat{H}^{\rm eff}(E) = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP} \tag{76}$$

$\hat{H}^{\rm eff}(E)$ effectively incorporates the effect of Q.

- Single-channel (not in matrix form) Schrödinger equation for P
- No approximations $\Rightarrow$ the solution of Eq. (75) is equivalent to $|P\rangle$ in Eq. (70)
- $\hat{H}^{\rm eff}(E)$ is energy dependent $\rightarrow$ self-consistent solutions for discrete eigenstates

5.3 Single-resonance approximation

- Eigenstates of $\hat{H}_{QQ}$ without the transition ($\hat{V}_t = 0$, $\hat{V}_Q \neq 0$, see Fig. 11):

$$\hat{H}_{QQ}|\phi_i\rangle = \epsilon_i|\phi_i\rangle, \quad \text{(bound states)}, \tag{77}$$

$$\hat{H}_{QQ}|\phi(\epsilon)\rangle = \epsilon|\phi(\epsilon)\rangle \quad \text{(continuum states labeled by energy } \epsilon) \tag{78}$$

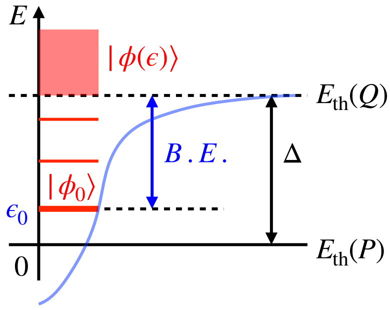

Figure 11: Schematic illustration of the eigenstates of $\hat{H}_{QQ}$.

- Spectral decomposition (continuum starts from $\epsilon = \Delta$)

$$\hat{I} = \sum_i |\phi_i\rangle \langle\phi_i| + \int_\Delta^\infty d\epsilon\, |\phi(\epsilon)\rangle \langle\phi(\epsilon)|, \tag{79}$$

$$\hat{H}^{\rm eff}(E) = \frac{\hat{\boldsymbol{p}}^2}{2\mu_P} + \hat{V}_P + \sum_i \frac{\hat{H}_{PQ}|\phi_i\rangle\langle\phi_i|\hat{H}_{QP}}{E-\epsilon_i} + \int_\Delta^\infty d\epsilon \frac{\hat{H}_{PQ}|\phi(\epsilon)\rangle\langle\phi(\epsilon)|\hat{H}_{QP}}{E-\epsilon+i0^+} \tag{80}$$

- $\hat{H}^{\rm eff}(E)$ acquires an imaginary part for $E > \Delta$

$$\int dx \frac{f(x)}{x-a+i0^+} = \mathcal{P}\int dx \frac{f(x)}{x-a} - i\pi f(a) \quad \text{(if } a \in \text{integration range)} \tag{81}$$

- When $|\epsilon_0|$ is sufficiently small, the dominant contribution for $E \ll \Delta$ is

$$\hat{H}^{\rm eff}(E) \approx \frac{\hat{\boldsymbol{p}}^2}{2\mu_P} + \frac{\hat{V}_t|\phi_0\rangle\langle\phi_0|\hat{V}_t}{E-\epsilon_0} \tag{82}$$

- ϵ_0 is measured from $E_{\rm th}(P) = 0$; binding energy $(B.E.)$ from $E_{\rm th}(Q) = \Delta$ is

$$B.E. = \Delta - \epsilon_0 \tag{83}$$

  $B.E.$ is fixed by $\hat{H}_{QQ}$ ⇒ If Δ is proportional to magnetic field, ϵ_0 can be tuned

5.4 Scattering amplitude and resonance

- $\hat{H}^{\rm eff}$ is a single-channel Hamiltonian for P ⇒ we can apply scattering theory in §3

$$\hat{H}^{\rm eff} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2\mu_P}, \quad \hat{V} = \frac{\hat{V}_t|\phi_0\rangle\langle\phi_0|\hat{V}_t}{E-\epsilon_0}, \quad \hat{H}_0|\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\mu_P}|\boldsymbol{p}\rangle \tag{84}$$

- **Lippmann–Schwinger (LS) equation** for the wave function $|P\rangle$

$$|P\rangle = |\boldsymbol{p}\rangle + \hat{G}\hat{V}|P\rangle \tag{85}$$

  Free Green's operator (resolvent)

$$\hat{G}(E) = (E - \hat{H}_0)^{-1} \tag{86}$$

- T operator and scattering amplitude

$$\hat{T} = \hat{V} + \hat{V}\hat{G}\hat{T} = \hat{V} + \hat{V}\hat{G}\hat{V} + \hat{V}\hat{G}\hat{V}\hat{G}\hat{V} + \cdots, \tag{87}$$

$$\langle \boldsymbol{p}'| \hat{T}(E + i0^+) |\boldsymbol{p}\rangle = t(\boldsymbol{p}' \leftarrow \boldsymbol{p}) = -\frac{1}{(2\pi)^2 \mu_P} f(E, \theta) \tag{88}$$

- LS equation for the T matrix (integral equation)

$$t(\boldsymbol{p}' \leftarrow \boldsymbol{p}) = \langle \boldsymbol{p}'| \hat{V} |\boldsymbol{p}\rangle + \int d\boldsymbol{q}\, \langle \boldsymbol{p}'| \hat{V} |\boldsymbol{q}\rangle \frac{1}{E - \boldsymbol{q}^2/(2\mu_P) + i0^+} t(\boldsymbol{q} \leftarrow \boldsymbol{p}) \tag{89}$$

- The potential $\hat{V}$ in Eq. (84) is **separable** (a product of functions of $\boldsymbol{p}$ and $\boldsymbol{p}'$):

$$\langle \boldsymbol{p}'| \hat{V}(E) |\boldsymbol{p}\rangle = \frac{\langle \boldsymbol{p}'| \hat{V}_t |\phi_0\rangle \langle \phi_0| \hat{V}_t |\boldsymbol{p}\rangle}{E - \epsilon_0} = \lambda F^*(\boldsymbol{p}')F(\boldsymbol{p}), \tag{90}$$

$$\lambda = \frac{1}{E - \epsilon_0}, \quad F(\boldsymbol{p}) = \langle \phi_0| \hat{V}_t |\boldsymbol{p}\rangle \quad \text{(form factor)} \tag{91}$$

In this case, the scattering amplitude can be written in a closed form:

$$f(E, \theta) = -(2\pi)^2 \mu_P \frac{F^*(\boldsymbol{p}')F(\boldsymbol{p})}{\frac{1}{\lambda} - \int d\boldsymbol{q} \frac{|F(\boldsymbol{q})|^2}{E - \boldsymbol{q}^2/(2\mu_P) + i0^+}} = \frac{-(2\pi)^2 \mu_P F^*(\boldsymbol{p}')F(\boldsymbol{p})}{E - \epsilon_0 - \Sigma(E)}, \tag{92}$$

$$\Sigma(E) = \int d\boldsymbol{q} \frac{|F(\boldsymbol{q})|^2}{E - \boldsymbol{q}^2/(2\mu_P) + i0^+} \quad \text{(self energy)} \tag{93}$$

- Pole condition:

$$0 = E - \epsilon_0 - \Sigma(E) \tag{94}$$

- When $\hat{V}_t = 0$ (no channel transition), $\Sigma(E) = 0$, and hence

$$E = \epsilon_0 \in \mathbb{R} \tag{95}$$

$\Rightarrow$ bound state by $\hat{H}_{QQ}$ (decoupled from channel P)

- When $\hat{V}_t \neq 0$, pole is a solution of $E - \epsilon_0 - \Sigma(E) = 0$ in general.
For weak $\hat{V}_t$, a perturbative approximation gives

$$E \approx \epsilon_0 + \Sigma(\epsilon_0) \in \mathbb{C} \quad \text{for } \epsilon_0 > 0 \tag{96}$$

$\Rightarrow$ **resonance with a complex eigenenergy** in channel P
Bound state of $\hat{H}_{QQ}$ acquires a decay width through coupling to the continuum of P

5.5 Summary of §5

- Coupled-channel Hamiltonian for P and Q
- Elimination of channel Q to obtain an effective Hamiltonian in P
- A bound state $|\phi_0\rangle$ in Q couples to P and generates a state with complex energy