9 Two-spin system and entanglement entropy

9.1 Representation of two-spin states

- Composite system of spin A and spin B: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
  (in quantum information, a two-qubit system of Alice and Bob)

- **Tensor-product states**

$$|\uparrow\uparrow\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle\,, \quad |\uparrow\downarrow\rangle\,, \quad |\downarrow\uparrow\rangle\,, \quad |\downarrow\downarrow\rangle \tag{189}$$

- Spin operators for each subsystem

$$\boldsymbol{s}_A = \underbrace{\boldsymbol{s}}_{\text{acts on } A} \otimes \underbrace{1}_{\text{acts on } B}\,, \quad \boldsymbol{s}_B = 1 \otimes \boldsymbol{s} \tag{190}$$

- Action of operators

$$(s_A)_z\,|\uparrow\uparrow\rangle = \frac{1}{2}\,|\uparrow\uparrow\rangle\,, \quad \ldots \tag{191}$$

- Total spin operator $\boldsymbol{S}$

$$\boldsymbol{S} = \boldsymbol{s}_A + \boldsymbol{s}_B \tag{192}$$

- Eigenstates of $\boldsymbol{S}$: labeled by total spin $S = 0, 1$ and z-component S_z

$$|S, S_z\rangle = \underbrace{|0,0\rangle}_{S=0},\quad \underbrace{|1,1\rangle\,, \quad |1,0\rangle\,, \quad |1,-1\rangle}_{S=1} \tag{193}$$

  Action of spin operators

$$S_z\,|1,-1\rangle = -\,|1,-1\rangle\,, \quad \boldsymbol{S}^2\,|1,-1\rangle = S(S+1)\,|1,-1\rangle = 2\,|1,-1\rangle\,, \quad \ldots \tag{194}$$

- Kronecker product of states: representation as 4-component vectors

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 \times \begin{pmatrix}1\\0\end{pmatrix}\\ 0 \times \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \tag{195}$$

$$|\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{196}$$

- Kronecker product of operators: 4×4 matrices

$$
\begin{aligned}
(s_A)_x = s_x \otimes 1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{197}
$$

$$
\begin{aligned}
s_x \otimes s_x &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1 \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1 \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{198}
$$

- Density matrix of a tensor-product state: $\rho = \rho_a \otimes \rho_b$ holds

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \underbrace{|\uparrow\rangle\langle\uparrow|}_{\rho_a} \otimes \underbrace{|\uparrow\rangle\langle\uparrow|}_{\rho_b} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{199}
$$

9.2 Complete orthonormal systems

- **Computational basis**: $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$
- **Bell basis**: $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$

$$
|\Phi^\pm\rangle = \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}} \quad \text{(not eigenstates of the total spin } \boldsymbol{S}\text{)} \tag{200}
$$

$$
|\Psi^\pm\rangle = \frac{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}} \quad \text{(states with } S_z = 0\text{, corresponding to } S = 1 \text{ and } S = 0\text{)} \tag{201}
$$

- General two-spin state (vector in $\mathbb{C}^4$)

$$
|\Psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \tag{202}
$$

Normalization

$$
|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \tag{203}
$$

- General state $|\Psi\rangle$: using CONS $\{|i_A\rangle\}$ of $\mathcal{H}_A$ and $\{|j_B\rangle\}$ of $\mathcal{H}_B$

$$|\Psi\rangle = \sum_{i,j} \alpha_{ij} \, |i_A\rangle \otimes |j_B\rangle \quad \text{(sum over } i \text{ and } j\text{)} \tag{204}$$

- **Schmidt decomposition**: there exist CONS $\{|i'_A\rangle\}$ and $\{|i'_B\rangle\}$ such that[5]

$$|\Psi\rangle = \sum_{i=1}^{l} \sqrt{p_i} \, |i'_A\rangle \otimes |i'_B\rangle \quad \text{(sum over a common } i\text{)}, \quad \sum_{i=1}^{l} p_i = 1 \tag{205}$$

  $p_i > 0$: Schmidt coefficients, l: Schmidt rank
  Shown by the singular value decomposition of α_{ij} (also valid for $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$)

9.3 Entropy of quantum systems

- **von Neumann entropy** by the density matrix ρ

$$S(\rho) = -\mathrm{Tr}\,[\rho \log \rho] \tag{206}$$

  Expanding $\rho = \sum_i \lambda_i \, |i\rangle \langle i|$ in its eigenbasis $\{|i\rangle\}$,

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{207}$$

  Shannon entropy of the eigenvalue distribution λ_i

- Example: canonical ensemble $\rho = e^{-\beta H}/Z$

$$\begin{aligned} S(\rho) &= -\mathrm{Tr}\left[\rho \log(e^{-\beta H}/Z)\right] \\ &= -\mathrm{Tr}\,[\rho(-\beta H - \log Z)] \\ &= \beta \mathrm{Tr}\,[\rho H] + \log Z \ \mathrm{Tr}\,[\rho] \\ &= \beta E + \log Z \\ &= \beta E - \beta F \\ F &= E - TS \end{aligned} \tag{208}$$

- **Entanglement (von Neumann) entropy**: quantifies correlations between A and B (see Sec. 10 for details)

$$\mathcal{E}_{\mathrm{vN}} = -\mathrm{Tr}_A\,[\rho_A \log \rho_A] \tag{209}$$

[5]Strictly speaking, these form an orthonormal set (ONS) in the full space; zero singular values are omitted.

- **Reduced density matrix**: for the general state in Eq. (202),

$$\rho_A = \mathrm{Tr}_B\left[\rho\right] = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\gamma^* + \beta\delta^* \\ \alpha^*\gamma + \beta^*\delta & |\gamma|^2 + |\delta|^2 \end{pmatrix} \tag{210}$$

Partial trace over spin B: gives the density matrix in $\mathcal{H}_A$

$$\mathrm{Tr}_B\left[\rho\right] = \sum_i (1 \otimes \langle i_B|)\ \rho\ (1 \otimes |i_B\rangle) \tag{211}$$

If $\rho = \rho_a \otimes \rho_b$ (product state),

$$\mathrm{Tr}_B\left[\rho_a \otimes \rho_b\right] = \rho_A \mathrm{Tr}_B\left[\rho_b\right] \tag{212}$$

- Using the Schmidt decomposition,

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{i,j} \sqrt{p_i p_j}\, |i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B| \tag{213}$$

$$\begin{aligned} \mathrm{Tr}_B\left[\rho\right] &= \sum_{i,j,k} \sqrt{p_i p_j}\, |i_A\rangle\langle j_A| \otimes \langle k_B|i_B\rangle\langle j_B|k_B\rangle = \sum_{i,j} \sqrt{p_i p_j}\, |i_A\rangle\langle j_A|\, \delta_{ij} \\ &= \sum_i p_i\, |i_A\rangle\langle i_A| \end{aligned} \tag{214}$$

Since $\mathrm{Tr}_A\left[\rho\right]$ has the same eigenvalues p_i,

$$\mathcal{E}_{\mathrm{vN}} = -\mathrm{Tr}_A\left[\rho_A \log \rho_A\right] = -\mathrm{Tr}_B\left[\rho_B \log \rho_B\right] \tag{215}$$

9.4 Measures of quantum correlations

- **Linear entropy** ($\log \rho_A = \log[1 + (\rho_A - 1)] = \rho_A - 1 + \cdots$)

$$\mathcal{E} = 1 - \mathrm{Tr}_A\left[\rho_A^2\right] \tag{216}$$

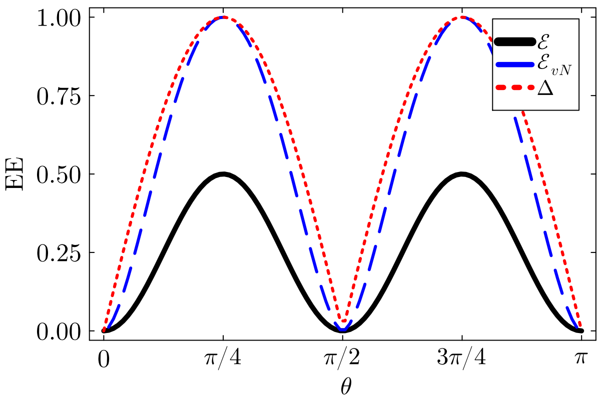
- Concurrence [51, 52]

$$\Delta = |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle| = 2|\alpha\delta - \beta\gamma| \tag{217}$$

Any entanglement measure defined from ρ_A can be expressed in terms of Δ:

$$\mathcal{E}_{\mathrm{vN}} = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-, \quad \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 - \Delta^2}), \tag{218}$$

$$\mathcal{E} = \frac{1}{2}\Delta^2 = 2|\alpha\delta - \beta\gamma|^2 \tag{219}$$



θ	0	$\pi/4$	$\pi/2$	$3\pi/4$
$\|\psi\rangle$	$\|\uparrow\downarrow\rangle$	$\|\Psi^+\rangle$	$\|\downarrow\uparrow\rangle$	$-\|\Psi^-\rangle$
$\mathcal{E}$	0	1/2	0	1/2
$\mathcal{E}_{\mathrm{vN}}$	0	1	0	1
Δ	0	1	0	1

Figure 20: Linear entropy $\mathcal{E}$, von Neumann entropy $\mathcal{E}_{\mathrm{vN}}$, and concurrence Δ for the state (228).

- Entanglement of a **product state** $|\uparrow\downarrow\rangle$

$$\beta = 1, \quad \alpha = \gamma = \delta = 0, \tag{220}$$
$$\Delta = 0, \tag{221}$$
$$\mathcal{E} = 0, \tag{222}$$
$$\lambda_+ = 1, \quad \lambda_- = 0 \quad \Rightarrow \quad \mathcal{E}_{\mathrm{vN}} = 0 \tag{223}$$

All measures vanish: 0

- Entanglement of a **Bell state** $|\Psi^+\rangle$

$$\alpha = \delta = 0, \quad \beta = \gamma = \frac{1}{\sqrt{2}}, \tag{224}$$
$$\Delta = 1, \tag{225}$$
$$\mathcal{E} = \frac{1}{2}, \tag{226}$$
$$\lambda_+ = \frac{1}{2}, \quad \lambda_- = \frac{1}{2} \quad \Rightarrow \quad \mathcal{E}_{\mathrm{vN}} = \log 2 = 1 \tag{227}$$

1 ebit (entanglement bit), the fundamental unit of entanglement[6]

- General superposition state

$$|\psi\rangle = \cos\theta\,|\uparrow\downarrow\rangle + \sin\theta\,|\downarrow\uparrow\rangle \tag{228}$$

Fig. 20: comparison of $\mathcal{E}$, $\mathcal{E}_{\mathrm{vN}}$, and Δ
All measures vanish for product states ($\theta = 0, \pi/2$), while they become **maximal** for Bell states ($\theta = \pi/4, 3\pi/4$)

[6] In quantum information, the logarithm is taken with base 2.