8 Spin-1/2 states

8.1 State representation

- Pauli spinor (quantum bit, **qubit**): two-component complex vector

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in \mathbb{C}, \tag{148}$$

$$\langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix} \tag{149}$$

Four degrees of freedom (Re a, Im a, Re b, Im b without any constraint, $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2$)

- Normalization and phase of the wave function

$$\langle\psi|\psi\rangle = 1 \quad \Rightarrow \quad |a|^2 + |b|^2 = 1 \tag{150}$$

(In the following, all state vectors are assumed to be normalized)
Phase transformation $|\psi\rangle \to e^{i\eta}|\psi\rangle$ ($\eta \in \mathbb{R}$) does not change expectation values
$\to |\psi\rangle$ is described by **two degrees of freedom**

- Bloch-sphere representation

$$|\theta, \phi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad 0 \le \theta < \pi, \quad 0 \le \phi < 2\pi \tag{151}$$

Two real parameters $\theta, \phi \in \mathbb{R}$: angular part of three-dimensional polar coordinates

- Fubini–Study measure (generalizable to higher-spin states)
Normalization and phase: equivalence class by complex scalar multiplication

$$|\psi\rangle \sim \lambda|\psi\rangle, \quad \lambda \in \mathbb{C}^\times \tag{152}$$

Quotient space $\mathbb{C}^2/\sim\, \simeq\, \mathbb{CP}^1$ (complex projective space)

$$|\theta_1, \nu_1\rangle = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 e^{i\nu_1} \end{pmatrix} \quad 0 \le \theta_1 < \frac{\pi}{2}, \quad 0 \le \nu_1 < 2\pi \tag{153}$$

Two degrees of freedom θ_1, ν_1

- Spin operator $\boldsymbol{s}$: represented by Pauli matrices $\boldsymbol{\sigma}$

$$\boldsymbol{s} = \frac{\boldsymbol{\sigma}}{2}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{154}$$

(In quantum information, $\sigma_x = X$, $\sigma_y = Y$, $\sigma_z = Z$)

- Eigenstates of s_z

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{155}$$

$$s_z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle, \quad s_z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle \tag{156}$$

(In quantum information, $|\uparrow\rangle = |0\rangle$, $|\downarrow\rangle = |1\rangle$)

- Eigenstates of s_x and s_y

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{157}$$

$$s_x |+\rangle = \frac{1}{2} |+\rangle, \quad s_x |-\rangle = -\frac{1}{2} |-\rangle, \tag{158}$$

$$|+i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{159}$$

$$s_y |+i\rangle = \frac{1}{2} |+i\rangle, \quad s_y |-i\rangle = -\frac{1}{2} |-i\rangle \tag{160}$$

8.2 State transformations (quantum gates)

- **Unitary operator** U

$$U^\dagger U = UU^\dagger = 1, \quad U^\dagger = U^{-1} \tag{161}$$

All Pauli matrices are Hermitian and unitary

- U transforms a state into another state: a **quantum gate** acting on a qubit

$$|\phi\rangle = U |\psi\rangle \tag{162}$$

- The norm (probability) is preserved under a unitary transformation

$$\langle\phi|\phi\rangle = \langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle \tag{163}$$

The time evolution of a system is described by a unitary operator

- Example: X gate

$$\sigma_x |\uparrow\rangle = |\downarrow\rangle, \quad \sigma_x |\downarrow\rangle = |\uparrow\rangle \tag{164}$$

Exchanges spin-up and spin-down (swaps 0 and 1: classical NOT gate)

- Hadamard gate

$$H = \frac{\sigma_x + \sigma_z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{165}$$

maps the z basis to the x basis (A 180° rotation about the axis $(1, 0, 1)/\sqrt{2}$)

$$H\left|\uparrow\right\rangle = \left|+\right\rangle, \quad H\left|\downarrow\right\rangle = \left|-\right\rangle, \tag{166}$$

$$H\left|+\right\rangle = \left|\uparrow\right\rangle, \quad H\left|-\right\rangle = \left|\downarrow\right\rangle \tag{167}$$

8.3 Projective measurement

- Finite-dimensional Hilbert space without degeneracy

- Physical observables are represented by **Hermitian operators**

- A Hermitian operator O can be diagonalized by a unitary operator U

$$U^{-1}OU = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \lambda_i \in \mathbb{R} \tag{168}$$

- **Spectral decomposition**: representation of O in terms of its eigenbasis $\{\left|i\right\rangle\}$

$$O = \sum_i \lambda_i \left|i\right\rangle \left\langle i\right| \tag{169}$$

- Complete orthonormal system (**CONS**) $\{\left|i\right\rangle\}$

$$\left\langle i|j\right\rangle = \delta_{ij}, \quad \sum_i \left|i\right\rangle \left\langle i\right| = 1 \tag{170}$$

The eigenvectors of a Hermitian operator can always be chosen to form a CONS
Example: $\{\left|\uparrow\right\rangle, \left|\downarrow\right\rangle\}$ (computational basis)

- **Projection operator** onto the eigenstate with eigenvalue λ_i

$$P_i = \left|i\right\rangle \left\langle i\right| \tag{171}$$

Orthonomality

$$P_i P_j = \delta_{ij} P_i, \quad \sum_i P_i = 1 \tag{172}$$

- Expansion of a state $|\psi\rangle$ in the basis $\{|i\rangle\}$

$$|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle = \sum_i C_i |i\rangle\,, \quad C_i = \langle i|\psi\rangle \tag{173}$$

Example:

$$|+\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle\,, \quad \langle\uparrow|+\rangle = \frac{1}{\sqrt{2}}, \quad \langle\downarrow|+\rangle = \frac{1}{\sqrt{2}} \tag{174}$$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|\uparrow\rangle + b|\downarrow\rangle\,, \quad \langle\uparrow|\psi\rangle = a, \quad \langle\downarrow|\psi\rangle = b \tag{175}$$

- **Born rule**: measurement of $O \to$ one of $\{\lambda_i\}$ is obtained with **probability**

$$\Pr(i) = |C_i|^2 = |\langle i|\psi\rangle|^2 \tag{176}$$

and the state collapses to $|i\rangle$

$$|\psi\rangle \to |i\rangle = \frac{P_i|\psi\rangle}{\|P_i|\psi\rangle\|} \tag{177}$$

- Example: $O = s_z$, with CONS $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$P_\uparrow = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_\downarrow = |\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{178}$$

$$\Pr(\uparrow) = |\langle\uparrow|\psi\rangle|^2 \tag{179}$$

- **Expectation value** of O

$$\langle\psi|O|\psi\rangle = \langle\psi|\left(\sum_i \lambda_i |i\rangle\langle i|\right)|\psi\rangle = \sum_i \Pr(i)\lambda_i \tag{180}$$

Sum of (probability)×(outcome)

8.4 Density matrix

- **Density matrix** (density operator) ρ corresponding to a state $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi| \tag{181}$$

- **Trace** of an operator O: sum of diagonal matrix elements in any CONS $\{|i\rangle\}$

$$\mathrm{Tr}\,[O] = \sum_i \langle i|O|i\rangle = \sum_i \lambda_i \tag{182}$$

The result is independent of the choice of CONS

- Normalization of the state

$$\mathrm{Tr}\left[\rho\right] = \sum_i \langle i|\psi\rangle \langle\psi|i\rangle = \langle\psi|\psi\rangle = 1 \tag{183}$$

- Probability

$$\Pr(i) = \mathrm{Tr}\left[P_i \rho\right] \tag{184}$$

- Expectation value

$$\langle\psi| O |\psi\rangle = \mathrm{Tr}\left[O\rho\right] \tag{185}$$

- Example: expectation values of Pauli matrices for the Bloch-sphere state $\rho = |\theta,\phi\rangle \langle\theta,\phi|$

$$\mathrm{Tr}\left[\sigma_x \rho\right] = \sin\theta\cos\phi \tag{186}$$

$$\mathrm{Tr}\left[\sigma_y \rho\right] = \sin\theta\sin\phi \tag{187}$$

$$\mathrm{Tr}\left[\sigma_z \rho\right] = \cos\theta \tag{188}$$

These correspond to the Cartesian coordinates of a point on the unit sphere