(付録 A) 特殊関数の公式

今回の集中講義で (部分的に) 使用する特殊関数の公式を、証明なしに掲げる。

球面調和関数の定義:

$$Y_{lm}(\theta,\phi) = (-)^{(m+|m|)/2}\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_{lm}(\cos\theta)\,e^{im\phi}, \tag{1}$$

$$P_{lm}(w) = \left(1-w^2\right)^{|m|/2}\frac{d^{|m|}}{dw^{|m|}}P_l(w) \quad (\text{ルジャンドルの陪関数}), \tag{2}$$

$$P_l(w) = \frac{1}{(2l)!!}\frac{d^l}{dw^l}\left(w^2-1\right)^l, \quad (-1\le w\le l) \quad (\text{ルジャンドルの多項式}), \tag{3}$$

$$(2n)!! = 2n(2n-2)(2n-4)\,...2, \quad 0!!=1, \tag{4}$$

$$(2n+1)!! = (2n+1)(2n-1)(2n-3)\,...1. \tag{5}$$

球面調和関数の規格直交性:

$$\int Y^*_{lm}(\Omega)\,Y_{l'm'}(\Omega)\,d\Omega = \delta_{ll'}\delta_{mm'}. \tag{6}$$

球面調和関数の完全性:

$$\sum_{lm} Y^*_{lm}(\Omega')\,Y_{lm}(\Omega) = \delta(\Omega'-\Omega). \tag{7}$$

ルジャンドルの多項式の規格直交性:

$$\int_{-1}^{1} P_l(w)\,P_{l'}(w)\,dw = \frac{2}{2l+1}\delta_{ll'}. \tag{8}$$

ルジャンドルの多項式と球面調和関数の関係:

$$P_l(\cos\eta) = \frac{4\pi}{2l+1}\sum_m Y_{lm}(\Omega_{\boldsymbol{R}_1})\,Y^*_{lm}(\Omega_{\boldsymbol{R}_2}) = \frac{4\pi}{2l+1}\sum_m Y^*_{lm}(\Omega_{\boldsymbol{R}_1})\,Y_{lm}(\Omega_{\boldsymbol{R}_2}), \tag{9}$$

$$\cos\eta = \frac{\boldsymbol{R}_1\cdot\boldsymbol{R}_2}{R_1R_2}. \tag{10}$$

特殊な場合:

$$Y_{l0}(\Omega) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta). \tag{11}$$

$$Y_{lm}(0,\phi) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m0}. \tag{12}$$

球面調和関数の具体形 ($l\le 3$):

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \tag{13}$$

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi), \tag{14}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right), \quad Y_{2\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi), \tag{15}$$

$$Y_{2\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi), \tag{16}$$

$$Y_{30}(\theta,\phi) = \sqrt{\frac{7}{16\pi}}\cos\theta\left(5\cos^2\theta-3\cos\theta\right), \quad Y_{3\pm1}(\theta,\phi) = \mp\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)\exp(\pm i\phi), \tag{17}$$

$$Y_{3\pm2}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta\exp(\pm 2i\phi), \quad Y_{3\pm3}(\theta,\phi) = \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(\pm 3i\phi). \tag{18}$$

ルジャンドルの多項式の具体形 ($l \leq 3$):

$$P_0(w) = 1, \quad P_1(w) = w, \quad P_2(w) = \frac{3}{2}w^2 - \frac{1}{2}, \quad P_3(w) = \frac{5}{2}w^3 - \frac{3}{2}w. \tag{19}$$

球ベッセル関数の具体形 ($l \leq 3$):

$$\begin{aligned} j_0(z) &= \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \quad j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z, \\ j_3(z) &= \left(\frac{15}{z^4} - \frac{6}{z^2}\right)\sin z - \left(\frac{15}{z^3} - \frac{1}{z}\right)\cos z. \end{aligned} \tag{20}$$

球ノイマン関数の具体形 ($l \leq 3$):

$$\begin{aligned} n_0(z) &= -\frac{\cos z}{z}, \quad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}, \quad n_2(z) = -\left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z, \\ n_3(z) &= -\left(\frac{15}{z^4} - \frac{6}{z^2}\right)\cos z - \left(\frac{15}{z^3} - \frac{1}{z}\right)\sin z. \end{aligned} \tag{21}$$

球ハンケル関数:

$$h_l^{(+)}(z) = -n_l(z) + ij_l(z), \quad h_l^{(-)}(z) = -n_l(z) - ij_l(z). \tag{22}$$

球ベッセル・球ノイ マン・球ハンケル関数の無限遠における漸近形:

$$j_l(z) \to \frac{1}{z}\sin\left(z - \frac{l}{2}\pi\right), \quad n_l(z) \to -\frac{1}{z}\cos\left(z - \frac{l}{2}\pi\right), \quad h_l^{(\pm)}(z) \to \frac{1}{z}\exp\left[\pm i\left(z - \frac{l}{2}\pi\right)\right]. \tag{23}$$