Section V

Relativistic Kinematics

Relativistic Kinematics

General Equations

A particle with total energy E_i and 3-momentum $\mathbf{p}_i$ has a 4-momentum vector $p_i = (E_i, \mathbf{p}_i)$. Let the rest mass of the particle be m and let its velocity be $\mathbf{v}_i$. Then the following relations hold:

$$\beta_i = v_i / c \tag{1}$$

$$\gamma_i = \frac{1}{(1-\beta_i^2)^{1/2}} \tag{2}$$

If we adopt units where c = 1, then

$$E_i = m_i + T_i = \gamma_i m_i \tag{3}$$

$$p_i = \beta_i E_i = \beta_i \gamma_i m_i = (E_i^2 - m_i^2)^{1/2} \tag{4}$$

where T_i is the kinetic energy of the particle.

If we consider any general 4-vectors $A = (A_0, \mathbf{A})$ and $B = (B_0, \mathbf{B})$, the scalar product is defined by

$$A \cdot B = A_0 B_0 = \mathbf{A} \cdot \mathbf{B} \tag{5}$$

Lorentz Transformations

Consider a particle with 4-momentum $p = (E, \mathbf{p})$ viewed from a second frame with velocity β_0 relative to the original frame and $\gamma = (1 - \beta_0^2)^{-1/2}$. The components of p in the second frame are denoted by $p^* = (E^*, \mathbf{p}^*)$ where

$$E^* = \gamma_0 (E - \mathbf{p} \cdot \beta_0) \tag{6}$$

$$\mathbf{p}^* = \mathbf{p} + \beta_0 \gamma_0 \left[\frac{\gamma_0}{\gamma_0 + 1} \beta_0 \cdot \mathbf{p} - E \right]. \tag{7}$$

The special case where $\beta_0 = \beta_0 \mathbf{k}$, with $\mathbf{k}$ the unit vector along the z-direction, can be written

$$E^* = \gamma_o E - \beta_o \gamma_o E , \tag{8}$$

$$p_z^* = \gamma_o p_z - \beta_o \gamma_o E , \tag{9}$$

$$p_x^* = p_x \;\; ; \;\; p_y^* = p_y . \tag{10}$$

It follows that the scalar product of any two 4-momenta, $p_1 \cdot p_2 = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2$ is invariant, that is frame independent.

Consider the case where the second frame is the center of mass of two colliding particles with 4-momenta p_1 and p_2 and the second particle is at rest in the first frame. This corresponds to the conventional laboratory system. The above equations give the transformation from the lab to the center of mass when the velocity of the center of mass β_{cm} is substituted for β_0.

$$\beta_{cm} = p_1/(E_1 + m_2) \tag{12}$$

and

$$\gamma_{cm} = \frac{E_1 + m_2}{\left[(E_1 + m_2)^2 - p_1^2\right]^{1/2}} \tag{13}$$

The transformation from the center of mass to the lab is obtained by substituting $-\beta_{cm}$ in the appropriate equations.

Two Body Kinematics Formulae

Some useful 2-body kinematics formulae are summarized in the following table. The particles have laboratory 4-momenta given by

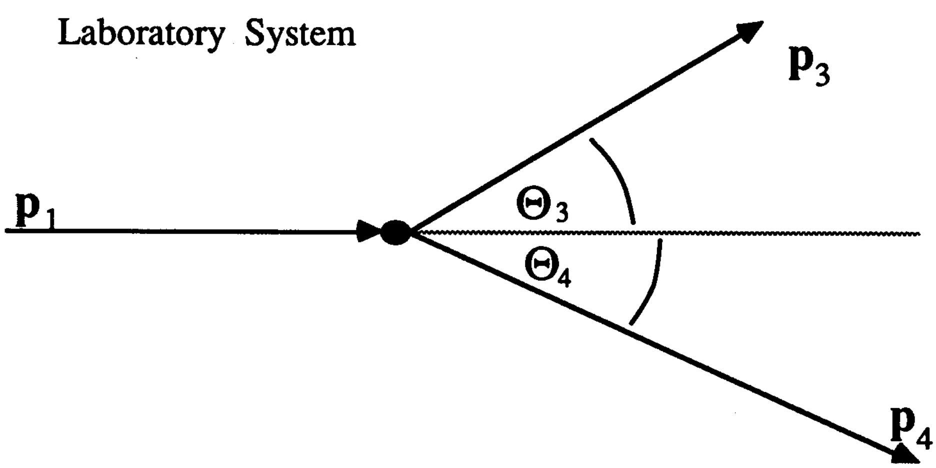
$p_1 = (E_1, \mathbf{p}_1)$ and $p_2 = (E_2, \mathbf{p}_2)$ for the initial state

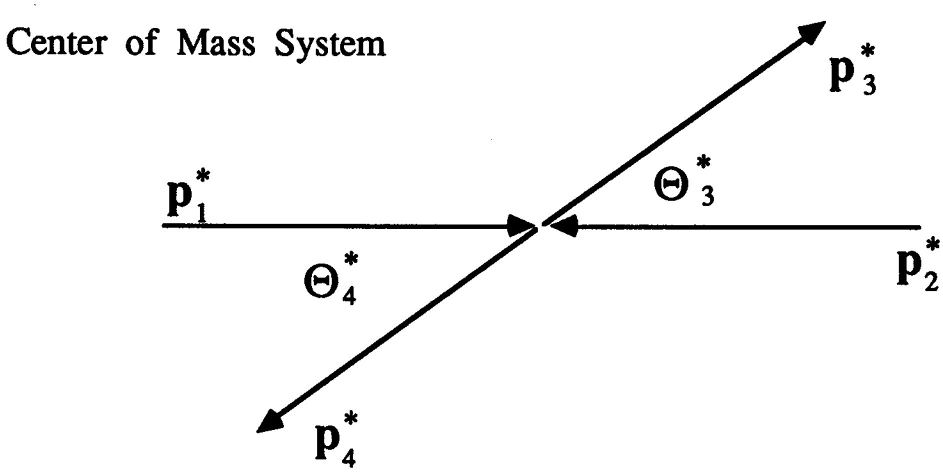
$p_3 = (E_3, \mathbf{p}_3)$ and $p_4 = (E_4, \mathbf{p}_4)$ for the final state.

The total energy in the center of mass is denoted by W. The relevant angle variables are shown in the figure below. In the table, the quantities δ_{ij} are defined by

$$\delta_{ij} = \beta_i / \beta_j, \tag{14}$$

where the subscripts refer to the particle. In the table, asterisked variables are in the center of mass, all others are in the lab frame.





Two Body Scattering Formulae

Quantity	General Formula	Elastic Scattering	N-N Scattering (equal mass)
1. Total c.m. energy	$W = [m_1^2 + m_2^2 + 2 m_2 E_1]^{1/2}$ $= [(E_1 + m_2)^2 - p_1^2]^{1/2}$ $= [(E_3 + E_4)^2 - (p_3 + p_4)^2]^{1/2}$	Same as the General formula	$W = [2 m_N E_1]^{1/2}$
2. c.m. momentum before the interaction	$p_1^* = \frac{1}{2W} \{[W^2 - (m_1 + m_2)^2] \cdot [W^2 - (m_1 - m_2)^2]\}^{1/2}$	"	$p_1' = \frac{1}{2}[W^2 - 4 m_N^2]^{1/2}$
3. c.m momentum after the interaction	$p_3^* = \frac{1}{2W} \{[W^2 - (m_3 + m_4)^2] \cdot [W^2 - (m_3 - m_4)^2]\}^{1/2}$	$p_3^* = p_1^*$	$p_3' = p_1' = \frac{1}{2}[W^2 - 4 m_N^2]^{1/2}$
4. velocity of the c.m.	$\beta^* = p_1 / (E_1 + m_2)$	Same as the General Formula	Same as the General Formula
5. γ of the c.m.	$\gamma^* = (E_1 + m_2) / W$	"	"
6. Maximum lab scattering angle	$\tan\theta_{3max} = \gamma_2^* \left[\delta_{23}^{*} - 1 \right]^{-1/2}$ $\delta^*_{23} \geq 1$ otherwise $\theta_{3max} = 180°$	"	$\theta_{3max} = 90°$
7. c.m. to lab angle	$\cos\theta_3 = \frac{\gamma^*_2 (\cos\theta^*_3 + \delta^*_{23})}{[\sin^2 + \gamma^{*2}_2 (\delta^*_{23} + \cos\theta^*_3)^2]^{1/2}}$	$\tan\theta_3 = \frac{\sin\theta^*}{\gamma^*_2 (\delta^*_{21} + \cos\theta^*_3)}$ $\tan\phi_3 = \frac{1}{\gamma^*_2} \cot\frac{\theta^*}{2}$	$\tan\theta_3 = \frac{\sin\theta^*_3}{\gamma^*_2 (1 + \cos\theta^*_3)}$

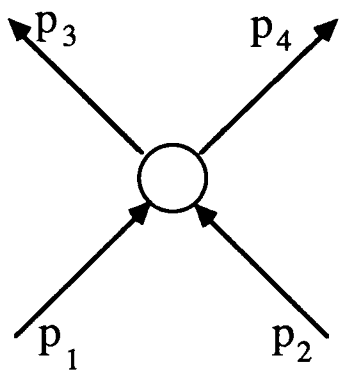
Two Body Scattering Formulae

Quantity	General Formula	N-N Scattering (equal mass)
8. lab to c.m. angle transformation	$\cos\theta_3 = \dfrac{\delta^*_{23}(\gamma_2\tan\theta)^2}{1+(\gamma_2\tan\theta)^2} \pm \left\{ \left[\dfrac{\delta^*_{23}(\gamma_2\tan\theta)^2}{1+(\gamma_2\tan\theta)^2}\right] - \dfrac{\delta^*_{23}(\gamma_2\tan\theta)^2-1}{(\gamma_2\tan\theta)^2+1} \right\}^{1/2}$	
9. Solid angle transformation (Jacobian)	$\dfrac{d\Omega}{d\Omega^*_3} = \dfrac{\gamma^*_2(1+\delta_{23}\cos\theta^*)}{[\sin^2\theta^* + \gamma^{*2}_2(\delta^*_{23}+\cos\theta_3)^2]^{3/2}}$ $\dfrac{d\Omega^*}{d\Omega_3} = \dfrac{\sin^3\theta_3}{\sin^3\theta^*}\gamma^*_2(\delta^*_{23}\cos\theta^*+1)$	$\dfrac{d\sigma}{d\Omega^*_3} = \dfrac{\gamma^*_2(1+\cos\theta^*)}{[\sin^2\theta^* + \gamma^{*2}_2(1+\cos\theta_3)^2]^{3/2}}$
10. Relations between the γ factors N.B. $k_{12} = m_1/m_2$	$(\gamma^{*2}_1 - 1) = k^2_{21}(\gamma^{*2}_2 - 1)$ $\gamma^*_1 = \dfrac{k_{12}+\gamma_1}{(1+k^2_{12}+2k_{12}\gamma_1)^{1/2}}$ $\gamma^*_2 = \dfrac{k_{21}+\gamma_1}{(1+k_{21}+2\gamma_1 k_{21})^{1/2}} = \gamma_{cm}$	$\gamma^*_1 = \gamma^*_2$ $\gamma^*_1 = \dfrac{(1+\gamma_1)^{1/2}}{2}$
11. Lab quantity relations	$2p_1p_3\cos\theta_3 = m_4^2 - m_1^2 - m_2^2 - m_3^2 + 2(E_1+m_2)E_3 - 2E_1m_2$ $2p_3p_4\cos(\theta_3+\theta_4) = m_3^2 + m_4^2 - m_1^2 m_2^2 - 2E_1m_2 + 2E_3E_4$	$p_3p_4\cos(\theta_3+\theta_4) = T_3T_4$
12. Maximum K.E. transfer to a stationary particle	$T_{max} = 2m_1^{-1}[k_{12}+k_{21}+2\gamma_1]p_1^2$ $\approx 2m_2\beta_1^2\gamma_1^2$	

Mandelstam Variables

In the general process 1 + 2 → 3 + 4 as shown in the figure below, the relativistically invariant Mandelstam variables are defined by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = W^2 \quad \text{(square of the total energy)}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \quad \text{(4-momentum transfer)}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$



The Mandelstam variables are not independent:

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Other useful relations are:

$$E_1^* = \sqrt{s} - E_2^* = \frac{1}{2\sqrt{s}} (s + m_1^2 - m_2^2)$$
$$E_3^* = \sqrt{s} - E_4^* = \frac{1}{2\sqrt{s}} (s + m_3^2 - m_4^2)$$
$$t = m_1^2 + m_3^2 - 2 E_1^* E_3^* + 2p_1^* p_3^* \cos\theta^* \ .$$

For elastic scattering

$$t = -2p_1^{*2} (1 - \cos\theta^*) = -2m_2 T_4 \ .$$

The 4-momentum transfer is negative for elastic scattering and → 0 as $\theta^* \to 0$. There is no similar simplification for u, since in general $m_1 \neq m_4$ and $m_2 \neq m_3$.

Other Useful Kinematic Variables

(1) Rapidity

Generally we write the total energy of a particle as $E^2 = m^2 + p^2$. If a particular direction is chosen for the z-axis, then define

$$m_\perp^2 = m^2 + p_x^2 + p_y^2 = E^2 - p_z^2 \ .$$

In this case we have

$$E = m_\perp \cosh y$$
$$p_z = m_\perp \sinh y$$

where

$$y = \frac{1}{2} \ln \left\{ \frac{E + p_z}{E - p_z} \right\} = \ln \left\{ \frac{E + p_z}{m_\perp} \right\} = \tanh^{-1} \left\{ \frac{p_z}{E} \right\} .$$

The variable y is called the rapidity.

(2) Pseudo-rapidity

A related variable is the pseudo-rapidity defined as

$$\eta = \ln \cot \frac{\theta}{2} \ .$$

Here θ is the angle between the particle and the beam direction. In the limit that $E \gg m$ we have $\eta \to y$.

(3) Feynman x Variable

If a particular direction is chosen for the z-axis, the Feynman x variable is defined using the z-momentum component in the overall c.m. system as follows:

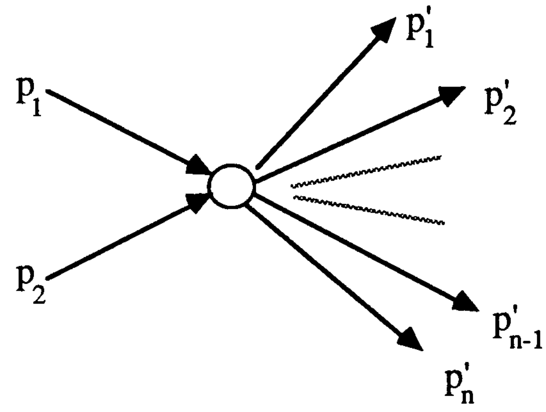
$$x = \left\{ \frac{p_z}{p_{z\,max}} \right\}_{cm} .$$

At high energy

$$x \approx \frac{2p_{z\,cm}}{\sqrt{s}} \approx \frac{2m_\perp \sinh y_{cm}}{\sqrt{s}} \ .$$

n-Body Phase Space

Consider the reaction shown in the figure below where there are n bodies in the final state.



The differential cross section for this reaction is given in either the lab or center of mass by the expression

$$d\sigma = \frac{(2\pi)^4 |M|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \, d\Phi_n(p_1, p_2; p_1', p_2', p_3', \ldots\ldots, p_n') \, .$$

Here $| M |^2$ is the square of the Lorentz invariant scattering amplitude, $[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2} = P_{1lab} m_2 = p_{1cm}\sqrt{s}$ is the Moller flux factor, and $d\Phi_n$ is the n-body phase space factor.

The n-body phase space factor can be written

$$d\Phi_n(p_1, p_2;\ p_1', p_2', \ldots\ldots, p_n') = \delta^4(p_1 + p_2 - \sum_{i=1}^{n} p_i') \prod_{i=1}^{n} \frac{d^3 p_i'}{(2\pi)^3 2 E_i'} \, .$$

A simple recursion relation is obtained if we consider particles 1' and 2' as a single system of momentum $p'_{12} = p_1' + p_2'$ and mass $m^2_{12} = p^2_{12}$

$$d\Phi_n(p_1, p_2; p_1', p_2', \ldots\ldots p_n') = d\Phi_{n-1}(p_1, p_2; p_{12}', p_3', \ldots\ldots p_n') \, d\Phi_2(p_{12}'; p_1', p_2') \, (2\pi)^3 dm_{12}$$