6 Nonrelativistic effective field theory

6.1 Effective field theories

- Microscopic quantum field theory $\mathcal{L}_{\text{micro}}$
- Λ: ultraviolet **cutoff** scale (see Fig. 12)
- Effective Field Theory, EFT $\mathcal{L}_{\text{EFT}}$
  - describes the same phenomena as $\mathcal{L}_{\text{micro}}$ up to Λ
  - can be constructed systematically

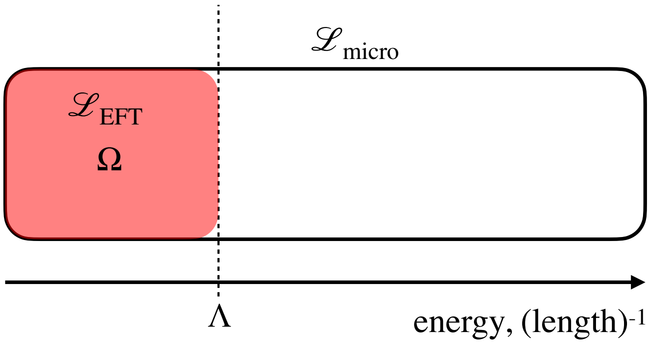

Figure 12: Schematic illustration of effective field theory.

- Electromagnetic interaction
  - Microscopic theory: QED
    photons ($m_\gamma = 0$), electrons ($m_e \neq 0$)
  - EFT: **Euler-Heisenberg theory** [39]
    only photons, "heavy" electrons are integrated out ($\Lambda \sim m_e$)
- Weak interaction
  - Microscopic theory: Weinberg-Salam theory
    leptons, neutrinos, $W^\pm, Z$
  - EFT: **Fermi theory** (Fig. 13)
    only leptons and neutrinos, "heavy" bosons are integrated out ($\Lambda \sim m_{W^\pm}, m_Z$)
    $W^\pm, Z$ exchange $\to$ contact four-Fermi interaction (Fig. 13)

$$\text{interaction} \propto \frac{g_w^2}{q^2 - m_W^2} = -\frac{g_w^2}{m_W^2}\left(1 + \mathcal{O}\left(\frac{q^2}{m_W^2}\right)\right) \tag{97}$$

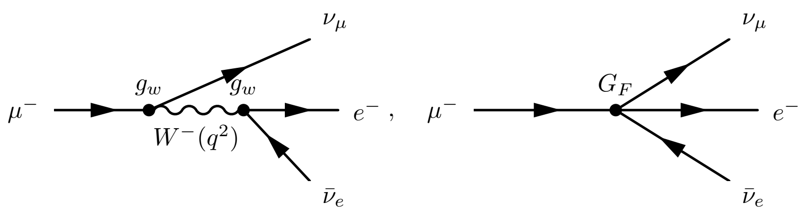

Figure 13: Schematic figure of μ^- decay. Left: Weinberg–Salam theory; Right: Fermi theory.

- Strong interaction
  - Microscopic theory: QCD
    low-energy d.o.f. (hadrons) are **different** from microscopic d.o.f. (quarks and gluons)
  - Weinberg's "theorem" [40]
    The **most general** $\mathcal{L}_{\mathrm{EFT}}$ consistent with the **symmetries** of $\mathcal{L}_{\mathrm{micro}}$ works
  - EFT: chiral perturbation theory $\leftarrow$ chiral symmetry of QCD [41]
    Infinitely many terms? $\to$ organized by power counting

$$\mathcal{L}_{\mathrm{ChPT}} = \mathcal{L}^{(\mathrm{LO})} + \mathcal{L}^{(\mathrm{NLO})} + \cdots$$

6.2 Zero-range model

- EFT for nonrelativistic two-body scattering processes
- R_{typ}: typical length scale of short-range interaction
  - Square well potential: $R_{\mathrm{typ}} = b$ (well width)
  - Yukawa potential $V(r) = g\dfrac{e^{-\kappa r}}{r}$: $R_{\mathrm{typ}} = 1/\kappa$
  - van der Waals potential $V(r) = -\dfrac{C_6}{r^6}$: $R_{\mathrm{typ}} \sim \ell_{\mathrm{vdW}} = \left(\dfrac{mC_6}{\hbar^2}\right)^{1/4}$
- **Zero-range model** $\mathcal{L}_{\mathrm{ZR}}$: s-wave scattering with $|a_0| \gg R_{\mathrm{typ}}$ [42]

$$f(p) = \frac{1}{-\dfrac{1}{a_0} - ip} \tag{98}$$

  - Nucleons: Yukawa potential due to pion (π) exchange

$$a_0(^1\mathrm{S}_0) \simeq 20 \text{ fm}, \quad a_0(^3\mathrm{S}_1) \simeq -4 \text{ fm}, \qquad |a_0| \gg R_{\mathrm{typ}} \sim \frac{1}{m_\pi} \sim 1 \text{ fm} \tag{99}$$

  - ^{4}He atoms: van der Waals potential due to polarization

$$a_0 \simeq 200 \text{ [Bohr radius]} \qquad |a_0| \gg R_{\mathrm{typ}} \sim \ell_{\mathrm{vdW}} \sim 10 \text{ [Bohr radius]} \tag{100}$$

- At low energies $p \ll 1/R_{\rm typ}$, both systems can be described by the same $\mathcal{L}_{\rm ZR}$ (**universality**)
- Lagrangian density of zero-range model

$$\mathcal{L}_{\rm ZR}(t,\boldsymbol{x}) = \underbrace{\psi^\dagger(t,\boldsymbol{x})\left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{2m}\right)\psi}_{\text{kinetic term}} - \underbrace{\frac{\lambda_0}{4}[\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})]^2}_{\text{interaction term}} \tag{101}$$

$\psi(t,\boldsymbol{x})$: bosonic field, $\quad m$: boson mass, $\quad \lambda_0$: (bare) coupling constant

(two fermions with an antisymmetric spin wave function $\sim$ two bosons)

- Quantization: equal-time commutation relations

$$[\psi(t,\boldsymbol{x}),\psi(t,\boldsymbol{x}')] = 0, \quad [\psi(t,\boldsymbol{x}),\psi^\dagger(t,\boldsymbol{x}')] = \delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{102}$$

- Interaction term: four-point contact interaction $\sim$ 3d δ function potential

$$-\mathcal{L}_{\rm int} = \frac{\lambda_0}{4}(\psi^\dagger\psi)^2 \sim \mathcal{H}_{\rm int} \sim (\text{energy}) \tag{103}$$

$$\begin{cases} \lambda_0 > 0 & \text{increase energy} \Rightarrow \text{repulsion} \\ \lambda_0 < 0 & \text{decrease energy} \Rightarrow \text{attraction} \end{cases} \tag{104}$$

- **Symmetries**: space-time translations, rotations, parity, Galilean boosts, phase symmetry

$$\psi \to e^{i\theta}\psi, \quad N = \int d\boldsymbol{x}\ \psi^\dagger\psi \quad (\text{particle number}) \tag{105}$$

$\Rightarrow \mathcal{L}_{\rm int}$ does not change the particle number (a two-body state remains a two-body state)

6.3 Feynman rules

- Calculation of physical quantities in quantum field theory

  1. Derive Feynman rules (pieces of Feynman diagrams)
  2. Sum over all possible Feynman diagrams (possible in two-body sector of $\mathcal{L}_{\rm ZR}$)

  2'. Perform perturbation theory (if step 2 is not feasible)

- Propagator: propagation of a particle (Fig. 14, left)

$$G(\omega,\boldsymbol{k}) = \frac{i}{\omega - \boldsymbol{k}^2/(2m) + i0^+} \tag{106}$$

only positive-energy component: propagation is forward in time only

- Vertex: interaction (Fig. 14, middle)

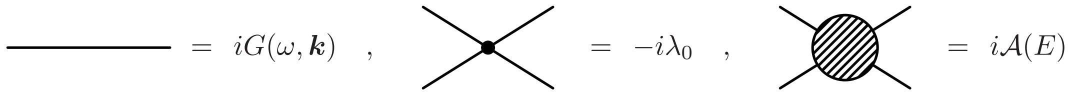

Figure 14: Feynman rules of the zero-range model (101). Left: boson propagator G; Middle: vertex $-i\lambda_0$; Right: four-point function $i\mathcal{A}$.

6.4 Two-boson scattering

- Two-body scattering amplitude ← four-point function $i\mathcal{A}(E)$ (2 in, 2 out, Fig. 14, right)
- Write down all diagrams from Feynman rules with fixed initial and final states (Fig. 15)

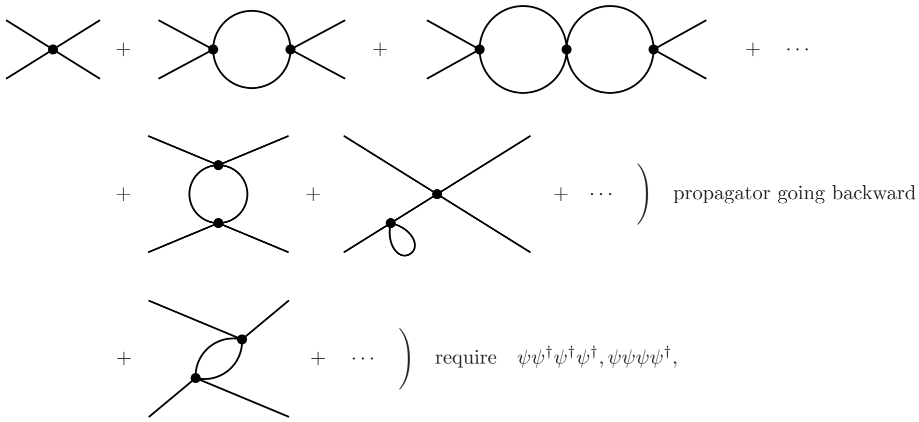

Figure 15: Candidates of Feynman diagrams.

- Eventually, the same structure as the Lippmann–Schwinger equation emerges (Fig. 16)

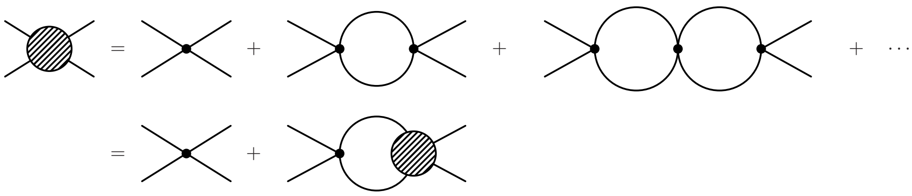

Figure 16: Possible Feynman diagrams.

- Terms of order $(\lambda_0)^n$ are summed to all orders: nonperturbative scattering amplitude
- Two-body scattering amplitude $\mathcal{A}(E)$

$$i\mathcal{A}(E) = -i\lambda_0 - i\lambda_0 \frac{1}{2} \int \frac{d\omega d\boldsymbol{q}}{(2\pi)^4} G(\omega, \boldsymbol{q}) G(E - \omega, -\boldsymbol{q}) i\mathcal{A}(E) \tag{107}$$

- 1/2: symmetry factor
- Completeness relation in QFT: $1 = \int \frac{d\boldsymbol{q}}{(2\pi)^3} |\boldsymbol{q}\rangle \langle \boldsymbol{q}|$
- dq integration diverges: introduce cutoff Λ (integral range $0 \le q \le \Lambda$)
- $i\mathcal{A}(E)$ in r.h.s. is not integrated over $\sim$ separable interaction in momentum space

- $\mathcal{A}(E)$ can be obtained algebraically

$$\mathcal{A}(E) = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2}\left(\Lambda - \sqrt{-mE - i0^+}\arctan\frac{\Lambda}{\sqrt{-mE-i0^+}}\right)\right]^{-1} \tag{108}$$

- Energy E and momentum p

$$E = \frac{p^2}{2\mu} = \frac{p^2}{m} \quad \leftarrow \quad \mu = \frac{mm}{m+m} = \frac{m}{2} \tag{109}$$

For physical scattering $E > 0$, $p > 0$,

$$\sqrt{-mE - i0^+} = -i\sqrt{m|E|} = -i\sqrt{p^2} = -ip \tag{110}$$

- For small momentum $p \ll \Lambda$

$$\arctan\left(\frac{\Lambda}{-ip}\right) = \frac{\pi}{2} + \mathcal{O}\left(\frac{p}{\Lambda}\right) \tag{111}$$

then, Eq. (106) is

$$\mathcal{A}(p) = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2}\left(\Lambda + ip\frac{\pi}{2}\right)\right]^{-1} = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2}\Lambda - ip\frac{m}{8\pi}\right]^{-1} \tag{112}$$

- Scattering amplitude

$$f(p) = \frac{m}{8\pi}\mathcal{A}(p) = \frac{1}{-\frac{8\pi}{m}\left(\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda\right) - ip} \tag{113}$$

Comparing with Eq. (98), the scattering length is given by

$$a_0 = \frac{m}{8\pi}\left(\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda\right)^{-1} \tag{114}$$

6.5 Summary of §6

- EFT: a description of low-energy physics

- Zero-range model: nonperturbative, unitary scattering amplitude

$$f(p) = \frac{1}{-1/a_0 - ip} \tag{115}$$