Theory of resonances and its application to hadron physics

Tetsuo Hyodo

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Abstract

Lecture note for 2022 YITP molecule workshop

Contents

1 Introduction

Resonance states

- Quasi-stable "state" which decays quantum mechanically
- History
  - 1928 : Complex eiegnenergy (Gamow [1])
  - 1939 : Outgoing boundary condition (Siegert [2])
  - 1965-1968 : Orthogonality, completeness, ... (Hokkyo [3], Berggren [4])
  - 1981 : Rigged Hilbert space (Bohm [5])

Resonances in hadron physics

- Hadrons $(p, n, \pi, \cdots)$: more than 380 species
- Most of hadrons are unstable against strong decay (light π as the NG boson).
- Structure of exotic hadrons (multiquarks, hadronic molecules, ...)
  $\leftrightarrow$ Structure of **unstable states**

Resonances in other fields

- Nuclear physics (unstable nuclei, halo structure, ...)
- Atomic physics (Feshbach resonance, ...)
- Astrophysics (black hole quasinormal mode, pole skipping, ...)

Plan of this lecture

- Basics
  Scattering wave functions, Jost functions, discrete eigenstates, poles of scattering amplitude, ...
- Bound-to-resonance transition
  Short range interaction [6], Coulomb plus short range interaction? three-body resonances? ...

References

- Resonances in quantum mechanics
  Textbook : A. Bohm [7], Kukulin-Krasnopol'sky-Horacek [8], N. Moiseyev [9]
  Review article : Ashida-Gong-Ueda [10]
- Scattering theory
  Textbook : J.R. Taylor [11], R.G. Newton [12]
  Review article : Hyodo-Niiyama [13]

2 Basics

2.1 Scattering wave functions

- Setup :
  - Quantum scattering of distinguishable particles with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$
  - Spatial 3D, nonrelativistic, $\hbar = 1$
  - No internal degrees of freedom (spin, flavor, etc.)
  - Elastic scattering (initial state = final state, no coupled channels)
  - Spherical real potential $V(r) \in \mathbb{R}$
  - $V(r) \to 0$ for $r \to \infty$ (not confining)

- Schrödinger equation

$$\left(-\frac{\boldsymbol{\nabla}^2}{2\mu} + V(r)\right)\psi_{\ell,m}(\boldsymbol{r}) = E\psi_{\ell,m}(\boldsymbol{r}) \tag{1}$$

$$\psi_{\ell,m}(\boldsymbol{r}) = \frac{u_\ell(r)}{r} Y_\ell^m(\hat{\boldsymbol{r}}) \tag{2}$$

- Radial Schrödinger equation

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r)\right) u_\ell(r) = E u_\ell(r), \quad 0 \le r < \infty \tag{3}$$

- Boundary condition at the origin

$$u_\ell(r) \sim r^{\ell+1} \quad (r \to 0) \tag{4}$$

  for $V(r) \sim r^{-2+\epsilon} (\epsilon > 0)$ at $r \to 0$
  $\Rightarrow u_\ell$ vanishes at the origin.

- Scattering solutions : no boundary condition at $r \to \infty$
  $\Rightarrow$ continuous spectrum for $E > 0$
  $\Rightarrow$ Normalization is not unique.

- For $E > 0$, the eigenmomentum $p > 0$ (physical region) is uniquely given by

$$p = \sqrt{2\mu E} \tag{5}$$

  $u_\ell(r;p)$: wave function at momentum p

- $r \to \infty$ asymptotic behavior for short range potentials

$$u_\ell(r;p) \to A\hat{j}_\ell(pr) + B\hat{n}_\ell(pr) = C\hat{h}_\ell^+(pr) + D\hat{h}_\ell^-(pr) \tag{6}$$

  $\hat{j}_\ell(z) = zj_\ell(z)$: Riccati-Bessel function ($\hat{j}_0(z) = \sin z$)
  $\hat{n}_\ell(z) = zn_\ell(z)$: Riccati-Neumann function ($\hat{n}_0(z) = \cos z$, no sign in Ref. [11])
  $\hat{h}_\ell^\pm(z) = \hat{n}_\ell(z) \pm i\hat{j}_\ell(z)$: Riccati-Hankel functions

$$\hat{h}_\ell^\pm(z) \to \exp\{\pm i(z - \ell\pi/2)\} \quad (z \to \infty) \tag{7}$$

  $\hat{h}_\ell^+(pr) \sim e^{+ipr}$ [$\hat{h}_\ell^-(pr) \sim e^{-ipr}$] is outgoing (incoming) wave.

- Partial wave scattering amplitude $f_\ell(p)$ and S-matrix $s_\ell(p)$

$$u_\ell(r;p) \to \hat{j}_\ell(pr) + p f_\ell(p) \hat{h}^+_\ell(pr) \quad (r \to \infty) \tag{8}$$

$$\leftarrow \quad \psi(\boldsymbol{r}) \to e^{i\boldsymbol{p}\cdot\boldsymbol{r}} + f(p,\theta)\frac{e^{ipr}}{r} \tag{9}$$

$$u_\ell(r;p) \to \hat{h}^-_\ell(pr) - s_\ell(p) \hat{h}^+_\ell(pr) \quad (r \to \infty) \tag{10}$$

From this, we have

$$f_\ell(p) = \frac{s_\ell(p) - 1}{2ip} \tag{11}$$

- Regular solution $u_{\ell,p}(r)$ (normalization is fixed in addition to b.c. (4).)

$$u_\ell(r;p) \to \hat{j}_\ell(pr) \quad (r \to 0) \tag{12}$$

Integral form (Lippmann-Schwinger eq.)

$$u_\ell(r;p) = \hat{j}_\ell(pr) + 2\mu \int_0^r dr' g_\ell(r,r';p) V(r') u_\ell(r';p) \tag{13}$$

$$g_\ell(r,r';p) = \frac{\hat{j}_\ell(pr)\hat{n}_\ell(pr') - \hat{n}_\ell(pr)\hat{j}_\ell(pr')}{p} \tag{14}$$

Regular solution $u_\ell(r;p)$ is real (for $p > 0$)

- Asymptotic form of $u_\ell(r;p)$

$$u_\ell(r;p) \to \frac{i}{2}\left[\mathscr{F}_\ell(p)\hat{h}^-_\ell(pr) - [\mathscr{F}_\ell(p)]^* \hat{h}^+_\ell(pr)\right] \quad (r \to \infty) \tag{15}$$

- **Jost function** $\mathscr{F}_\ell(p) \in \mathbb{C}$: amplitude of incoming wave

2.2 Properties of the Jost functions

- Integral representation $\leftarrow$ Eqs. (13) and (15)

$$\mathscr{F}_\ell(p) = 1 + \frac{2\mu}{p}\int_0^\infty dr\ \hat{h}^+_\ell(pr) V(r) u_\ell(r;p) \tag{16}$$

- Small $p > 0$ expansion

$$\hat{j}_\ell(pr) \sim u_\ell(r;p) \sim p^{\ell+1}, \quad \hat{n}_\ell(pr) \sim p^{-\ell} \tag{17}$$

therefore

$$\mathscr{F}_\ell(p) = 1 + 2\mu \int_0^\infty dr\ \frac{\hat{n}(pr) + i\hat{j}(pr)}{p} V(r) u_\ell(r;p) \tag{18}$$

$$= 1 + \underbrace{[\alpha_\ell + \beta_\ell p^2 + \mathcal{O}(p^4)]}_{\text{even powers of } p} + i\underbrace{[\gamma_\ell p^{2\ell+1} + \mathcal{O}(p^{2\ell+3})]}_{\text{odd powers of } p}, \quad \alpha_\ell, \beta_\ell, \gamma_\ell, \cdots \in \mathbb{R} \tag{19}$$

- Region of analyticity

$$|\mathscr{F}_\ell(p) - 1| \le \frac{\text{const}}{|p|}\int_0^\infty dr\ |V(r)| \frac{|pr|}{1 + |pr|} e^{(|\text{Im } p| - \text{Im } p)r} \tag{20}$$

$V(r)$ with compact support : $\mathscr{F}_\ell(p)$ is analytic for all $p \in \mathbb{C}$.
$V(r) = e^{-mr}/r$: $\mathscr{F}_\ell(p)$ has a branch cut from $p = -im/2$ to $-i\infty$.
In the following we consider the region of p on which Jost functions are analytic.

- Complex conjugate of Jost function

$$\left[\mathscr{f}_\ell(p)\right]^* = \mathscr{f}_\ell(-p^*) \quad p \in \mathbb{C} \tag{21}$$

Schwarz reflection principle with $\mathscr{f}_\ell(p)$ being real on the imaginary p axis
$\to$ reflection symmetry with respect to the imaginary p axis

- For real p:

$$u_\ell(r;p) \to \frac{i}{2}\left[\mathscr{f}_\ell(p)\hat{h}_\ell^-(pr) - \mathscr{f}_\ell(-p)\hat{h}_\ell^+(pr)\right] \quad (r \to \infty) \tag{22}$$

- Explicit form for attractive square well potential with $\ell = 0$ ($V(r) = -V_0$ for $0 \le r \le b$)

$$\mathscr{f}_0(p) = \left[\cos(\sqrt{p^2 + 2\mu V_0}b) - i\frac{p}{\sqrt{p^2 + 2\mu V_0}}\sin(\sqrt{p^2 + 2\mu V_0}b)\right]e^{ipb} \tag{23}$$

Entirely analytic (polynomial in p, no pole)

2.3 Discrete eigenstates

- Bound state solution : eigenenrgy $E < 0 \Leftrightarrow$ pure imaginary eigenmomentum $p = \sqrt{2\mu E}$

$$p = i\kappa, \quad \kappa > 0 \tag{24}$$

Wave function at $r \to \infty$ behaves as

$$u_\ell(r;p) \propto \mathscr{f}_\ell(i\kappa)e^{+\kappa r} - \mathscr{f}_\ell(-i\kappa)e^{-\kappa r} \quad (r \to \infty) \tag{25}$$

- Boundary condition : $u_\ell(r;p)$ is square integrable $\to$ eliminate diverging component $e^{+\kappa r}$

$$\mathscr{f}_\ell(i\kappa) = 0 \tag{26}$$

Eq. (23) $\Rightarrow \kappa = -k\cot(kb)$ with $k = \sqrt{2\mu V_0 - \kappa^2}$.

- **Outgoing boundary condition** : zero of the Jost function

$$\mathscr{f}_\ell(p) = 0 \tag{27}$$

Incoming wave (e^{-ipr}) vanishes, leaving outgoing wave (e^{+ipr}) only
Bound state solution is obtained by analytic continuation of (27).

- Resonance states : solution of Eq. (27) with **complex** p

- Attractive square well potential have infinitely many resonance solutions
Eq. (27) with $V_0 = 10b^{-2}\mu^{-1}$
Fig. 1 : Poles of $1/|\mathscr{f}_\ell(p)|$ in complex p plane
Table 1 : numerical solutions

- Imaginary part of eigenmomentum is negative :

$$p = p_R - ip_I, \quad p_R, p_I > 0 \tag{28}$$

behavior of wave function

$$u_\ell(r;p) \propto \mathscr{f}_\ell(-p)e^{ipr} \propto \underbrace{e^{ip_R r}}_{\text{oscillation}}\underbrace{e^{+p_I r}}_{\text{increasing}} \tag{29}$$

$u_\ell(r;p)$ diverges with oscillation for $r \to \infty$, not square integrable

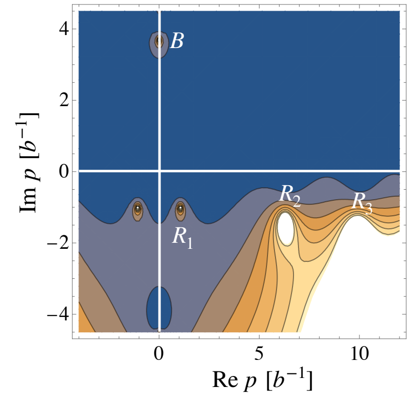

Figure 1: Contour plot of inverse of the Jost function of square well potential with $V_0 = 10b^{-2}\mu^{-1}$.

Table 1: Discrete eigenstates of attractive square well potential with $V_0 = 10b^{-2}\mu^{-1}$.

	p $[b^{-1}]$	$E = p^2/2\mu$ $[b^{-2}\mu^{-1}]$
Bound state B	$+3.68i$	-6.78
1st resonance R_1	$1.06 - 1.02i$	$0.05 - 1.08i$
2nd resonance R_2	$6.29 - 1.41i$	$18.8 - 8.86i$
3rd resonance R_3	$9.90 - 1.69i$	$47.6 - 16.8i$
$\vdots$		

- S matrix : Eqs. (22) and (10)

$$s_\ell(p) = \frac{\not{f}_\ell(-p)}{\not{f}_\ell(p)} \tag{30}$$

$\Rightarrow$ discrete eigenstates are represented by **poles of S matrix**

- Scattering amplitude : from Eq. (11)

$$f_\ell(p) = \frac{s_\ell(p) - 1}{2ip} = \frac{\not{f}_\ell(-p) - \not{f}_\ell(p)}{2ip\not{f}_\ell(p)} \tag{31}$$

$\Rightarrow$ discrete eigenstates are represented by **poles of scattering amplitude**
(p in the denominator cancels with $\not{f}_\ell(-p) - \not{f}_\ell(p) \sim \mathcal{O}(p)$)

- $s_\ell(p)$ and $f_\ell(p)$ are meromorphic functions of p ($\sim$ no singularity except for poles)

2.4 Classification of eigenstates

- Analytic continuation of $\not{f}_\ell(p)$, $s_\ell(p)$, $f_\ell(p)$ defined in physical region $p > 0$ to complex plane
- Complex momentum p, complex energy E

$$p = |p|e^{i\theta_p}, \quad E = |E|e^{i\theta_E} \tag{32}$$

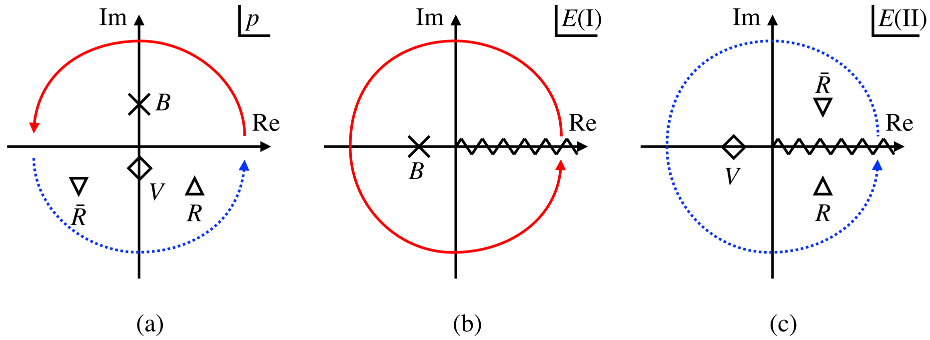

Figure 2: Poles in complex plane. (a) : p plane, (b) : E plane (1st Riemann sheet), (c) : E plane (2nd Riemann sheet). B, V, R, and $\bar{R}$ represent bound state, virtual state, resonance, and anti-resonance.

- Relations

$$E = \frac{p^2}{2\mu} = \frac{|p|^2}{2\mu} e^{2i\theta_p} \tag{33}$$

$$\Rightarrow \quad |E| = \frac{|p|^2}{2\mu}, \quad 2\theta_p = \theta_E \tag{34}$$

  - When θ_p varies $0 \to 2\pi$, θ_E moves $0 \to 4\pi$
  - p and $-p$ (θ_p and $\theta_p + \pi$) are mapped onto the same E

- Meromorphic functions of p ($s_\ell(p)$, $f_\ell(p)$) are defined on two-sheeted Riemann surface of E
  $0 \leq \theta_E < 2\pi$: 1st Riemann sheet of E (upper half plane of p, $0 \leq \theta_p < \pi$)
  $2\pi \leq \theta_E < 4\pi$: 2nd Riemann sheet of E (lower half plane of p, $\pi \leq \theta_p < 2\pi$)

- Complex p and E planes : Fig. 2
  Cut on real axis of E plane (branch point at $E = 0$)

- Eigenstate of Hamiltonian : zero of Jost function $\not{f}_\ell(p) = 0$

- From Eq. (21), when $\not{f}_\ell(p) = 0$,

$$\not{f}_\ell(-p^*) = [\not{f}_\ell(p)]^* = 0 \tag{35}$$

  $\Rightarrow$ If p is a solution, $-p^*$ (point which is symmetric about imaginary axis) is also a solution

- Solutions with $p = -p^*$ (on imaginary axis)

  - bound state (B) : $\times$ in Fig. 2

$$\mathrm{Re}\,[p_B] = 0, \quad \mathrm{Im}\,[p_B] > 0 \tag{36}$$

    Energy E_B is real and negative (1st Riemann sheet)

  - Virtual state (anti-bound state, V) : $\diamondsuit$

$$\mathrm{Re}\,[p_V] = 0, \quad \mathrm{Im}\,[p_V] < 0 \tag{37}$$

    Energy E_V is real and negative (2nd Riemann sheet)
    Residue of pole ($\sim$ norm) is negative [14] : non-physical degree of freedom?

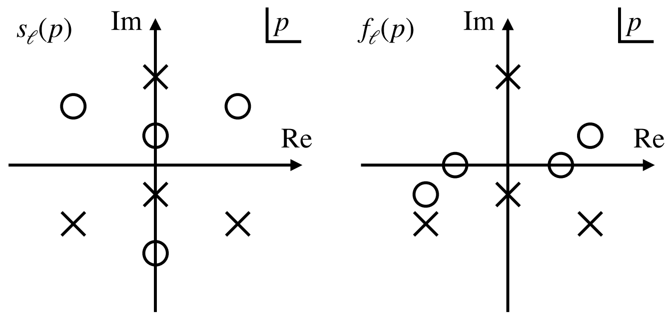

Figure 3: Poles and zeros in the complex p plane. Left : poles (crosses) and zeros (circles) of the S matrix $s_\ell(p)$. Right : poles (crosses) and zeros (circles) of the scattering amplitude $f_\ell(p)$.

- Solutions with $p \neq -p^*$ (always appear in pairs)
  - Solutions exist only in lower half plane of p
    $\leftarrow$ complex E is allowed only for non-square-integrable wave function
  - Resonance (R) : $\triangle$

$$\text{Re}\,[p_R] > 0, \quad \text{Im}\,[p_R] < 0 \tag{38}$$

    Energy Re $[E_R] > 0$, Im $[E_R] < 0$ (2nd Riemann sheet)
  - Anti-resonance ($\bar{R}$) : $\triangledown$

$$\text{Re}\,[p_{\bar{R}}] < 0, \quad \text{Im}\,[p_{\bar{R}}] < 0 \tag{39}$$

    appears together with resonance
    Growing solution with time [15] ("conjugate" of resonance)

2.5 Poles and zeros

- Motivation : pole trajectory with respect to a potential parameter λ
- Meromorphic function $\mathcal{F}(p)$ can be expressed by poles $\{p_j\}$ and zeros $\{z_i\}$.

$$\mathcal{F}(p) = \frac{(\text{polynomial in } p)}{(\text{polynomial in } p)} = \frac{\prod_i (p - z_i)}{\prod_j (p - p_j)} \tag{40}$$

  (assume that $z_i(\lambda)$ and $p_i(\lambda)$ are continuous in λ)

- Poles of $s_\ell(p)$ and $f_\ell(p)$ are the same [zero of $\not{f}_\ell(p)$] but zeros are different.
- Zeros of S matrix (Fig. 3, left)

$$s_\ell(z_i) = \frac{\not{f}_\ell(-z_i)}{\not{f}_\ell(z_i)} = 0 \quad \Leftrightarrow \quad \not{f}_\ell(-z_i) = 0 \quad \Leftrightarrow \quad z_i = -p_i \tag{41}$$

  z_i does not appear in the physical region $p > 0 \leftarrow |s_\ell(p)| = 1$ for $p > 0$

- Zeros of scattering amplitude : (Castillejo-Dalitz-Dyson) **CDD zero** [16]

$$f_\ell(z_i^{\rm CDD}) = 0 \quad \Leftrightarrow \quad \not{f}_\ell(-z_i^{\rm CDD}) = \not{f}_\ell(z_i^{\rm CDD}) \quad \Leftrightarrow \quad f_\ell(-z_i^{\rm CDD}) = 0 \tag{42}$$

$z_i^{\rm CDD}$ appears symmetric with $p = 0$, not correlated with p_i
Physical interpretation

$$f_\ell(z_i^{\rm CDD}) = 0 \quad \Leftrightarrow \quad s_\ell(z_i^{\rm CDD}) = 1 \text{ (no scattering)} \tag{43}$$

Realizable for physical scattering $p > 0$: Ramsauer-Townsend effect

- Argument principle (assuming simple zeros and poles)

$$\frac{1}{2\pi}\oint_C dp \frac{d \arg \mathcal{F}(p)}{dp} = n_Z - n_P \equiv n_C \tag{44}$$

$$\begin{aligned} C &: \text{counterclockwisely closed contour} \\ n_Z &: \text{number of zeros in } C \\ n_P &: \text{number of poles in } C \end{aligned}$$

- n_C is stable against the continuous variation of parameters
$\leftarrow$ Topological invariant $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$ (vortices/antivortices in 2d XY model)

- A pole can disappear only when it encounter a zero (Fig. 4).

$$(n_Z, n_P) = (1, 0) \quad \not\leftrightarrow \quad (n_Z, n_P) = (0, 0) \tag{45}$$

$$(n_Z, n_P) = (1, 1) \quad \leftrightarrow \quad (n_Z, n_P) = (0, 0) \tag{46}$$

- When $z_i \to p_i$:

$$\mathcal{F}(p) \sim \frac{(p - z_i)}{(p - p_j)} \tag{47}$$

$\to$ pole skipping (indefinite of 0/0) [17]

- Example : s-wave S matrix in the zero range limit

$$s_0(p; a) = \frac{-\frac{1}{a} + ip}{-\frac{1}{a} - ip} \tag{48}$$

$$\lim_{a\to\infty}\lim_{p\to 0} s_0(p; a) = \lim_{a\to\infty} \frac{-\frac{1}{a}}{-\frac{1}{a}} = 1 \tag{49}$$

$$\lim_{p\to 0}\lim_{a\to\infty} s_0(p; a) = \lim_{p\to 0} \frac{ip}{-ip} = -1 \tag{50}$$

zero energy limit / unitary limit do not commute

- $f_\ell(p)$ in the zero coupling limit (decoupling limit of bare state) [18]

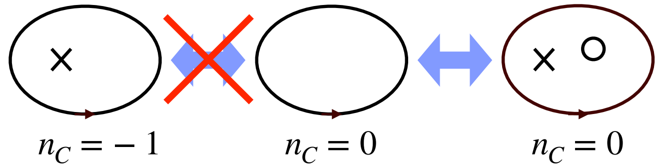

Figure 4: Poles and zeros in the complex p plane. Left : poles (crosses) and zeros (circles) of the S matrix $s_\ell(p)$. Right : poles (crosses) and zeros (circles) of the scattering amplitude $f_\ell(p)$.

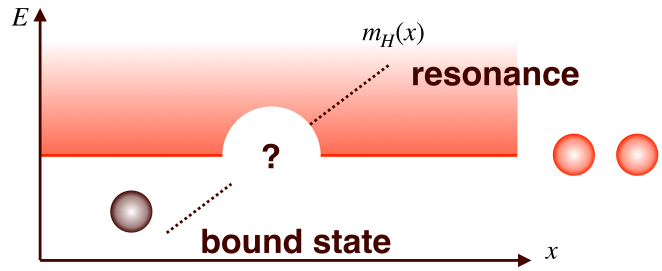

Figure 5: Schematic figure of bound-to-resonance transition.

3 From bound states to resonances

3.1 Motivation in hadron physics

- Hadron mass scaling with respect to QCD parameters (Fig. 5)

$$m_H(x), \quad x = m_q,\ 1/N_c,\ T,\ \mu, \cdots \tag{51}$$

What happens when $m_H(x)$ crosses a threshold?

- Coulomb plus short range potential
  - Kaonic atoms (K^-+nuclei) : interplay between nuclear bound states and atomic bound states
  - $p\Omega^-$ system : shallow bound state, $B \sim 1.54$ MeV, $a_0 \sim 5.30$ fm [19]

3.2 Short range interaction

- Eigenstate at $p = 0$: $1 + \alpha_\ell = 0$ in Eq. (19)

$$\not{f}_\ell(p) = \begin{cases} i\gamma_0 p + \mathcal{O}(p^2) & \ell = 0 \\ \beta_l p^2 + \mathcal{O}(p^3) & \ell \neq 0 \end{cases}. \tag{52}$$

$\Rightarrow$ zero of $\not{f}_\ell(p)$ at $p = 0$ is simple (double) for $\ell = 0$ ($\ell \neq 0$)

- Double zero : exceptional point (degeneracy of two eigenstates)
- Perturbation around $p = 0$:

$$V \to (1 + \delta\lambda)V \tag{53}$$

$$\alpha_\ell(\delta\lambda),\ \beta_\ell(\delta\lambda),\ \gamma_\ell(\delta\lambda), \cdots \tag{54}$$

$$\alpha_\ell(0) = 0,\ \beta_\ell(0) \neq 0,\ \gamma_\ell(0) \neq 0 \tag{55}$$

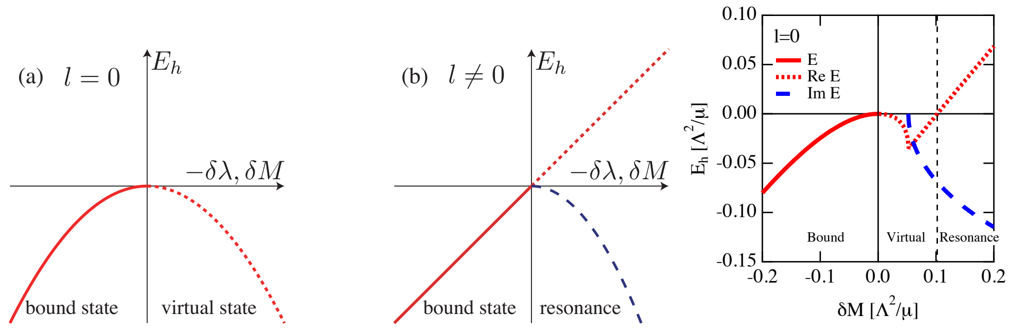

Figure 6: Schematic illustration of the near-threshold eigenenergy. Left : $\ell = 0$. Middle : $\ell \neq 0$. Right : $\ell = 0$ in larger δM region

Eigenmomentum

$$\begin{cases} p = i\frac{\alpha_0'(0)}{\gamma_0(0)}\delta\lambda & \ell = 0 \\ p^2 = -\frac{\alpha_\ell'(0)}{\beta_\ell(0)}\delta\lambda & \ell \neq 0 \end{cases} \tag{56}$$

Eigenenergy (stronger attraction)

$$E = \begin{cases} -F_0\delta\lambda^2 & \ell = 0 \\ -F_\ell\delta\lambda & \ell \neq 0 \end{cases} \quad \delta\lambda > 0 \tag{57}$$

$$F_0 = \frac{[\alpha_0'(0)]^2}{2\mu[\gamma_0(0)]^2}, \quad F_\ell = \frac{\alpha_\ell'(0)}{2\mu\beta_\ell(0)} \tag{58}$$

Eigenenergy (weaker attraction)

$$\begin{cases} E = -F_0\delta\lambda^2 & \ell = 0 \\ \text{Re } E = -F_\ell\delta\lambda & \ell \neq 0 \quad \delta\lambda < 0 \\ \text{Im } E \propto (\delta\lambda)^{\ell+1/2} & \end{cases} \tag{59}$$

s-wave : bound to virtual, other partial waves : bound to resonance
Schematic illustration (Fig. 6)

- Slope at $E = 0$: field renormalization constant [6]
- s-wave case with larger energy region : virtual state turns into resonance
- Effective range expansion

$$f_\ell(p) = \frac{p^{2\ell}}{-\frac{1}{a_\ell} + \frac{r_\ell}{2}p^2 + \cdots - ip^{2\ell+1}} \tag{60}$$

Pole condition up to p^2 [14]

$$\begin{cases} 0 = -\frac{1}{a_0} - ip + \frac{r_0}{2}p^2 & \ell = 0 \\ 0 = -\frac{1}{a_\ell} + \frac{r_\ell}{2}p^2 & \ell \neq 0 \end{cases} \tag{61}$$

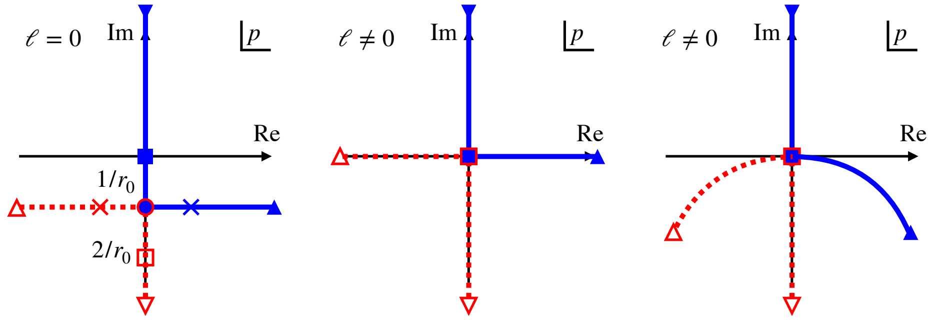

Figure 7: Pole trajectory in the effective range expansion with $r_\ell < 0$. Left : $\ell = 0$. Middle : $\ell \neq 0$. The right panel shows the result with $\ell \neq 0$ with the $-ip^{2\ell+1}$ term. Inverted triangles, squares, circles, crosses, and triangles correspond to $1/a_\ell = +\infty, 0, 2/r_\ell, 1/r_\ell, -\infty$, respectively. Solid (dashed) line with filled (empty) symbols stands for p^- (p^+).

Solution

$$p_\pm = \begin{cases} \frac{i}{r_0} \pm \frac{1}{r_0}\sqrt{\frac{2r_0}{a_0} - 1} & \ell = 0 \\ \pm\sqrt{\frac{2}{a_\ell r_\ell}} & \ell \neq 0 \end{cases} \tag{62}$$

Trajectory along with $1/a_\ell$ (r_ℓ fixed) : Fig. 7

3.3 Discussion

- Coulomb plus short range (strong) interaction
  - Strong interaction (MeV) scale : same with §3.2
  - EM (eV) scale : infinitely many Coulomb bound states
  - Coulomb plus square well [R.R. Lucchese, T. N. Rescigno, C.W. McCurdy, J. Phys. Chem. A 123, 82 (2019)]

$$S_\ell(p) = \frac{\Gamma(\ell + 1 + iZ/p)}{\Gamma(\ell + 1 - iZ/p)} S_\ell^{\rm SR}(p) \tag{63}$$

$$S_\ell^{\rm SR}(p) = \frac{D_\ell(-p)}{D_\ell(p)} \tag{64}$$

$$D_\ell(p) = ... \tag{65}$$

- Three-body resonances
  - Efimov state (s-wave three-body bound state) directly goes to resonance [20] (Fig. 8).
  - General classification?

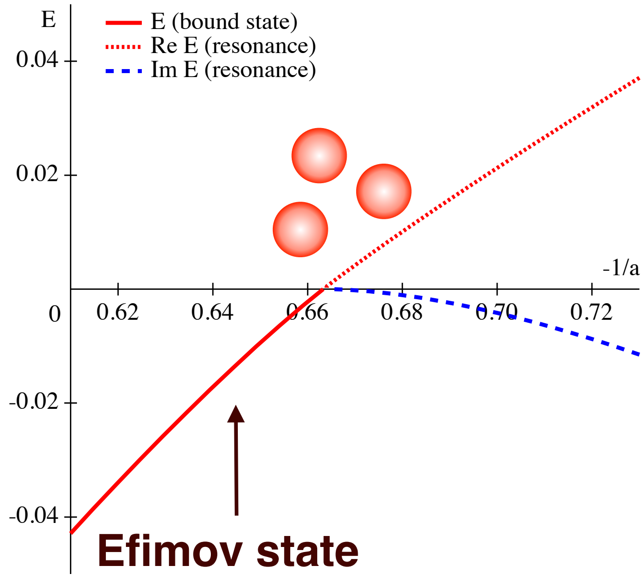

Figure 8: Bound-to-resonance transition of the Efimov state

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