7 Compositeness and weak-binding relation

7.1 Introduction

- Internal structure of hadron resonances: superposition of all possible configurations

$$|\Lambda(1405)\rangle = C_{3q}\,|uds\rangle + C_{5q}\,|uds\bar{q}q\rangle + C_{MB}\,|MB\rangle + \cdots \tag{116}$$

- C_i (weights of each component) can be determined in specific models
  $\to$ model-dependent result (the wavefunction is not an observable)

- A **model-independent** relation that connects observables with the internal structure?

7.2 Weak-binding relation

- **Compositeness X** of a stable bound state (deuteron) [43, 20, 21]

$$|d\rangle = \sqrt{X}\,|NN\rangle + \sqrt{Z}\,|\text{others}\rangle\,, \quad X + Z = 1 \tag{117}$$

  $|d\rangle$: wavefunction of the deuteron
  $|NN\rangle$: two-nucleon (s-wave) component
  (labeled by continuous variables, and X is the sum over all components)
  $|\text{others}\rangle$: all other components
  ($\sim 6q$ states, $N\Delta$ states, NN (d-wave) states, ...)
  Z: elementarity (elementariness)

- $(X, Z) \approx (1, 0)$: composite-dominant state
  $(X, Z) \approx (0, 1)$: elementary-dominant state

- **Weak-binding relation**

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}, \quad R = \frac{1}{\sqrt{2\mu B}} \tag{118}$$

  a_0: NN scattering length ($^3\mathrm{S}_1$ channel)
  B: deuteron binding energy, $\mu = M_N/2$: NN reduced mass
  R: deuteron radius (length scale associated with the binding energy)
  R_{typ}: typical interaction range

- When B is small (R_{typ}/R is negligible), the compositeness X is determined by observables.

- Conditions:

  – X is the compositeness in the s-wave scattering channel (not applicable to $\ell \neq 0$)

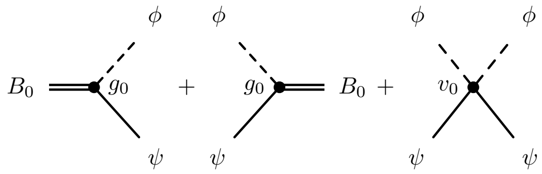


Figure 17: Feynman rules for the interaction Hamiltonian (122).

- Stable bound state (no decay width)

- Empirical values: $B = 2.22$ MeV, $a_0 = 5.42$ fm, $R_{\rm typ} \sim 1/m_\pi = 1.43$ fm [44]

$$X = 1.68^{+3.18}_{-0.943} \quad \text{(including finite-range corrections)} \tag{119}$$

By definition, $X \leq 1 \Rightarrow 0.74 \leq X \leq 1$: $\sim 80\%$ of the deuteron is an NN composite

- No input on the nuclear force or wave function is required.

7.3 Compositeness in effective field theory

- Hamiltonian of EFT [33, 34]

$$H = H_{\rm free} + H_{\rm int} \tag{120}$$

$$H_{\rm free} = \int d\boldsymbol{r} \left[\frac{1}{2M}\nabla\psi^\dagger \cdot \nabla\psi + \frac{1}{2m}\nabla\phi^\dagger \cdot \nabla\phi + \frac{1}{2M_0}\nabla B_0^\dagger \cdot \nabla B_0 + \omega_0 B_0^\dagger B_0\right] \tag{121}$$

$$H_{\rm int} = \int d\boldsymbol{r} \left[g_0 \left(B_0^\dagger\phi\psi + \psi^\dagger\phi^\dagger B_0\right) + v_0\psi^\dagger\phi^\dagger\phi\psi\right] \tag{122}$$

$\phi(\boldsymbol{r})$, $\psi(\boldsymbol{r})$, $B_0(\boldsymbol{r})$: field operators, M, m, M_0: masses, ω_0: bare energy
Each term in the free Hamiltonian $H_{\rm free}$ is equivalent to the kinetic term of $\mathcal{L}_{\rm eff}$ in §6
g_0, v_0: coupling constants
$H_{\rm int}$: contact interactions (Fig. 17), $\psi\phi \leftrightarrow \psi\phi$ and $\psi\phi \leftrightarrow B_0$

- Eigenstates of $H_{\rm free}$

$$H_{\rm free}\,|B_0\rangle = \omega_0\,|B_0\rangle \quad \text{(discrete state } B_0\text{, elementary component)} \tag{123}$$

$$H_{\rm free}\,|\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\mu}\,|\boldsymbol{p}\rangle \quad (\psi\phi \text{ scattering states with } \boldsymbol{p}\text{, composite component}) \tag{124}$$

$$\langle B_0|B_0\rangle = 1, \quad \langle B_0|\boldsymbol{p}\rangle = 0, \quad \langle \boldsymbol{p}'|\boldsymbol{p}\rangle = (2\pi)^3\delta^3(\boldsymbol{p}' - \boldsymbol{p}) \tag{125}$$

- Complete set: (from particle number conservation)

$$1 = |B_0\rangle\langle B_0| + \int \frac{d\boldsymbol{p}}{(2\pi)^3}\,|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \tag{126}$$

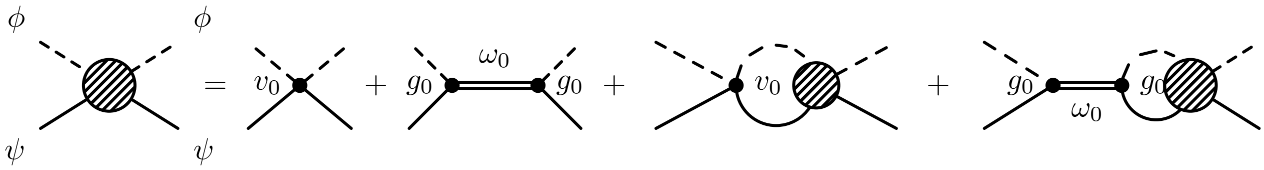


Figure 18: Feynman diagrams of the $\psi\phi$ scattering amplitude.

Relation to Feshbach projection [45] ($\hat{P} + \hat{Q} = \hat{I}$)

$$\hat{P} = \int \frac{d\boldsymbol{p}}{(2\pi)^3} |\boldsymbol{p}\rangle \langle \boldsymbol{p}| \quad \text{(projection onto } \psi\phi \text{ scattering states)} \tag{127}$$

$$\hat{Q} = |B_0\rangle \langle B_0| \quad \text{(projection onto discrete state } B_0\text{)} \tag{128}$$

Field theory equivalent to the single-resonance approximation of Q channel in §5

- Bound state $|B\rangle$ with binding energy B as an eigenstate of H (physical state)

$$H |B\rangle = -B |B\rangle \tag{129}$$

Compositeness X, elementarity Z

$$X \equiv \langle B| \hat{P} |B\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} |\langle \boldsymbol{p}|B\rangle|^2 \geq 0 \quad \text{(overlap with scattering states)} \tag{130}$$

$$Z \equiv \langle B| \hat{Q} |B\rangle = |\langle B_0|B\rangle|^2 \geq 0 \quad \text{(overlap with discrete state)} \tag{131}$$

- From the normalization $\langle B|B\rangle = 1$ and the completeness relation (126),

$$Z + X = 1 \tag{132}$$

X and Z can be interpreted as **probabilities** $\leftarrow$ (130), (131), and (132)

7.4 Weak-binding relation in effective field theory

- $\psi\phi$ scattering amplitude (derived as in §6, see Fig. 18)

$$f(E) = -\frac{\mu}{2\pi} \frac{1}{1/v(E) - G(E)} \tag{133}$$

$$v(E) = v_0 + \frac{g_0^2}{E - \omega_0}, \tag{134}$$

$$G(E) = \frac{1}{2\pi^2} \int_0^\Lambda dq \frac{q^2}{E - q^2/(2\mu) + i0^+} \tag{135}$$

- Cutoff Λ: momentum scale at which the hadron interaction can be regarded as pointlike

$$\Rightarrow R_{\text{typ}} = \frac{1}{\Lambda} \tag{136}$$

- Expression for the compositeness in terms of the scattering amplitude

$$X = \left.\frac{G'(E)}{G'(E) - [1/v(E)]'}\right|_{E=-B}, \quad A'(E) = \frac{dA(E)}{dE} \tag{137}$$

- Expansion of the scattering length in powers of (R_{typ}/R)

$$a_0 = -f(E=0) = \frac{\mu}{2\pi}[1/v(0) - G(0)]^{-1} \tag{138}$$

$$= \cdots = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}, \quad R = \frac{1}{\sqrt{2\mu B}} \tag{139}$$

- If $\mathcal{O}(R_{\text{typ}}/R)$ is negligible, **compositeness** $X \leftarrow (a_0, B)$

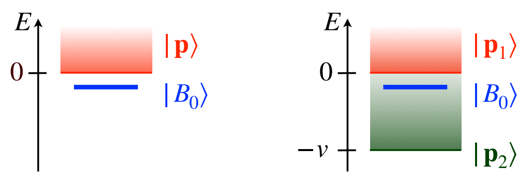
7.5 Generalization to unstable states

- For applications to hadron resonances, a generalization to unstable states is necessary
- Inclusion of a decay channel (Fig. 19)
  Fields: ϕ_1, ψ_1, ϕ_2, ψ_2, B_0
  Four-point contact interactions: $\psi_1\phi_1 \leftrightarrow \psi_1\phi_1$, $\psi_1\phi_1 \leftrightarrow \psi_2\phi_2$, $\psi_2\phi_2 \leftrightarrow \psi_2\phi_2$
  Three-point contact interactions: $\psi_1\phi_1 \leftrightarrow B_0$, $\psi_2\phi_2 \leftrightarrow B_0$
- When the threshold energy $-\nu$ of the additional channel 2 ($\psi_2\phi_2$) is lower than B (§5),

$$H\,|R\rangle = E_h\,|R\rangle\,, \quad E_h \in \mathbb{C} \tag{140}$$

- Normalization condition for resonances ($\langle R|R\rangle$ diverges)

$$\langle\tilde{R}|R\rangle = 1 \tag{141}$$



(a) (b)

Figure 19: Spectrum of free Hamiltonian. (a): EFT in Sec. 7.3 and 7.4, (b): EFT in Sec. 7.5.

$\langle\tilde{R}|$ is the left eigenvector with the same complex eigenvalue

$$\langle\tilde{R}|\, H = E_h \langle\tilde{R}| \tag{142}$$

(Because of the non-hermiticity, left eigenvector is not conjugate of right eigenvector)

- Complete set

$$1 = \hat{P} + \hat{Q} \tag{143}$$

$$\hat{P} = \int \frac{d\boldsymbol{p}}{(2\pi)^3} |\boldsymbol{p}_1\rangle \langle \boldsymbol{p}_1| \quad \text{(projection onto } \psi_1\phi_1 \text{ scattering states)} \tag{144}$$

$$\hat{Q} = |B_0\rangle\langle B_0| + \int \frac{d\boldsymbol{p}}{(2\pi)^3} |\boldsymbol{p}_2\rangle \langle \boldsymbol{p}_2| \quad \text{(projection onto } B_0 \text{ and } \psi_2\phi_2) \tag{145}$$

- Compositeness X, elementarity Z

$$X = \langle\tilde{R}|\, P\, |R\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle\tilde{R}|\boldsymbol{p}_1\rangle \langle \boldsymbol{p}_1|R\rangle \in \mathbb{C}$$

$$Z = \langle\tilde{R}|\, Q\, |R\rangle = \langle\tilde{R}|B_0\rangle\langle B_0|R\rangle + \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle\tilde{R}|\boldsymbol{p}_2\rangle \langle \boldsymbol{p}_2|R\rangle \in \mathbb{C}$$

- While $X+Z=1$ still holds, each term is **complex** and cannot be interpreted as a probability
- Expand the scattering length of channel 1 by $(R_{\rm typ}/R)$ (R, a_0 are complex)

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\left|\frac{R_{\rm typ}}{R}\right|\right) + \mathcal{O}\left(\left|\frac{l}{R}\right|^3\right)\right\}, \quad R = \frac{1}{\sqrt{-2\mu E_h}}, \quad l = \frac{1}{\sqrt{2\mu\nu}} \tag{146}$$

ν: threshold energy difference, l: corresponding length scale

- If $\mathcal{O}(R_{\rm typ}/R)$ and $\mathcal{O}(|l/R|^3)$ are negligible, **compositeness** $X \leftarrow (a_0, E_h)$
- Several interpretation of complex X [33, 34, 46, 47, 48, 29], but no consensus is reached
- $\Lambda(1405)$ case:
  - Channel 1 is $\bar{K}N$, channel 2 is $\pi\Sigma$
  - Inputs: $E_h = -10 - 26i$ MeV, $a_0 = 1.39 - 0.85i$ fm [49, 50]
  - For $|R| \sim 2$ fm, $R_{\rm typ} \sim 0.25$ fm (ρ-meson exchange), $l \sim 1.08$ fm ($\pi\Sigma$ channel)

Neglecting the correction terms, we obtain

$$X = 1.2 + i0.1 \tag{147}$$

X is close to 1 $\Rightarrow$ **the $\bar{K}N$ molecular component is dominant**