Two-nucleon systems in a relativistic Bethe-Salpeter approach

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Abstract

We study two-nucleon systems using a relativistic Bethe-Salpeter approach. In order to solve the Bethe-Salpeter equation (BSE), we use a separable ansatz for the interaction kernel of the integral equation. The interaction kernel is expanded using the Yamaguchi type function of suitable rank $N$ for various two-nucleon channels, with which it becomes possible to solve the Bethe-Salpeter equation easily. Then we study the scattering problems and deuteron properties in the relativistic framework.

To start with we investigate the relation between the rank I separable potential for the covariant Bethe-Salpeter equation and the one-boson-exchange potential. Parameters of the two potentials are related in the long wave length limit. It turns out that it is difficult to reproduce properties of a single term of the one-boson-exchange potential, separately by the rank I separable potential. Instead, it is shown that the separable potential is useful to parameterize the total nucleon-nucleon interaction.

Second we solve the nucleon-nucleon ($NN$) $T$ matrix for the several lower partial waves of $J = 0$ and $J = 1$. Then we have performed the phase shift analysis for the following two-body channels: $^1S_0$ and $^3P_0$ for $J = 0$ and $^3S_1$, $^1P_1$ and $^3P_1$ for $J = 1$. When we use the rank I separable potential, we can reproduce $^1S_0$ and $^3S_1$ phase shift up to about 300 MeV and for $^3P_0$, $^1P_1$ and $^3P_1$ up to about 100 MeV. When we use the extended rank I separable potential, we can reproduce $^1P_1$ and $^3P_1$ phase shift up to about 500 MeV and for $^1P_1$ up to 100 MeV. When we use the rank II separable potential, we can reproduce $^1S_0$, $^3P_0$, $^1P_1$, $^3P_1$ and $^3S_1$ phase shifts up to 600 MeV.

Finally, we investigate the electromagnetic properties of the deuteron such as the charge and magnetic form factors. In solving the deuteron bound state solution to the BSE, we include negative energy components of $P$-wave in addition to the $^3S_1$ and $^3D_1$ states. We found that the inclusion of the negative energy components improves systematically above physical quantities which are not described in the conventional impulse approximation.
1 Introduction

One of the most important problems of the modern nuclear particle physics is the understanding of the strong interaction. The interactions of elementary particles consist of the four interactions, namely, the strong interaction, the electromagnetic interaction, the weak interaction and the gravity interaction.

Nuclear force is the strong interaction which governs the nucleus. Quantum Chromodynamics (QCD) is the fundamental theory of the strong interaction. In principle, we can describe the dynamics of nuclei from the quark and gluon degree of freedom. However, by now, it is not possible to treat the nuclear dynamics quantitatively directly from QCD, due to the non-perturbative nature of QCD in the low energy region. Therefore, it is still common to treat the nucleus as a many body system of the nucleons for the study of nuclear physics, although the nucleon is not an elementary particle.

The nucleon-nucleon interaction has been built in many cases in a non-relativistic way. The discovery of the neutron in 1932 proved that nucleus is the many body system which consists of protons and neutrons. Since then, the understanding of the nuclear force has become one of main problems of nuclear physics. In 1935 Yukawa introduced a new particle, which we call meson now, to solve the problem of the nuclear force [1]. Yukawa thought that the nuclear force, the strong interaction, comes from the process where one nucleon absorbs a meson coming from the other nucleon, that is the exchange of the meson. His idea has been established for the study of the nucleon-nucleon interaction. Today, based on his idea we have many realistic nuclear forces, for example, Bonn, Paris, Nijmegen and Argonne [2, 3, 4, 5].

The nuclear force are in many cases developed and applied to the non-relativistic methods of nuclear physics problem. Among them, the simplest is the two-nucleon system, where nucleon-nucleon scattering and deuteron properties such as electromagnetic process are investigated. For example, Gari and Hyuga succeeded in explaining the electromagnetic property of the deuteron up to momentum transfer 2.4 GeV$^2$ [7, 8]. They used the non-relativistic potential, Hamada-Johnston and Reid soft-core potentials [9, 10], and considered the meson-exchange currents in a non-relativistic framework. In 90’s, Tamura et al., and Blunden et al.and Wringa et al. [11, 12, 13] also investigated the electromagnetic properties of the deuteron in a non-relativistic framework.

However, all non-relativistic treatments which are based on the impulse approximation with relativistic corrections and meson exchange currents are sensitive to the model of the meson exchange currents.

In principle nucleons are described by the Dirac equation, which is the relativistic equation. Therefore, the traditional non-relativistic framework may not be appropriate, especially, for the $NN$ scattering as well as nuclear reactions of, for instance, the deuteron and light nuclei with large momentum transfer.

In the relativistic framework, on the contrary, we expect the following advantage:

- Relativistic kinematic effects are automatically included.
- Some exchange currents are included through the $Z$-graph process.
- Relativistic dynamical effect such as the LS-force, spin-spin interaction and the negative energy components are naturally included.
In this thesis, in order to see the above effects of the relativistic framework, we study the scatterings of two-nucleons and the deuteron properties as the simplest nuclear system. For this purpose, we treat the Bethe-Salpeter equation in the fully relativistic manner [14].

The Bethe-Salpeter equation is a relativistic equation not only for scattering problems but also for bound state problems. In 1972, J. Kubis first formulated the Bethe-Salpeter equation for a two-nucleon system, which consists from two spin 1/2 particles in completely relativistic way [15]. He introduced the new quantum number $\rho$-spin, which distinguishes the positive and negative energy states of particles. He expanded the Bethe-Salpeter equation by the complete basis, in terms of not only the spin-angular momentum but also the $\rho$-spin. Then, he obtained coupled integral equations for each partial wave.

In 70’s, Fleischer and Tjon adopted Kubis’s formalism for $J = 0, 1$ channel [16, 17]. They solve the Bethe-Salpeter equation numerically by iteration and summation of the perturbation series by pade approximation. They used the $\pi$-, $\eta$-, $\epsilon$-, $\rho$-, $\omega$- relativistic exchange potential for the interaction kernel, and reproduced phase shifts in various channels. However they made an approximation which did not keep relativistic framework for complexity of the calculation. In 1989, Tjon and Rupp solved the Bethe-Salpeter equation by using the separable ansatz with no approximation, except that they ignored the negative energy components [25].

In 1989, Tjon and Rupp introduced the separable ansatz in order to solve the integral equation of the Bethe-Salpeter equation exactly. The separable ansatz is the way to solve the Bethe-Salpeter equation exactly. The separable ansatz is an assumption that the potential can be expressed in the following way.

$$ V(k', k) = \sum_{i,j=1}^{N} \lambda_{ij} g_i(k') g_j(k) \tag{1.1} $$

Here $k$ expresses the momentum of the incoming particle while $k'$ that of outgoing particle. The Separable ansatz was first introduced in 1954 by Yamaguchi et al. [18, 19]. They introduced the separable ansatz in order to obtain the exact solution of the Schrödinger equation for the deuteron. This is the non-local potential while the traditional potential is local potential.

For the function of separable ansatz, they put Yamaguchi type function

$$ g(k) = (k^2 + \beta^2)^{-1}. \tag{1.2} $$

This is the Fourier transformation of the Yukawa type function. By using this function, they succeeded in describing the property of the deuteron and photo-disintegration.

In 1973 Ernst and his collaborators formulated the method (EST method) [20]. They generalized the separable representation of the two-body interactions, so that they could approximate any form of potential by the separable ansatz.

In 80’s J. Haidenbauer and his collaborators applied this method in the case of $^3S_1-^3D_1$ $N-N$ potential. They approximated Reid and Paris potential by EST method [21, 22, 23, 24]. And they succeeded in obtaining the off-shell and on-shell properties of deuteron in non-relativistic framework.

In 1989, G. Rupp and J. A. Tjon and solved the Bethe-Salpeter equation by the covariant Graz II potential which is simple covariantization of Graz II potential which was
built by L. Mthelisch and W. Plessas and W. Schweiger in 1981, so that they reproduced the $^1S_0$ and $^3S_1-^3D_1$ phase shift up to 600 MeV and the low energy parameter [25, 24]. They succeeded in reproducing $^1S_0$ and $^3S_1-^3D_1$ phase shifts up to 600 MeV, keeping the relativistic framework. However they did not include the negative energy components. For the investigation of fully relativistic framework, we should include the negative energy states appropriately. This is one of main purposes of this thesis.

This thesis consists of five sections. In section 2, we show the general formalism for the two-nucleon system in BS approach. In section 3, we try to interpret the physical meaning of the separable potential by relating to the OBEP. Then we discuss applications of the separable ansatz to the phase shift analysis in section 4. Finally, we include the negative energy states properly and estimate how much negative components in order to explain some deuteron properties. We emphasize the importance of the negative energy components, although the rate in the probability is small.
2 The general formalism of the Bethe-Salpeter approach

2.1 Basic Properties of the Two–Nucleon BS equation

We discuss the general properties of the Bethe-Salpeter equation (BSE) for a two-nucleon (particle) system. Let us start from a scattering equation for a four-point Green’s function in the coordinate system given by

\[
G_{\alpha\beta,\gamma\delta}(x_1, x_2, x'_1, x'_2) = S_{\alpha\beta,\gamma\delta}(x_1, x_2, x'_1, x'_2)
\]

\[+ i \int \prod_i d^4 w_i S_{\alpha\beta,\sigma\rho}(x_1, x_2, w_1, w_2) V_{\sigma\rho,\lambda\omega}(w_1, w_2, w_3, w_4) G_{\lambda\omega,\gamma\delta}(w_3, w_4, x'_1, x'_2).\]

In this equation, Greek letters denote spinor indices, and the repeated indices are assumed to be summed up. Index \(i\) is the number of \(i\)-th nucleon. The function \(G\) is the full Green’s function, \(V\) the interaction kernel and \(S\) is the Green’s function for the two free nucleons, which is written as a product of free one-nucleon propagators. \(S^{(i)}\) is the free one-nucleon propagators for the \(i\)-th nucleon:

\[
S_{\alpha\beta,\gamma\delta}(x_1, x_2, x'_1, x'_2) = S^{(1)}_{\alpha\gamma}(x_1 - x'_1) S^{(2)}_{\beta\delta}(x_2 - x'_2),
\]

\[
S^{(i)}(x_i - x'_i) = -\frac{1}{(2\pi)^4} \int d^4 p \frac{p \cdot \gamma + m}{p^2 + m^2} e^{ip(x_i - x'_i)} \quad (i = 1, 2).
\]

It is convenient to express the BSE in the momentum space. For that purpose, we introduce a Fourier transform of the interaction kernel and the full Green’s function:

\[
V(P, k', k) = \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 V(x_1, x_2, y_1, y_2) \times \exp \left( iP \left( \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} \right) + ik' \frac{x_1 - x_2}{2} - ik \frac{y_1 - y_2}{2} \right),
\]

\[
G(P, k', k) = \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 G(x_1, x_2, y_1, y_2) \times \exp \left( iP \left( \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} \right) + ik' \frac{x_1 - x_2}{2} - ik \frac{y_1 - y_2}{2} \right),
\]

where \(P\) is the total momentum, \(k\) and \(k'\) are the relative 4-momenta of the two nucleons before and after the interaction. They are related to the 4-momenta of the first \((p_1)\) and second \((p_2)\) particles:

\[
P = p_1 + p_2, \quad k = (p_1 - p_2)/2,
\]

\[
p_1 = P/2 + k, \quad p_2 = P/2 - k.
\]

\[
P' = p'_1 + p'_2, \quad k' = (p'_1 - p'_2)/2,
\]

\[
p'_1 = P'/2 + k', \quad p'_2 = P'/2 - k'.
\]
We can write similar formula for $S^{(2)}(x_1, x_2, x'_1, x'_2)$, where $P'$ total momentum. Due to the conservation of the total momentum, we have $P = P'$. Diagrammatic representations with definition of various moments are shown in Fig. 1 and Fig. 2.

Figure 1: The two nucleon system

$P = p_1 + p_2$

$P' = p'_1 + p'_2$

$k = (p_1 - p_2)/2$

$k' = (p'_1 - p'_2)/2$

Figure 2: The two nucleon system expressed by the Bethe-Salpeter equation.

Using these momentum, the BS equation for the full Green's function of the two-nucleon system can be written as

$$G_{\alpha\beta,\gamma\delta}(P, k' - k) = S^{(1)}_{\alpha\gamma}(P/2 + k)S^{(2)}_{\beta\delta}(P/2 - k)\delta^{(4)}(k' - k)$$

$$+ iS^{(1)}_{\alpha\epsilon}(P/2 + k')S^{(2)}_{\beta\delta}(P/2 - k') \int \frac{d^4k''}{(2\pi)^4} V_{\epsilon\lambda,\rho\omega}(P, k', k'')G_{\rho\omega,\gamma\delta}(P, k'', k).$$

(2.8)

Here one-nucleon propagator $S^{(i)}_{\alpha\beta}(p)$ is expressed as

$$S^{(i)}_{\alpha\beta}(p) = [1/(p \cdot \gamma - m + i\epsilon)]_{\alpha\beta},$$

(2.9)

where $m$ is the mass of the nucleon, $p \cdot \gamma$ denotes $p_{\mu}\gamma^\mu$, and $\gamma_{\mu}$ are Dirac matrices. The Greek letters ($\alpha, \beta, \cdot \cdot \cdot$) denote components of the Dirac $4 \times 4$ matrices. Introducing the two-nucleon $T$-matrix by

$$S^{(1)}S^{(2)}T = GV,$$

(2.10)
Therefore, we can express the $T$-matrix as

\[ T_{\alpha\beta,\gamma\delta}(P, k', k) = V_{\alpha\beta,\gamma\delta}(P, k', k) \]

\[ + i \int \frac{d^4 k''}{(2\pi)^4} V_{\alpha\beta,\epsilon\lambda}(P, k', k'') S_{\epsilon\eta}^{(1)}(P/2 + k'') S_{\lambda\rho}^{(1)}(P/2 - k'') T_{\eta\rho,\gamma\delta}(P, k'') , \]

(2.12)

If the two-nucleon system has a bound state, $T$-matrix has a simple pole in the total momentum squared ($s = P^2$) at the point corresponding to the mass of the bound state $s = M^2 = (2m - E_B)^2$, where $E_B$ is the binding energy of the two-nucleon system. Therefore, we can express the $T$-matrix by

\[ T_{\alpha\beta,\gamma\delta}(P, k', k) = \frac{\Gamma_{\alpha\beta}(P, k') \overline{\Gamma}_{\gamma\delta}(P, k)}{P^2 - M^2} + R_{\alpha\beta,\gamma\delta}(P, k', k) . \]

(2.13)

Here $\Gamma_{\alpha\beta}(P, k')$ is the vertex function for the bound state. $R_{\alpha\beta,\gamma\delta}(P, k', k)$ is a regular function at $P^2 = M^2$. In order to obtain the equation which is satisfied by the vertex function, we substitute Eq. (2.13) into Eq. (2.12). Then we have

\[ \frac{\Gamma_{\alpha\beta}(P, k') \overline{\Gamma}_{\gamma\delta}(P, k)}{P^2 - M^2} + R_{\alpha\beta,\gamma\delta}(P, k', k) = V_{\alpha\beta,\gamma\delta}(P, k', k) \]

\[ + i \int \frac{d^4 k''}{(2\pi)^4} V_{\alpha\beta,\epsilon\lambda}(P, k', k'') S_{\epsilon\eta}^{(1)}(P/2 + k'') S_{\lambda\rho}^{(1)}(P/2 - k'') \]

\[ \times \left( \Gamma_{\eta\rho}(P, k'') \overline{\Gamma}_{\gamma\delta}(P, k) \right) \]

(2.14)

Multiplying $P^2 - M^2$ to the both sides of Eq. (2.14) and taking the limit $P^2 \to M^2$, we obtain

\[ \frac{\Gamma_{\alpha\beta}(P, k') \overline{\Gamma}_{\gamma\delta}(P, k)}{P^2 - M^2} \]

\[ = i \int \frac{d^4 k''}{(2\pi)^4} V_{\alpha\beta,\epsilon\eta}(P, k', k'') S_{\epsilon\eta}^{(1)}(P/2 + k'') S_{\lambda\rho}^{(1)}(P/2 - k'') \Gamma_{\eta\rho}(P, k'') \overline{\Gamma}_{\gamma\delta}(P, k) . \]

(2.15)

This leads to the integral equation for the vertex function $\Gamma_{\alpha\beta}(P, k')$

\[ \Gamma_{\alpha\beta}(P, k') = i \int \frac{d^4 k''}{(2\pi)^4} V_{\alpha\beta,\epsilon\eta}(P, k', k'') S_{\epsilon\eta}^{(1)}(P/2 + k'') S_{\lambda\rho}^{(1)}(P/2 - k'') \Gamma_{\eta\rho}(P, k'') \overline{\Gamma}_{\gamma\delta}(P, k) . \]

(2.16)

Now let us define the Bethe-Salpeter amplitude by

\[ \Phi_{\alpha\beta}(P, k) = S_{\alpha\gamma}^{(1)}(\frac{P}{2} + k) S_{\beta\delta}^{(1)}(\frac{P}{2} - k) \Gamma_{\gamma\delta}(P, k) . \]

(2.17)

Then, the equation for the Bethe-Salpeter amplitude in the momentum space is written as

\[ \Phi_{\alpha\beta}(P, k) = i S_{\alpha\gamma}^{(1)}(\frac{P}{2} + k) S_{\beta\delta}^{(1)}(\frac{P}{2} - k) \int \frac{d^4 k''}{(2\pi)^4} V_{\eta\rho,\epsilon\lambda}(P, k, k'') \Phi_{\epsilon\lambda}(P, k'') . \]

(2.18)
The amplitude must satisfy a normalization condition which is given as
\[
\int \frac{d^4k'}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \Phi(P, k') \left[ \delta^{(4)}(k - k')S^{(1)}(P/2 + k)S^{(1)}(P/2 - k) \right.
\]
\[
\left. \frac{\partial}{\partial P_{\mu}} \left\{ S^{(1)}(P/2 + k)S^{(1)}(P/2 - k) \right\} S^{(1)}(P/2 + k)S^{(1)}(P/2 - k) + \frac{\partial}{\partial P_{\mu}} V(P, k', k) \right] \Phi(P, k) = -2iP_{\mu}. \tag{2.19}
\]

This condition will be proved in Appendix C. Especially, if the interaction kernel does not depend on the total momentum \(P\), then Eq. (2.19) becomes
\[
\int \frac{d^4k}{(2\pi)^4} \Gamma(P, k) \frac{\partial}{\partial P_{\mu}} \left\{ S^{(1)}(P/2 + k)S^{(1)}(P/2 - k) \right\} \Gamma(P, k) = -2P_{\mu}. \tag{2.20}
\]

## 2.2 Partial–Wave Decomposition of the BS Amplitude

In order to solve the BS equation for a few body system, it is convenient to perform the partial wave decomposition of the BS amplitude, and separate the radial and the spin-angular parts. For this purposes we consider two representations for the spin part, one is “Direct Product Representation” and the other is “Matrix Representation”.

### 2.2.1 Direct Product Representation

In the direct product representation, we express two particle spinors in the center of mass frame of the two particle as \(U_{\rho_1}^{s_1}(k) \otimes U_{\rho_2}^{s_2}(-k)\), where \(s_i\) is the spin projection, \(\rho_{1,2}\) is the \(\rho\)-spin [15] which distinguishes the positive and negative energy states. Both the positive and negative energy states are necessary in order to prepare a complete set for a two-particle bound state in the relativistic formalism. The spinors \(U_{\rho}^{s}(k)\) are related with the ordinary Dirac free spinors, \(u_{s}(k)\) and \(v_{s}(k)\), as
\[
U_{\rho}^{s}(k) = \begin{cases} u_{s}(k), & \rho = +, \\ v_{-s}(-k), & \rho = -. \end{cases} \tag{2.21}
\]

The Dirac spinors are given by
\[
u_{s}(k) = L(k)u_{s}(0), \quad v_{s}(k) = L(k)v_{s}(0), \tag{2.22}
\]
where the boost operator for a particle with spin 1/2 and a mass \(m\) is defined by
\[
L(k) = \frac{m + k \cdot \gamma \gamma_0}{\sqrt{2E_k (m + E_k)}}. \tag{2.23}
\]

Here \(k = (E_k, \mathbf{k})\) is the 4-momentum of a particle on mass shell, and therefore \(E_k = \sqrt{m^2 + k^2}\) is the energy of the particle. In Eq. (2.22) the spinors are defined in the rest frame which are written as
\[
u_{s}(0) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}, \quad v_{s}(0) = \begin{pmatrix} 0 \\ \chi_{-s} \end{pmatrix},
\]
where $\chi_s$ are two-component Pauli spinors. The normalization conditions are:

$$\bar{u}_s(k) u_{s'}(k) = \frac{m}{E_k} \delta_{ss'}, \quad \bar{v}_s(k) v_{s'}(k) = -\frac{m}{E_k} \delta_{ss'}. \quad (2.24)$$

Now let us expand the general BS amplitude by the BS amplitudes in the partial waves $\Phi^{JM}(P, k)$

$$\Phi(P, k) = \sum_{JM} a_{JM} \Phi^{JM}(P, k), \quad (2.25)$$

where $a_{JM}$ are the expansion coefficients. The BS amplitude $\Phi^{JM}(P, k)$ of the two-particle system with the total angular momenta $J$ and projection $M$ in the laboratory frame can be written as

$$\Phi^{JM}(P, k) = \sum_{LS\rho_1\rho_2} \phi_{JLS\rho_1\rho_2}(k_0, |k|) \gamma_{JLS\rho_1\rho_2,M}(k), \quad (2.26)$$

where $P = (M, 0)$ is the total momentum, $k$ is the relative momentum of the two-particle system $(k = (k_0, k), k_0 \neq E_k)$. Here $JLS\rho_1\rho_2$ are the quantum numbers of the total angular momentum $J$, orbital momentum $L$, spin $S$ and $\rho$-spins $\rho_1, \rho_2$. We define the spin angular function $\gamma_{JLS\rho_1\rho_2,M}(k)$ by

$$\gamma_{a\beta}^{JLS\rho_1\rho_2,M}(k) \equiv i^L \sum_{s_1 s_2 m L m_S} (L m_L S m_S | J M ) \left( \frac{1}{2} s_1 \frac{1}{2} s_2 | S m_S \right) Y_{Lm}(k) \times \left( U_{s_1}^\rho(k) \right)_\alpha \left( U_{s_2}^\rho(-k) \right)_\beta, \quad (2.27)$$

where $(.,.)$ are the Clebsch-Gordan coefficients and $\hat{k} = k/|k|$. The spin-angular function $\gamma_{JLS\rho_1\rho_2,M}(k)$ is a matrix, $(1 \times 4) \otimes (1 \times 4) = (1 \times 16)$ in a spinor space. The spinor indices $\alpha \beta$ specify components of this matrix.

The orthogonality condition for the spin-angular functions is

$$\int d\Omega_k \gamma_{a\beta}^{JLS\rho_1\rho_2,M}(k) \gamma_{a'\beta'}^{J'L'S'\rho_1'\rho_2',M'}(k) = \delta_{LL'} \delta_{MM'} \delta_{SS'} \delta_{\rho_1 \rho_1'} \delta_{\rho_2 \rho_2'}, \quad (2.28)$$

where $d\Omega_k \equiv d\phi_k \ d\cos \theta_k$ and the conjugate spin-angular function can be obtained by the substitution, $U_{s_1}^\rho(k) \rightarrow \bar{U}_{s_2}^\rho(k)$.

In order to proceed the calculation, it is convenient to write the inverse propagators $[S(P/2 + k)]^{-1}, [S(P/2 - k)]^{-1}$ in the laboratory frame as:

$$\begin{align*}
[S(P/2 + k)]^{-1} &= P \cdot \gamma/2 + k \cdot \gamma - m \\
&= \frac{1}{2E_k} \left[ (p_1 \cdot \gamma - m) S_+^{(1)-1} + (p_2 \cdot \gamma + m) S_-^{(1)-1} \right], \\
[S(P/2 - k)]^{-1} &= P \cdot \gamma/2 - k \cdot \gamma - m \\
&= \frac{1}{2E_k} \left[ (p_2 \cdot \gamma - m) S_-^{(2)-1} + (p_1 \cdot \gamma + m) S_+^{(2)-1} \right],
\end{align*} \quad (2.29)$$

where $p_1 = (E_k, k), p_2 = (E_k, -k), S = P^2$, and the scalar functions $S(i)_\rho$, with $\rho = \pm$ are

$$\begin{align*}
S^{(1)}_\rho &= 1/(\sqrt{s}/2 + k_0 - \rho E_k), \\
S^{(2)}_\rho &= 1/(\sqrt{s}/2 - k_0 - \rho E_k).
\end{align*} \quad (2.30)$$
Using the Eqs. (2.21), (2.22), (2.29) and the Dirac equation, we can write

\[
\begin{align*}
[S(P/2 + k)]^{-1} U^\rho_\mu(k) &= \rho S^{(1)}_\rho U^\rho_\mu(-k), \\
[S(P/2 - k)]^{-1} U^\rho_\mu(-k) &= \rho S^{(2)}_\rho U^\rho_\mu(k).
\end{align*}
\]

Now we expand the BS vertex function \( \Gamma^{JM}_{\alpha\beta}(P, k) \) as

\[
\Gamma^{JM}_{\alpha\beta}(P, k) = \sum_{LS\rho_1\rho_2} \rho_1 \rho_2 g_{JLS\rho_1\rho_2}(k_0, |k|) \mathcal{Y}^{JLS\rho_1\rho_2,M}_{\alpha\beta}(k).
\] (2.31)

Here the radial part \( g_{JLS\rho_1\rho_2}(k_0, |k|) \) of the vertex function is related with the radial part \( \phi_{JLS\rho_1\rho_2}(k_0, |k|) \) of the BS amplitude through

\[
\phi_{JLS\rho_1\rho_2}(k_0, |k|) = S^{(1)}_{\rho_1} S^{(1)}_{\rho_2} g_{JLS\rho_1\rho_2}(k_0, |k|).
\] (2.32)

For the discussions of two-nucleon states, it is convenient to introduce the symmetrical notation for the positive and negative energy states. We define the states with total \( \rho \)-spin number \( \rho \):

\[
\begin{align*}
|+\rangle &\equiv |+\rangle \otimes |+\rangle, \\
|\rangle &\equiv \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle), \\
|\epsilon\rangle &\equiv \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle).
\end{align*}
\] (2.33)

Then Eq. (2.32) can be written in the following way

\[
\phi_{JLS\rho}(k_0, |k|) = S_{\rho}(k_0, |k|; s) g_{JLS\rho}(k_0, |k|),
\] (2.34)

where \( S_{\rho}(k_0, |k|; s) \) is

\[
\begin{align*}
S_+ &= \left(\frac{\sqrt{s}}{2} + k_0 - E_k + i\epsilon\right)^{-1}\left(\frac{\sqrt{s}}{2} - k_0 - E_k + i\epsilon\right)^{-1}, \\
S_- &= \left(\frac{\sqrt{s}}{2} + k_0 + E_k - i\epsilon\right)^{-1}\left(\frac{\sqrt{s}}{2} - k_0 + E_k - i\epsilon\right)^{-1}, \\
S_{+-} &= \left(\frac{\sqrt{s}}{2} + k_0 - E_k + i\epsilon\right)^{-1}\left(\frac{\sqrt{s}}{2} - k_0 + E_k - i\epsilon\right)^{-1}, \\
S_{-+} &= \left(\frac{\sqrt{s}}{2} + k_0 + E_k - i\epsilon\right)^{-1}\left(\frac{\sqrt{s}}{2} - k_0 - E_k + i\epsilon\right)^{-1}, \\
S_e &= \frac{S_{+-} + S_{-+}}{2}, \\
S_o &= \frac{S_{+-} - S_{-+}}{2}.
\end{align*}
\]
2.2.2 Matrix Representation

In the matrix representation, we replace the spinor of a second particle by the transposed spinor and then calculate the direct product,

\[ U_{s_1}^{\rho_1}(k) \otimes U_{s_2}^{\rho_2}(-k) \longrightarrow U_{s_1}^{\rho_1}(k) \otimes U_{s_2}^{\rho_2 T}(-k). \] (2.36)

We use the matrices of \((4 \times 1) \otimes (1 \times 4) = (4 \times 4)\) instead of the matrices \((1 \times 4) \otimes (1 \times 4) = (1 \times 16)\) in the spinor space. Then, \((4 \times 4)\) matrices can be expanded by the Dirac \(\gamma\)-matrices.

The BS amplitude in the rest frame can be written as:

\[ \Phi^{JM}(P, k) = \chi^{JM}(P, k) C = \sum_{LS\rho} \phi_{JLS\rho}(k_0, |k|) \Gamma^{JLS\rho,M}(k) C, \] (2.37)

where \(\Gamma^{JLS\rho,M}(k)\) is the vertex function after the partial wave decomposition and \(C\) is the charge-conjugate matrix

\[ C = i \gamma_2 \gamma_0, \] (2.38)

and the spin-angular part \(\Gamma^{JLS\rho,M}(k)\) has a structure similar to \(\gamma^{JLS\rho,M}(k)\), however, with the replacement of Eq. (2.36).

\[ \Gamma^{JSL,\rho,M}(k) \equiv i^L \sum_{s_1 s_2 M L M S} (Lm_L Sm_S |JM\rangle (\frac{1}{2} s_1 \frac{1}{2} s_2 |Sm_S\rangle Y_{LM}(\hat{k}) \times (U_{s_1}^{\rho_1}(k)) (U_{s_2}^{\rho_2}(-k))^T. \] (2.39)

We show here, as an example, the function \(\Gamma^{3S_1^+,M}(k)\), with the notation \(2^{S+1}L^\rho_j\) for the partial states \((JLS\rho)\):

\[ \Gamma^{3S_1^+,M}(k)C \]
\[ = \sum_{s_1 s_2} (\frac{1}{2} s_1 \frac{1}{2} s_2 |1M\rangle u_{s_1}(k)u_{s_2}^T(-k) \]
\[ = L(k) \sum_{s_1 s_2} (\frac{1}{2} s_1 \frac{1}{2} s_2 |1M\rangle \chi_{s_1} (\chi_{s_2}^T 0) L^T(-k) \]
\[ = L(k) \left( \sum_{s_1 s_2} (\frac{1}{2} s_1 \frac{1}{2} s_2 |1M\rangle \chi_{s_1} \chi_{s_2}^T 0 \right) L^T(-k) \]
\[ = L(k) \frac{1 + \gamma_0}{2\sqrt{2}} \left( \begin{array}{cc} 0 & -\sigma \cdot \xi_M \\ \xi_M & 0 \end{array} \right) \left( \begin{array}{cc} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{array} \right) L^T(-k) \]
\[ = L(k) \frac{1 + \gamma_0}{2\sqrt{2}} (-\gamma \xi_M) L(k) U_C \]
\[ = \frac{1}{2E_k} (m + p_1 \cdot \gamma) \frac{1 + \gamma_0}{2\sqrt{2}} \xi_M \cdot \gamma (m - p_2 \cdot \gamma) U_C. \]

Here we make use of the following equation for the Pauli spinors and the Clebsch–Gordan coefficients:

\[ \sqrt{2} \sum_{s_1 s_2} (\frac{1}{2} s_1 \frac{1}{2} s_2 |1M\rangle \chi_{s_1} \chi_{s_2}^T = (\sigma \cdot \xi_M) (i\sigma_2), \] (2.40)
where \( \xi_M \) is a 3-vector of the polarization of a particle with spin 1 and the components in the rest frame,

\[
\xi_{+1} = (1, -i, 0)/\sqrt{2}, \quad \xi_{-1} = (-1, i, 0)/\sqrt{2}, \quad \xi_0 = (0, 0, 1).
\] (2.41)

The polarization 4-vector \( \xi_M = (0, \xi_M) \) is defined in the rest frame of the two-particle system.

In general, we can separate a \( \rho \)-dependence and rewrite the spin-angular functions

\[
\Gamma^{JLS\rho,M}(k) \equiv \Gamma^{JLS\rho,M}(k).
\] as:

\[
\begin{align*}
\Gamma^{JLS++.,M}(k) &= \frac{p_1 \cdot \gamma + m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k) - \frac{p_2 \cdot \gamma - m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k), \\
\Gamma^{JLS--,M}(k) &= \frac{p_1 \cdot \gamma - m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k) - \frac{p_2 \cdot \gamma + m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k), \\
\Gamma^{JLS+,M}(k) &= \frac{p_1 \cdot \gamma + m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k) - \frac{p_2 \cdot \gamma - m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k), \\
\Gamma^{JLS-,M}(k) &= \frac{p_1 \cdot \gamma - m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k) - \frac{p_2 \cdot \gamma + m}{\sqrt{2E_k(m + E_k)}} \frac{1 + \gamma_0}{2} \Gamma^{JLS,M}(k).
\end{align*}
\]

The conjugate functions can be written in the following form:

\[
\Gamma^{JLS\rho,M}(k) = \gamma_0 \left[ \Gamma^{JLS\rho,M}(p) \right]^\dagger \gamma_0.
\] (2.43)

Then the orthogonal condition can be presented as

\[
\int d\Omega\, \text{Tr}\{\Gamma^{JLS\rho,M}(k) \Gamma^{JLS\rho,M'}(k) \} = \delta_{MM'}\delta_{LL'}\delta_{SS'}\delta_{\rho\rho'},
\] (2.44)

which is corresponding to Eq. (2.28)

Using the Pauli principle for identical particles

\[
\Phi^{JM}(P, k) = -P_{12}\Phi^{JM}(P, k),
\] (2.45)

where \( P_{12} \) is the permutation operator of a two-particles, we can write the BS amplitude \( \chi^{JM}(P, k) \) in the rest frame as:

\[
\chi^{JM}(P, k) = (-1)^{I+1} U_C [\chi^{JM}(P, -k)]^T U_C,
\] (2.46)

where \( I \) is the isospin of the system. This relation gives us symmetry properties of the radial functions \( \phi_{JLS\rho}(k_0, |k|) \) for the transformations, when we replace \( k_0 \rightarrow -k_0 \). Radial functions for different \( LS\rho \) will be odd or even under this transformation.

### 2.3 The BS amplitude for \( J = 0, 1 \).

#### \( J = 0 \).

The BS amplitude of the two-nucleon system in the \( J = 0 \) channel has four states: \( ^1S_0^+, ^1S_0^-, ^3P_0^-, ^3P_0^+ \). In order to obtain these states, we have to consider the states which conserves \( J^P = 0^+ \). Here \( P = (-1)^I P_{12} \).

For \( NN \)-scattering we take \( \sqrt{s} = \sqrt{P^2} \). The corresponding spin-angular parts are shown in Table 2 where \( p_1 = (E_k, k) \) and \( p_2 = (E_k, -k) \).

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Using the Lorentz invariant expressions for $|k|$, $k_0$ and $E_k$:

$$k_0 = \frac{(P \cdot k)}{M}, \quad E_k = \sqrt{\frac{(P \cdot k)^2}{M^2} - k^2 + m^2}, \quad |k| = \sqrt{\frac{(P \cdot k)^2}{M^2} - k^2},$$

we can rewrite the equations for the BS amplitude in a covariant form. It is more convenient to use the direct covariant form of the BS amplitude in the $J = 0$ channel which is written in the $4 \times 4$ matrix form. For this purpose, we introduce four Lorentz covariant functions, $h_i(P \cdot k, k^2)$:

$$\sqrt{4\pi} \chi^0(P, k) = h_1 \gamma_5 + h_2 \frac{1}{m} (p_1 \cdot \gamma \gamma_5 + \gamma_5 p_2 \cdot \gamma) + h_3 \frac{(p_1 \cdot \gamma - m)}{m} \gamma_5 - \frac{(p_2 \cdot \gamma + m)}{m} \gamma_5 + h_4 \frac{p_1 \cdot \gamma - m}{m} \gamma_5 \frac{p_2 \cdot \gamma + m}{m},$$

where 4-momentum of the two particles, $p_1$ and $p_2$, are determined by (2.7), and the functions $h_i(P \cdot k, k^2)$ can be expressed by the radial functions $\phi_{JLS\rho}(k_0, |k|)$ defined in Eq.(2.37):

$$\sqrt{4\pi} h_1 = \sqrt{2} D_0 (-\phi_{1S_0^+} + \phi_{1S_0^-}) - \sqrt{2}/4 (\phi_{1S_0^+} - \phi_{1S_0^-}) - \mu k_0 |k|^{-1} \phi_{3p_0} - 8m|k|^{-1} D_1 \phi_{3p_0},$$

$$\sqrt{4\pi} h_2 = i \frac{1}{4} m |k|^{-1} \phi_{3p_0},$$

$$\sqrt{4\pi} h_3 = -8\sqrt{2} a_0 m^2 (\phi_{1S_0^+} - \phi_{1S_0^-}) - \frac{1}{2} \frac{k_0 |k|^{-1} \phi_{3p_0} - 8a_0 m |k|^{-1} (2m^2 - E_k^2) \phi_{3p_0}},$$

$$\sqrt{4\pi} h_4 = 4a_0 \sqrt{2} m^2 (\phi_{1S_0^+} - \phi_{1S_0^-}) + 8a_0 m^3 |k|^{-1} \phi_{3p_0}.$$

Here

$$a_0 = 1/(16ME_k), \quad \varepsilon = 2m + E_k, \quad \mu = m/M,$$

$$D_0 = a_0 (4k_0^2 + 16m^2 - 4E_k^2 - M^2 - \frac{8m^2 M}{E_k}),$$

$$D_1 = a_0 (-M^2/4 + k_0^2 - 3E_k^2 + 4m^2 - \frac{2Mm^2}{E_k}).$$
In order to obtain these states, we have to consider the state $s$ which conserves $J$. The BS amplitude for the deuteron has eight states: $3S^+_1$, $3S^-_1$, $3D^+_1$, $3D^-_1$, $3P^+_1$, $3P^-_1$, $1P^+_1$, $1P^-_1$, (or $3P^+_1$, $3P^-_1$, $1P^+_1$, $1P^-_1$), which are numbered as $1, \ldots, 8$. In order to obtain these states, we have to consider the states which conserves $J^P = 1^+$. Here $P = (-1)^J\rho_1\rho_2$. The corresponding spin-angular parts $\tilde{\Gamma}^{JLS,M}(k)$ are tabulated in the Table 3.

The BS amplitude has the following covariant matrix form,

$$
\sqrt{4\pi} \chi^{JLM}(P,k) = h_1 \xi_M \cdot \gamma + h_2 \frac{k\xi_M}{m} +
$$

$$
+ h_3 \left( \frac{p_1 \cdot \gamma - m}{m} \xi_M \cdot \gamma + \xi_M \cdot \gamma \frac{p_2 \cdot \gamma + m}{m} \right) +
$$

$$
+ h_4 \left( \frac{p_1 \cdot \gamma + p_2 \cdot \gamma}{m} \right) \frac{k\xi_M}{m} +
$$

$$
+ h_5 \left( \frac{p_1 \cdot \gamma - m}{m} \xi_M \cdot \gamma - \xi_M \cdot \gamma \frac{p_2 \cdot \gamma + m}{m} \right) +
$$

$$
+ h_6 \left( \frac{p_1 \cdot \gamma - p_2 \cdot \gamma - 2m}{m} \right) \frac{k\xi_M}{m} +
$$

$$
+ \frac{p_1 \cdot \gamma - m}{m} \left( h_7 \xi_M \cdot \gamma + h_8 \frac{k\xi_M}{m} \right) \frac{p_2 \cdot \gamma + m}{m}.
$$

Table 3: Spin-angular parts $\tilde{\Gamma}^{1LS,M}$ for the deuteron

<table>
<thead>
<tr>
<th>$^{1LS}$</th>
<th>$\sqrt{8\pi} \tilde{\Gamma}^{1LS,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3S^+_1$</td>
<td>$\xi_M \cdot \gamma$</td>
</tr>
<tr>
<td>$3D^-_1$</td>
<td>$\frac{1}{\sqrt{2}} \left[ \xi_M \cdot \gamma + \frac{3}{2}(p_1 \cdot \gamma - p_2 \cdot \gamma)(k\xi_M)</td>
</tr>
<tr>
<td>$3P^-_1$</td>
<td>$\sqrt{2} \left[ \frac{1}{2}\xi_M \cdot \gamma(p_1 \cdot \gamma - p_2 \cdot \gamma) - (k\xi_M) \right]</td>
</tr>
<tr>
<td>$1P^-_1$</td>
<td>$\sqrt{3}(k\xi_M)</td>
</tr>
</tbody>
</table>

We note that the functions $h_2$ and $\phi_{3P_0^+}$ are odd under $k_0 \rightarrow -k_0$, while the other functions are even.

$J = 1$(Deuteron). The BS amplitude for the deuteron has eight states: $3S^+_1$, $3S^-_1$, $3D^+_1$, $3D^-_1$, $3P^+_1$, $3P^-_1$, $1P^+_1$, $1P^-_1$, (or $3P^+_1$, $3P^-_1$, $1P^+_1$, $1P^-_1$), which are numbered as $1, \ldots, 8$. In order to obtain these states, we have to consider the states which conserves $J^P = 1^+$. Here $P = (-1)^J\rho_1\rho_2$. The corresponding spin-angular parts $\tilde{\Gamma}^{JLS,M}(k)$ are tabulated in the Table 3.

Here we have introduced eight covariant functions $h_i(P \cdot k, k^2)$, which are connected with the radial functions $\phi_{JLSM}(k_0, |k|)$ in the rest frame via relations:

$$
\sqrt{4\pi} h_1 = -D_1^+ (\phi_{3D_1^+} - \sqrt{2}\phi_{3S^+_1}) - D_1^- (\phi_{3D^-_1} - \sqrt{2}\phi_{3S^-_1})
$$

$$
- \frac{\sqrt{6}}{2} \mu k_0 ||k||^{-1} \phi_{3P_0^+},
$$

$$
\sqrt{4\pi} h_2 = - \sqrt{2}(D_2^- \phi_{3S^-_1} - D_2^+ \phi_{3S^+_1}) - D_3^+ \phi_{3D_1^+} + D_3^- \phi_{3D^-_1}
$$

$$
+ \frac{\sqrt{6}}{|k|} D_4 (i\phi_{3P_0^+}) + \frac{\sqrt{6}}{2} \mu k_0 ||k||^{-1} \phi_{3P_0^+} - \sqrt{3}m^2 ||k||^{-1} E_{-1}(i\phi_{3P_0^+}),
$$

$$
\sqrt{4\pi} h_3 = \frac{\sqrt{6}}{8m} ||k||^{-1} (i\phi_{3P_0^+}),
$$

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The coefficients in these equations are:

\[ \sqrt{4\pi} h_4 = -4a_1 \sqrt{2} mk_0 (\phi_{3s^+_i} - \phi_{3s^-_i}) - 4a_2 \varepsilon mk_0 (\phi_{3d^+_i} - \phi_{3d^-_i}) + 8a_0 \sqrt{3m^2 |k|}^{-1} i(k_0 \phi_{1p^-} - E_k \phi_{1p^+}), \]
\[ \sqrt{4\pi} h_5 = -8a_0 m^2 [\phi_{3d^+_i} + \phi_{3d^-_i} - \sqrt{2} (\phi_{3s^+_i} + \phi_{3s^-_i})] - 4a_0 \sqrt{6m|m|}^{-1} i[k_0 E_k \phi_{3p^0} + (2m^2 - E_k^2) \phi_{3p^0}] \]
\[ \sqrt{4\pi} h_6 = -2a_1 \sqrt{2} m [D^+_6 \phi_{3s^+_i} + D^-_6 \phi_{3s^-_i}] + 4m^2 |k|^{-2} i[D^+_5 \phi_{3d^+_i} + D^-_5 \phi_{3d^-_i}] - 8a_0 \sqrt{6m^2 |k|}^{-1} (m \phi_{3p^0} - M \phi_{3p^0}), \]
\[ \sqrt{4\pi} h_7 = -4a_0 m^3 [\sqrt{2} (\phi_{3s^+_i} + \phi_{3s^-_i}) - (\phi_{3d^+_i} + \phi_{3d^-_i})] + 4a_0 \sqrt{6m^3 |k|}^{-1} i\phi_{3p^0}, \]
\[ \sqrt{4\pi} h_8 = -4a_0 m^3 |k|^{-2} [\sqrt{2} a_1 (\phi_{3s^+_i} + \phi_{3s^-_i}) - a_2 (2E_k + m)(\phi_{3d^+_i} + \phi_{3d^-_i})] - 4a_0 \sqrt{6m^3 |k|}^{-1} i\phi_{3p^0}. \]

The coefficients in these equations are:

\[ a_0 = \frac{1}{16ME_k}, a_1 = \frac{m}{16(m + E_k)}, a_2 = \frac{m}{16ME_k(m^2 - E_k^2)}, \]
\[ D_1 = a_0 (-\frac{M^2}{4} + k_0^2 - 3E^2 + 4m^2) \]
\[ D_1^+ = a_0 (4k_0^2 + 16m^2 - M^2 - 4E_k^2 \pm 4ME_k), \]
\[ D_2^+ = a_1 (16m^2 + 16mE_k^2 + 4E_k^2 \pm M^2 - 4k_0^2 \pm 4m\varepsilon), \]
\[ D_3^+ = a_2 [-12mE_k^2 + 2M^2 E_k^2 - 8k_0^2 E_k^2 + 16m^3 + mM^2 - 4m^3k_0^2 + 8E_k^3 \pm (16m^2 M + 4mME_k^2 - 8E_k^2 M), \]
\[ D_4 = (16m^2 - 4E_k^2 - 4k_0^2 + m^2), \]
\[ D_5^+ = a_0 (-2E_k^2 \pm 4m^2 + 4mE_k \pm \varepsilon M), \]
\[ D_6^+ = 2\varepsilon \pm M, \]

and \(a_0, \varepsilon, \mu, D_0\) are given in Eq. (2.50). Here the functions \(h_3, h_4\) and \(\phi_{3p^0}, \phi_{1p^0}\) are odd and others are even under \(k_0 \rightarrow -k_0\). 

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2.4 General solution to the BS equation

In this section, we consider the solution of the BS equation with a separable ansatz. The separable interaction is well known in a non-relativistic approach [18]. By using this ansatz, we can solve the integral equation rigorously. In this case, the $T$-matrix can be also expressed in a separable form in accordance with the interaction $V$. The BS equation for $T$-matrix Eq. (2.13) after the partial wave decomposition (see subsection 2.2) is written as:

\[
T_{JLL'S';JLS\rho}(k'_0, |k'|, k_0, |k|; s) = \sum_{i,j=1}^{N} \lambda_{ij} g_{i}^{JL'S';L's}(k'_0, |k'|) g_{j}^{JLS\rho}(k_0, |k|), \quad \lambda_{ij} = \lambda_{ji}(2.55)
\]

where $N$ is the number of terms of expansion, $\lambda_{ij}$ parameters of the interaction kernel, and $g_{i}^{JLS\rho}(k_0, |k|)$ are arbitrary functions which we will discuss in the next section. Then according to Eq.(2.54), the $T$-matrix can be also expressed in a separable form. We assume the following separable form for it

\[
T_{JLL'S';JLS\rho}(k'_0, |k'|, k_0, |k|; s) = \sum_{i,j=1}^{N} \tau_{ij}(s) g_{i}^{JL'S';L's}(k'_0, |k'|) g_{j}^{JLS\rho}(k_0, |k|). \quad (2.56)
\]

Substituting Eqs. (2.55) and (2.56) in Eq. (2.54), we can obtain an equation for $\tau_{ij}(s)$,

\[
(\tau^{-1}(s))_{ij} = (\lambda^{-1})_{ij} - H_{ij}(s), \quad (2.57)
\]

where $H_{ij}(s)$ is determined by the equation:

\[
H_{i\kappa}(s) = \frac{i}{2\pi^2} \sum_{L'S\rho} dk' \int d|k'| S_{\rho}(k_0, |k|; s) g_{i}^{JL'S\rho}(k_0, |k|) g_{j}^{JL'S\rho}(k_0, |k|). \quad (2.58)
\]

The solution to the radial part of the BS amplitude can be presented in the following form,

\[
\phi_{JLS\rho}(k_0, |k|) = \sum_{i,j=1}^{N} S_{\rho}(k_0, |k|; s) \lambda_{ij} g_{i}^{JLS\rho}(k_0, |k|) c_{j}(s), \quad (2.59)
\]

where the coefficients $c_{j}(s)$ satisfy the following system of linear homogeneous equations:

\[
c_{i}(s) - \sum_{k,j=1}^{N} H_{i\kappa}(s) \lambda_{kj} c_{j}(s) = 0. \quad (2.60)
\]
2.5 The form of the $g$-function

In sections 4.1, 4.2 and 4.3, we will use a separable potential which is the simple covariantization of the Yamaguchi type potential. For example, in the previous works

$$g(k) = \frac{1}{(k^2 + \beta^2)}, \quad k^2 = -k_0^2 + k^2$$  \hspace{1cm} (2.61)

was taken for $S$-wave, and

$$g(k) = \frac{|k|}{(k^2 + \beta^2)^2}$$  \hspace{1cm} (2.62)

for $P$-wave, and

$$g(k) = \frac{k^2}{(k^2 + \beta^2)^2}$$  \hspace{1cm} (2.63)

for $D$-wave. These functions are manifestly Lorentz invariant.

Now if we consider the amplitude in the center of mass system of the two-nucleon, it is natural to expect that they behave as $|k|/|k| \to 0$ for partial waves of $l$-th angular momentum. However, the functions Eqs. (2.62) and (2.63) do not satisfy this condition. In this thesis we consider the $g$-function satisfying this. For example, we shall take

$$g(k) = \frac{|k|}{(k^2 + \beta^2)^2}, \quad k^2 = -k_0^2 + k^2$$  \hspace{1cm} (2.64)

$$g(k) = \frac{k_0 k}{(k^2 + \beta^2)^2}.$$  \hspace{1cm} (2.65)

At first glance, these expressions manifestly break the Lorentz invariance. However, they are appropriate in the CM frame as the angular momentum depends on the choices of a reference frame. At an arbitrary frame, one can transfer the expressions by the Lorentz transformations. In this thesis, we consider the amplitude and calculate physical quantities in the CM system using amplitude of the form Eqs. (2.64) and (2.65).
3 Relation between the separable and one-boson-exchange potential for the covariant Bethe-Salpeter equation

3.1 Separable ansatz

In this section we investigate the relation between the rank I separable potential for the covariant Bethe-Salpeter equation and the one-boson-exchange potential (OBEP).

Traditional approach for describing the nucleon-nucleon (NN) interaction is based on the Schrödinger equation with a one-boson-exchange potential. It is a non-relativistic approach and widely used. But for the phenomena of high momentum transfer, such an approach should be improved by taking account relativistic effects. In our approach, the BS equation is constructed in a fully relativistic manner. Generally, solving the relativistic BS equation is a difficult problem. Separable ansatz for the interaction kernel makes it easy to solve BS equation keeping relativistic covariance. Using the separable ansatz, we can express the potential by the product of the functions of relative incoming and outgoing momenta (See section 2). If we can fix the parameters included the separable potential, we can describe the nucleon-nucleon (NN) interaction.

In the previous study, the separable ansatz has been considered as a mathematical technique to solve the BS equation, with little physical interpretation. On the contrary, the OBEP is constructed on a physical picture of boson exchange.

The separable presentation of the NN-interaction is written as following way,

\[
\langle p' | V_{sp} | p \rangle = f(p') f(p). \tag{3.1}
\]

Here \( p, p' \) are the relative incoming and outgoing two-nucleon momentum. It is a non-local interaction, as expressed in the coordinate space by a function of \( r \) and \( r' \) (\( r \) and \( r' \) are conjugate to \( p \) and \( p' \).)

\[
\langle r' | V_{sp} | r \rangle = V(r', r). \tag{3.2}
\]

On the other hand the OBE potential is assumed to depend only on the momentum transfer \( (q \equiv p' - p) \), which is intermediated by bosons exchanged between the nucleons. The ‘local’ one-boson-exchange potential is expressed as in momentum space as

\[
\langle p' | V_{OBE} | p \rangle = V(q), \quad (q \equiv p' - p). \tag{3.3}
\]

In coordinate space, it is written as

\[
\langle r | V_{OBE} | r' \rangle = \delta(r' - r)V(r). \tag{3.4}
\]

We can expand approximately the OBE potential into the series of separable functions as

\[
\langle p' | V_{OBE} | p \rangle = \sum_{i,j=1}^{N} \lambda_{ij} f_{i}(p') f_{j}(p). \tag{3.5}
\]
The accuracy of this expansion depends on how many terms of the series one can take into account. In this section we use the rank I separable ansatz and we use Yamaguchi type for the function,

$$g(p) = \frac{1}{p^2 - \beta^2},$$ (3.6)

where $p$ is a relative momentum in the deuteron and $\beta$ is a free parameter. And we consider the $^1S_0$ state only for simplicity.

We investigate whether the parameters of the separable potential may be related to those of the OBEP, since the latter is considered to be physically more fundamental, at least for longer range part of the NN interaction.

By doing this, we expect that the separable potential can be understood with physics ground, not just a mathematically convenient tool.

### 3.2 The determination of the interaction strength

Let us consider a single channel equation for $^1S_0$. This is sufficient for our present qualitative discussions. After angular integration, we can immediately write the BS equation from the discussion of section (2.4),

$$T(p', p; s) = V(p', p) + \frac{i}{4\pi^3} \int dk_0 |k|^2 d|k| \frac{V(p', k)T(k, p; s)}{(\sqrt{s} - E_k + i\epsilon)^2 - k_0^2},$$ (3.7)

where $T(p', p; s)$ is the $T$-matrix, $V(p', p)$ the interaction kernel, $s = (p_1 + p_2)^2$ and $E_k = \sqrt{|k|^2 + M^2}$ with $M$ being the mass of the nucleon. We omit all state which have at least one negative energy state. The momentum variables are for four momentum, e.g., $p = (p_0, \mathbf{p})$, etc. In Eq. (3.7), we have considered the equation in the center of mass system. The rank I separable ansatz assumes to write the interaction

$$V_{\text{sep}}(p', p) = \lambda g(p') g(p)$$ (3.8)

with a coupling constant $\lambda$ and a function $g(p)$ as a scalar function of $p$ and $p'$. The $T$-matrix is then obtained in the separable form as

$$T(p', p; s) = \tau(s) g(p') g(p),$$

$$\tau(s) = \frac{1}{\lambda + h(s)},$$ (3.9)

where

$$h(s) = -\frac{i}{4\pi^3} \int dk_0 |k|^2 d|k| \frac{g(k)^2}{(\sqrt{s} - E_k + i\epsilon)^2 - k_0^2}.$$ (3.10)

These equations are $N=1$ case of section (2.4), which the formalism will be given in the next section in detail. Phase shifts are then given by the relation

$$T(p', p; s) = -\frac{16\pi}{\sqrt{s} \sqrt{s - 4M^2}} e^{i\delta} \sin \delta,$$ (3.11)

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where the relative momenta are \( p = (0, p) \), \( p' = (0, p') \) for on-shell nucleons. Finally the phase shift \( \delta(s) \) can be presented by
\[
\cot \delta(s) = -\frac{\lambda^{-1} + \text{Re}(h(s))}{\text{Im}(h(s))},
\]  
(3.12)

Now important terms of the OBEP can be written as
\[
V_\sigma(q) = -g_\sigma^2 \frac{1}{q^2 + m_\sigma^2} \left( \frac{\Lambda^2_\sigma - m^2_\sigma}{\Lambda^2_\sigma - q^2} \right)^2, \quad V_\omega(q) = g_\omega^2 \frac{1}{q^2 + m_\omega^2} \left( \frac{\Lambda^2_\omega - m^2_\omega}{\Lambda^2_\omega - q^2} \right)^2,
\]
(3.13)
\[
V_\pi(q) = \frac{g_\pi^2}{4M_N^2} \frac{q^2}{q^2 + m^2_\pi} \left( \frac{\Lambda^2_\pi - m^2_\pi}{\Lambda^2_\pi - q^2} \right)^2, \quad V_\rho(q) = \frac{g_\rho^2}{2M_N^2} \frac{q^2}{q^2 + m^2_\rho} \left( \frac{\Lambda^2_\rho - m^2_\rho}{\Lambda^2_\rho - q^2} \right)^2,
\]
(3.14)
where \( q = p - p' = (0, p - p') \). Here the masses \( m_\alpha \), the coupling constants \( g_\alpha \) and the cutoff parameters \( \Lambda_\alpha (\alpha = \sigma, \omega, \pi, \rho) \) are given in Ref. [2], and are summarized in Table. 4. In Eqs. (3.13) and (3.14) we picked up the dominant piece of the one boson exchange potential. The higher order terms are proportional to the initial and final relative momenta \( p \) and \( p' \). For the \( \rho \)-exchange potential, we use only the tensor coupling term, where the correction from the vector term is about 5%. The coupling strength given in Table. 4 (in the second row) produces only the \( f \)-coupling in Ref. [2].

For the parameterization of the separable potential, we assume the Yukawa function for \( g(p) \) with the same mass parameter \( m \) as in the OBEP, \( g(p) = 1/(p^2 - m^2_0) \). Then we try to impose that \( V_{\text{sep}} \) equals \( V_b \) in the long wave length limit [26]1:
\[
V_{\text{sep}}(0, 0) = V_b(0, 0),
\]
(3.15)
which determines the strength \( \lambda \). In this way, we have a separable potential approximately related to the OBEP
\[
V_{\text{sep}}(p', p) = \lambda_b \frac{1}{p^2 - m_b^2} \frac{1}{p'^2 - m_b^2},
\]
(3.16)
where \( \lambda_b \) are given by
\[
\lambda_\sigma = -g_\sigma^2 m_\sigma^2 \left( 1 - \frac{m^2_\sigma}{\Lambda^2_\sigma} \right)^2, \quad \lambda_\omega = g_\omega^2 m_\omega^2 \left( 1 - \frac{m^2_\omega}{\Lambda^2_\omega} \right)^2,
\]
(3.17)
\[
\lambda_\pi = -\frac{g_\pi^2 m_\pi^4}{4M_N^2} \left( 1 - \frac{m^2_\pi}{\Lambda^2_\pi} \right)^2, \quad \lambda_\rho = \frac{g_\rho^2 m_\rho^4}{2M_N^2} \left( 1 - \frac{m^2_\rho}{\Lambda^2_\rho} \right)^2.
\]
(3.18)
In the case of \( \pi \) and \( \rho \), we excluded \( q^2 \) dependence in the numerator of the Eqs. (3.19), otherwise we can not determine the \( \lambda \) parameter. It corresponds to excluding the \( \delta \)-function term in the \( r \) space, namely for the Eq. (3.15) we have used
\[
V_\pi(q) = -\frac{g_\pi^2}{4M_N^2} \frac{m^2_\pi}{q^2 + m^2_\pi} \left( \frac{\Lambda^2_\pi - m^2_\pi}{\Lambda^2_\pi - q^2} \right)^2, \quad V_\rho(q) = -\frac{g_\rho^2}{2M_N^2} \frac{m^2_\rho}{q^2 + m^2_\rho} \left( \frac{\Lambda^2_\rho - m^2_\rho}{\Lambda^2_\rho - q^2} \right)^2.
\]
(3.19)
The numerical values of the \( \lambda \)'s are also given in the Table. 4 (last column).

1The relations shown here differ from those of Ref [26], where extra factor of \( 4\pi^2 \) was erroneously included.
3.3 Comparison of phase shifts

We obtain the phase shifts for the two cases. First we solved the Shrodinger equation for the $^1S_0$ state, using one-boson-exchange potential,

$$\{-\frac{1}{m} \nabla^2 + V_\alpha\} \psi = E \psi,$$

(3.20)

where the index $\alpha$ represents $\sigma, \omega, \rho$ and $\pi$ for the corresponding one-boson-exchanges.

For later use, we show the one boson exchange potential of the $^1S_0$ channel as a function of $r$ in Figs. 4, where various terms of the OBEP are shown separately. The thick solid line in Fig. 4-(a) is the total potential including the $\sigma, \omega, \pi$ and $\rho$ exchange potentials. Second, we solve the Bethe-Salpeter equation using the separable interaction with the parameters determined by the Eqs. (3.17), (3.18). (See Table. 4) The resulting phase shifts are shown in Fig.3.

Now we discuss the phase shifts calculated from each meson exchange potential.

- Fig. 3-(a) shows the phase shifts calculated from the potentials of the $\sigma$ channel as functions of $T_{lab}$, the kinetic energy in the laboratory frame. Here $T_{lab}$ is related to $s$ by

$$T_{lab} = \frac{s - 4M^2}{2M}.$$

(3.21)

The thick solid line represents the phase shift for the separable potential, and the thin solid line for the OBEP. As shown in Fig. 3-(a), both phase shifts start from 180 degrees, indicating a strong attraction as accommodating one bound state. Indeed, as shown in Fig. 4-(a), the depth of the $\sigma$-exchange potential of the OBEP reaches about 200 MeV at 0.75 fm. The strong attraction of the OBEP causes the raising behavior at $T_{lab=0}$, which turns to decreasing at $T_{lab} = 20$ MeV. On the other hand, the separable potential can not be that strongly attractive. This can be checked by analyzing the scattering matrix of Eq. (3.9), which will be discussed later. In fact, the phase shift approaches the upper limit as indicated by the dashed line in Fig. 3-(a) in the limit $\lambda \to -\infty$. Furthermore we have checked that it is not possible to reproduce the strong attraction of the $\sigma$-exchange potential, whatever $m_b$ value of the separable potential we choose. The fact that the separable potential can not be too strong has been discussed previously [36].

- Fig. 3-(b) shows the phase shifts calculated from the potentials of the $\omega$ channel as functions of $T_{lab}$. The phase shifts calculated from the two interactions (Separable and OBEP) show repulsive nature as it starts from 0 degree and decreases as the energy increases. As shown in Fig. 4-(a), the repulsive force of the $\omega$-exchange potential is very strong. As shown in Fig. 3-(b), the result of the separable potential resembles that of OBEP. However, that strong repulsion of the OBEP can not be reproduced by the separable potential, which is similar to the case of the $\sigma$-exchange potential. Once again, as shown by the dashed line in Fig. 3-(b) there is a lower bound of the phase shift of the separable potential in the limit $\lambda \to \infty$. However if we allow $m_b$ to change, it is possible to reproduce the phase shift of the $\omega$-exchange...
potential by using a parameter set of, for instance, $m_b = 630$ MeV and $\lambda = 81.5 \times 10^6$ MeV$^2$. To make $m_b$ small corresponds to the increase of repulsion. At this point, we recognize that the physical meaning of the mass parameter $m_b$ in the separable potential is different from that in the OBEP.

- Fig. 3-(c) shows the phase shifts calculated from the potentials of the $\pi$ channel as functions of $T_{lab}$. The phase shifts calculated from the two interactions look very different. Fig. 4-(b) shows that $\pi$-exchange potential is attractive at long distances $r > 1$ fm and repulsive at middle and short distances $r < 1$ fm. Therefore the phase shift calculated from OBEP starts from 0 degree, raising at first, then turns to decrease at $T_{lab} \sim 5$ MeV and becomes repulsive at $T_{lab} \sim 20$ MeV. Due to the form factor, the one-pion-term of the OBEP here is written as a sum of the long range attraction and the short range repulsion. On the other hand, the phase shift of the separable interaction is weakly attractive. Because there is only one term in the rank I separable potential, the contributions of attraction and repulsion can not be reproduced simultaneously. In particular, the coupling strength $\lambda_\pi$ determined from the relation (3.18) at low momentum region ($p = p' = 0$) is too attractive, which therefore can not reproduce the repulsive behavior at higher $T_{lab}$. However, one can fit the repulsive behavior at higher energies by changing the range parameter $m_\pi$ and the coupling constant $\lambda$. For instance if we choose $m_\pi = \Lambda_\pi = 1300$ MeV and $\lambda = 135 \times 10^6$ MeV$^2$, we can reproduce the phase shift at around $T_{lab} \approx 200$ MeV as indicated by the dashed line of Fig. 3-(c).

- Fig. 3-(d) shows the phase shifts calculated from the potentials of the $\rho$ channel as functions of $T_{lab}$. The phase shifts calculated from the two interactions look very different. With $\lambda = -37.2 \times 10^6$ MeV$^2$ which is determined by Eq. (3.18), the result of the separable potential is too attractive, such that it generates one bound state and the phase shift starts from 180 degrees. In contrast, as shown in Fig. 4-(b) the $\rho$-exchange piece of the OBEP is attractive at long distances $r > 0.6$ fm and repulsive at middle and short distances $r < 0.6$ fm. The attractive interaction here, however, is not very large due to the cancellation by the repulsive component. Therefore the phase shift calculated from OBEP starts raising from 0 degree, turns to decrease at $T_{lab} \sim 40$ MeV, and change into repulsion at $T_{lab} \sim 160$ MeV. Just as in the $\pi$ case, we can re-fit the strength of the separable potential $\lambda_\rho$. By reducing the strength by about factor 8, $\lambda = -4.52 \times 10^6$ MeV$^2$, we obtain the phase shift of the separable potential as shown by the dashed line of Fig. 3-(d), which looks rather close to the result of OBEP.

These results show that it is difficult to reproduce the phase shifts of each terms of the one-boson-exchange potential separately by the rank I separable potential when we use the parameters determined in the long wave length limit. As explained above in detail, the separable potential can not be stronger than a certain strength both for attractive and repulsive cases if we do not change the $m_b$ parameter.

The fact that the separable potential can not be stronger than a certain strength may be understood from Eq. (3.9) where the factor $1/\lambda$ vanishes in the limit $|\lambda| \to \infty$. 26
Interestingly, in this limit there is no distinction between attractive \((\lambda \to -\infty)\) and repulsive \((\lambda \to +\infty)\) interactions. In order to find the maximum strength of the separable potential, we plot in Fig. 6 the real and imaginary parts of \(h(s)\) from which we can calculate the phase shift by using Eq. (3.12). The result is shown in Fig. 6. The real part monotonically decreases from 0.276 GeV\(^{-2}\), while the imaginary part starts from 0, reaches the maximum value at some \(s\) and turns to decrease monotonically. This behavior resembles what is familiar in the non-relativistic scattering theory where the phase shift varies from 0 to 180 degrees when there is one bound state. In a relativistic theory, however, a naive argument in the non-relativistic theory cannot be applied, since in the large \(s\) region particle production may occur and the discussion within a fixed particle number can not be applied. In the present separable potential model, the treatment will break down at and beyond \(s = 4(M_N + m_b)^2\) where an unphysical pole of mass \(m_b\) appears.

In our calculation of the phase shift, in order to determine the initial value \(\delta(T_{lab} = 0)\), for the attractive interaction we increased \(\lambda\) gradually from a small value and verified that there is a jump from \(\delta(T_{lab} = 0) = 0\) to \(\delta(T_{lab} = 0) = 180\) degrees at certain strength of \(\lambda\) only once. Therefore, we conclude that the maximum strength of the separable potential in our method is what allows one bound state for an attractive interaction. Similarly for the repulsive case, it is also possible to show that there is the maximum strength of the interaction if \(m_b\) is fixed.

Now turning to the full result of the \(1S_0\) channel, as indicated in Fig. 5, the separable potential can reproduce rather well the result of the total nuclear force of OBEP when we take \(\lambda = -0.294 \times 10^6\) MeV\(^2\) and \(m_b = 224\) MeV [26]. The very strong attractive and repulsive forces of the \(\sigma\)- and \(\omega\)-exchange potentials are largely canceled, yielding a rather mild nuclear force. This is the reason that the separable potential for nuclear reaction have been successful.

### Table 4: Parameters of the OBEP from the Bonn potential [2].

<table>
<thead>
<tr>
<th>(m_b) (MeV)</th>
<th>(g^2/4\pi)</th>
<th>(\Lambda_b) (MeV)</th>
<th>(\lambda) (MeV(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>550</td>
<td>7.78</td>
<td>2000</td>
</tr>
<tr>
<td>(\rho)</td>
<td>769</td>
<td>34.77</td>
<td>1300</td>
</tr>
<tr>
<td>(\omega)</td>
<td>783</td>
<td>20.0</td>
<td>1500</td>
</tr>
<tr>
<td>(\pi)</td>
<td>138</td>
<td>14.9</td>
<td>1300</td>
</tr>
</tbody>
</table>

### 3.4 Summary

We have studied the relation between the rank I separable potential for the covariant Bethe-Salpeter equation and the one-boson-exchange potential (OBEP). Individual channels of \(\sigma\)-, \(\omega\)-, \(\pi\)- and \(\rho\)-exchanges were investigated separately. As a result, it turned out that the rank I separable potential could not reproduce the phase shift calculated from each component of the OBEP when we use the parameters determined in the long wave length limit. As for the \(\sigma\) channel, where the potential is strongly attractive, we could not reproduce the phase shift of OBEP even if we take the limit \(\lambda \to -\infty\) and we change \(m_b\) parameter. Similarly as for the \(\omega\) channel with strong attraction, the separable potential...
Figure 3: Phase shifts of the $^1S_0$ channel calculated from the separable potentials (thick solid lines) and those from the OBEP (thin solid lines). Dashed lines (a) and (b) represent the upper limit and the lower limit of the phase shift calculated from the separable potential with $|\lambda| \to \infty$. The dashed line of (c) represents the phase shift fitted to the phase shift of the OBEP around $T_{lab} \approx 200\text{MeV}$. The dashed line of (d) represents the phase shift calculated by the separable potential which fits the phase shift of the OBEP around $T_{lab} \approx 0\text{MeV}$.
Figure 4: The separable contributions to the Bonn potential of the $^1S_0$ channel from the various meson exchange terms as denoted by the labels. (a): Thick solid, solid and thin dashed lines are for the total nuclear force, $\sigma$- and $\omega$-exchange potentials, respectively. (b): Dashed and dotted lines are $\pi$- and $\rho$-exchange potentials of the $^1S_0$ channel, respectively.

Figure 5: The phase shifts of the $^1S_0$ channel calculated from the best fitted separable potential as a function of the kinetic energy in the laboratory frame as compared with the experimental data (data are taken using SAID program http://gwdac.phys.gwu.edu/).
Figure 6: The real and imaginary part of $h(s)$ when $m_b = 500$(MeV) as a function of the total momentum square.
could not reproduce again the phase shift of OBEP even in the limit $\lambda \to \infty$. However we could reproduce the strong repulsion, if we change the $m_b$ parameter. These observations imply that the physical meaning of the mass parameters in the separable potential and OBEP are different. The mass parameter of the OBEP represents the interaction range of a local potential, while that of the separable potential could mimic, for instance, the range of a non-local interaction. The non-locality of the nuclear force is related to the structure of the nucleon at short ranges $r < 0.5$ fm [32]. Concerning the $\pi$ and $\rho$ channels, where the potential consists of attraction at long distances and repulsion at short distances, the rank 1 separable potential could not reproduce the mixed nature of the interaction, although the interaction strengths are not as strong as the $\sigma$ and $\omega$ channels. Despite the above fact, the rank 1 separable potential can reproduce the experimental data of the $^1S_0$ phase shift up to the energy $T_{lab} \sim 200$ MeV where a mild attractive interaction dominates.

These results show that the rank 1 separable potential is not suited to the description of very strong attraction. For instance, phase shifts calculated from the separable potential can not become larger than 180 degrees, no matter how large the attraction coupling constant takes. Rather, the separable potential can describe relatively mild attraction and all repulsion. In the realistic nuclear force, such a mild strength is obtained by the sum of the strongly attractive $\sigma$-exchange and the strongly repulsive $\omega$-exchange potentials.

In this section, we have shown that the separable potential works well for the two nucleon system if parameters are chosen suitably, although the decomposition into components of physical OBEP does not make sense. In a sense, the different nature of the two potential should have been expected. The main purpose of the present paper was to see whether it is possible to make physical meaning of the separable potential in comparison with the OBEP by using a simple parameterization of one term of rank 1. In order to perform a good description of phenomena in the covariant Bethe-Salpeter formalism, we can introduce a higher rank form. This is discussed in section 3, where the use of the improved rank one ansatz and of higher rank interactions are tested.
4 Two-nucleon scattering

4.1 Rank I Yamaguchi type separable potential

In the previous chapter we have discussed that it is not always possible to reproduce the phase-shifts calculated from a single term of the one-boson-exchange potential especially of the \( \sigma \)-exchange term, separately by the rank I separable potential. It implies that the separable ansatz may be just a mathematical technique. It has been shown, however, that the separable ansatz is suited to the description of two-nucleon problem with the realistic nucleon force when it is parameterized appropriately.

In this section we solve the Bethe-Salpeter equation for the two-nucleon system using the Yamaguchi type separable ansatz for the \( J = 0 \) \( ^1S_0 \), \( ^3P_0 \) and \( J = 1 \) \( ^3S_1 - ^3D_1 \), \( ^3P_1 \), \( ^3P_1 \) channels. In order to understand the analytic properties of the solution to the \( T \)-matrix with the separable interaction, let us first consider a single channel case especially, of \( ^1S_0 \) \( (J = 0, L = 0, S = 0) \). Furthermore, for simplicity, we neglect the negative energy states, expecting the negative energy components do not play important roles for scattering problems. In this case the BS Eq. (2.54) which we should solve is

\[
t(k'_0, |k'|, k_0, |k|; s) = v(k'_0, |k'|, k_0, |k|; s) + i \frac{1}{4\pi^3} \int dk'' \int |k''|^2 |k''| \frac{v(k'_0, |k'|, k''_0, |k''|; s)t(k''_0, |k''|, k_0, |k|; s)}{(\sqrt{s}/2 - E_{k''} + i\epsilon)^2 - k''_0^2}.
\]  

(4.1)

In this equation we omitted all partial wave indices, \( E_k = \sqrt{|k|^2 + m^2} \) \( (m \) is a nucleon mass), \( t \) the \( T \)-matrix and \( v \) interaction kernel \( V \). Using a rank I separable ansatz, the interaction kernel \( v \) can be expressed as

\[
v(k'_0, |k'|, k_0, |k|; s) = \lambda g(k'_0, |k'|)g(k_0, |k'|),
\]  

(4.2)

which corresponds to Eq. (2.55) with \( N = 1 \), and \( \lambda \) is the parameter which expresses the strength of the potential. Then the \( T \)-matrix can be expressed as

\[
t(k'_0, |k'|, k_0, |k|; s) = \tau(s)g(k'_0, |k'|)g(k_0, |k'|)
\]  

(4.3)

where the function \( \tau(s) \) is

\[
\tau(s) = 1/(\lambda^{-1} + h(s)),
\]  

(4.4)

and function \( h(s) \) is

\[
h(s) = -i \frac{1}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{|g(k_0, |k|)|^2}{(\sqrt{s}/2 - E_k + i\epsilon)^2 - k_0^2}.
\]  

(4.5)

Eqs. (4.3), (4.4) and (4.5) corresponds to Eqs. (2.56), (2.57) and (2.58), respectively. As a result, the \( T \)-matrix can be rewritten in the following form:

\[
t(k'_0, |k'|, k_0, |k|; s) = \frac{g(k'_0, |k'|)g(k_0, |k|)}{\lambda^{-1} + h(s)}.
\]  

(4.6)
For the on-mass-shell $T$-matrix is expressed:

$$t(s) = \frac{n(s)}{d(s)} = \frac{[g(0, \tilde{p})]^2}{\lambda^{-1} + h(s)}$$  \hspace{1cm} (4.7)$$

which can be related to the phase shifts $\delta(s)$ from

$$t(s) \equiv t(0, \tilde{p}, 0, \tilde{p}; s) = -\frac{16\pi}{\sqrt{s} \sqrt{s - 4m^2}} e^{i\delta(s)} \sin \delta(s),$$  \hspace{1cm} (4.8)$$

with $\tilde{p} = \sqrt{s/4 - m^2} = \sqrt{mT_{lab}/2}$. We assume that the imaginary part of the function $n(s)$ satisfies the following condition:

$$\text{Im} \, n(s) = 0,$$  \hspace{1cm} (4.9)$$

which is related with the specific choice of the $g$-functions for the $NN$-vertex which will be discussed later. Then, taking into account Eqs. (4.7) and (4.8), the phase shift $\delta(s)$ can be given as

$$\cot \delta(s) = \frac{\text{Re} \, t(s)}{\text{Im} \, t(s)} = -\frac{\lambda^{-1} + \text{Re} \, h(s)}{\text{Im} \, h(s)}.$$  \hspace{1cm} (4.10)$$

In order to express the low-energy parameters in terms of the $T$-matrix solution, we expand the function $h(s)$ in a series of $\tilde{p}$ terms:

$$h(s) = h_0 + i\tilde{p} h_1 + \tilde{p}^2 h_2 + i\tilde{p}^3 h_3 + \mathcal{O}(\tilde{p}^4),$$  \hspace{1cm} (4.11)$$

$$\text{Re} \, h(s) = h_0 + \tilde{p}^2 h_2 + \mathcal{O}(\tilde{p}^4),$$  \hspace{1cm} (4.12)$$

$$\text{Im} \, h(s) = \tilde{p}(h_1 + \tilde{p}^2 h_3 + \mathcal{O}(\tilde{p}^3)).$$  \hspace{1cm} (4.13)$$

The low-energy parameters of $NN$-scattering are introduced by expanding the $T$-matrix into series of $\tilde{p}$ terms by the following way:

$$\tilde{p} \cot \delta(s) = -a_0^{-1} + \frac{r_0}{2} \tilde{p}^2 + \mathcal{O}(\tilde{p}^3).$$  \hspace{1cm} (4.14)$$

We consider only the first two terms of the Eq. (4.14).

Now, using the definition Eq. (4.14) and Eqs. (4.10)-(4.13) we can find the parameters $a_0$ and $r_0$ as follows:

$$a_0 = \frac{h_1}{\lambda^{-1} + h_0},$$  \hspace{1cm} (4.15)$$

$$r_0 = \frac{2}{h_1} \left[ (\lambda^{-1} + h_0) \frac{h_3}{h_1} - h_2 \right].$$  \hspace{1cm} (4.16)$$

In this work, we use the Yamaguchi type function [18] for $g(k_0, |\vec{k}|)$, which is expressed by

$$g(k_0, |\vec{k}|) = \frac{1}{(k_0^2 - |\vec{k}|^2 - \beta^2 + i\epsilon)},$$  \hspace{1cm} (4.17)$$

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which is a covariantization $|k|^2 \rightarrow -k_0^2 + |k|^2$ of the non-relativistic Yamaguchi type function

$$g(|k|) = \frac{1}{(|k|^2 + \beta^2)} ,$$

(4.18)

and the numerator reflects the character of the $S$-wave, that is, $1 = |k|^{l=0}$. In this case, the function $h$ can be rewritten as follows:

$$h(s, \beta) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k^2 + i\epsilon})^2 - k_0^2} \frac{1}{(k_0^2 - E^2_{\beta} + i\epsilon)^2} ,$$

(4.19)

where $E_{\beta} = \sqrt{|k|^2 + \beta^2}$. Here we introduced the second parameter $\beta$, whose inverse $\beta^{-1}$ correspond to the range of the interaction.

Analyzing Eq. (4.19), we can see that it has the four poles in the $k_0$-complex plane at

$$k_0^{(1)}(s) = \frac{\sqrt{s}}{2} - E_k + i\epsilon \quad \quad k_0^{(2)}(s) = -\frac{\sqrt{s}}{2} + E_k - i\epsilon$$

$$k_0^{(3)}(s) = -E_\beta + i\epsilon \quad \quad k_0^{(4)}(s) = E_\beta - i\epsilon$$

(4.20)

The variation of $s$ results in the move of the poles $k_0^{(1)}$ and $k_0^{(2)}$, and we have the situation where two poles “pinch” on the real $k_0$ axis. An end point at which this condition starts to be satisfied (the branch point) can be found from the following equations:

$$k_0^{(1)}(s) = k_0^{(2)}(s) \quad \Rightarrow \quad s_0 = 4m^2,$$

$$k_0^{(1)}(s) = k_0^{(4)}(s) \quad \Rightarrow \quad s_1 = 4(m + \beta)^2.$$

(4.21) (4.22)

Summarizing the situation, we can say that the function $h(s)$ has two cuts starting in points $s_0$ and $s_1$ respectively, and therefore can be written in a dispersion form:

$$h(s, \beta) = \int_{4m^2}^{+\infty} \frac{\rho(s', \beta) ds'}{s' - s - i\epsilon},$$

(4.23)

$$\rho(s', \beta) = \theta(t - 4m^2)\rho_{el}(s', \beta) + \theta(t - 4(m + \beta)^2)\rho_{in}(s', \beta)$$

with two spectral functions $\rho_{el, in}$ (el stands for elastic and in for inelastic) which are connected with the imaginary parts as follows:

$$\rho(s', \beta) = \frac{1}{\pi} \text{Im} h(s', \beta) = \frac{1}{2\pi i} (h - h^*).$$

(4.24)

The solution for $T$-matrix we obtained has some analytic properties as follows:

1. In the region $s > 4m^2$, $T$-matrix has the cut corresponding to the elastic $NN$ scattering (Eq. (4.21));

2. In the region $s > 4(m + \beta)^2$, $T$-matrix has the cut corresponding to the inelastic $NN$ scattering (Eq. (4.22));

3. $T$-matrix has no left-hand cuts, however there is a pole of second order at the point $s = 4(m^2 - \beta^2)$. 34
In order to find the spectral functions, we should perform \( k_0 \)-integration in Eq. (4.19) which results in:

\[
h(s, \beta) = -\frac{1}{2\pi^2} \frac{1}{\partial \beta^2} \int |k|^2 d|k| \frac{1}{s/4 - \sqrt{s}E_k + m^2 - \beta^2 + i\epsilon} \left[ \frac{1}{\sqrt{s - 2E_k + i\epsilon}} + \frac{1}{2E\beta} \right] (4.25)
\]

Taking into account the following symbolic equation

\[
\frac{1}{x - x_0 \pm i\epsilon} = \mathcal{P} \frac{1}{x - x_0} \mp i\pi \delta(x - x_0), \quad (4.26)
\]

it is easy to find the spectral functions:

\[
\begin{align*}
s' &\geq 4m^2, \\
\rho_{ei}(s', \beta) &= \sqrt{s'}\sqrt{s' - 4m^2}/(\pi^2(s' - 4m^2 + 4\beta^2)^2), \\
s' &\geq 4(m + \beta)^2, \\
\rho_{in}(s', \beta) &= -(64\beta^6 + 16\beta^4s' - 192\beta^4m^2 - 20\beta^2s'^2 \\
&\quad + 192\beta^2m^4 - 32\beta^2s'm^2 + 16m^4s' + 3s'^3 - 64m^6 - 12m^2s'^2)/ \\
&(2\pi^2s'(s' - 4m^2 + 4\beta^2)^2\sqrt{s' - 4(m + \beta)^2}\sqrt{s' - 4(m - \beta)^2}). (4.28)
\end{align*}
\]

We take the following conditions.

\[
\begin{align*}
m &> \beta > 0 \quad (4.29) \\
4(m + \beta)^2 &> s > 4m^2 \quad (4.30)
\end{align*}
\]

The condition (4.30) means that the second (inelastic) imaginary part does not contribute to the function \( \text{Im} \ h(s) \) when the phase shifts are calculated in the region \( 4(m + \beta)^2 > s > 4m^2 \) and, therefore

\[
\text{Im} \ h(s, \beta) = \text{Im} \ h_{el}(s, \beta) = \frac{\sqrt{s - 4m^2}\sqrt{s}}{(s - 4m^2 + 4\beta^2)^2\pi} \quad \text{if} \quad 4(m + \beta)^2 > s > 4m^2. (4.31)
\]

### 4.1.1 Calculation

At this moment, we can determine the two parameters of the separable interaction \((\lambda, \beta)\) to reproduce experimental values for low-energy parameters \(a_{0s}^{exp} = -23.748 \pm 0.010 \) fm, \(r_{0s}^{exp} = 2.75 \pm 0.05 \) fm for singlet channel \((^1S_0)\).

We use Eq. (4.16) with \( a_0 \equiv a_{0s}^{exp} \) to find \( \lambda \):

\[
\lambda^{-1} = (a_{0s}^{exp})^{-1} h_1(\beta) - h_0(\beta). (4.32)
\]

Substituting the above equation into Eq. (4.16) with \( r_0 \equiv r_{0s}^{exp} \) we find:

\[
r_{0s}^{exp} = \frac{2}{h_1(\beta)} [(a_{0s}^{exp})^{-1} h_3(\beta) - h_2(\beta)]. (4.33)
\]

Solving nonlinear Eq. (4.33), we find the value of \( \beta \), and then, using Eq. (4.32), we can find the value of \( \lambda \). We show the result at Table. 5.
4.1.2 Formalism in a coupled problems

For coupled channel problem, we have coupled channel equations which are expressed by the partial wave amplitudes. For given $J$, two channel of different $L = J \pm 1$ is possible, ignoring the $\rho$-spin. In this case, we have $2 \times 2$ $T$-matrix. For $L = J$, it is sufficient to consider a single channel problem, as we have done in the previous section. In coupled channel problem, BS equation we should solve is expressed as

$$t_{L/L'}(k_0', |k'|, k_0, |k|; s) = v_{L/L'}(k_0', |k'|, k_0, |k|; s) + \frac{i}{4\pi^3} \sum_{L''} \int dk_0'' \int |k''| d|k''| \frac{v_{L/L''}(k_0', |k'|, k_0, |k|; s)t_{L''/L}(k_0'', |k''|, k_0, |k|; s)}{(\sqrt{s}/2 - E_{k''} + i\epsilon)^2 - k_0''^2},$$

where $L, L', L''$ can be only $J - 1$ and $J + 1$ (for the $^3S_1, ^3D_1$ channel, 0 and 2).

Assuming the separable form for partial wave kernels: $T$-matrix can be expressed as

$$v_{L/L'}(k_0', |k'|, k_0, |k|; s) = \lambda g^{L'}(k_0', |k'|)g^L(k_0, |k|).$$

Then, we can solve Eq. (4.35) and write for $T$-matrix:

$$t_{L/L'}(k_0', |k'|, k_0, |k|; s) = \tau(s)g^{L'}(k_0', |k'|)g^L(k_0, |k|),$$

with function $\tau(s)$ being:

$$\tau(s) = 1/(\lambda^{-1} + h(s)),$$

and the function $h(s)$ is:

$$h(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \sum_L \frac{|g^L(k_0, |k|)|^2}{(\sqrt{s}/2 - E_{k} + i\epsilon)^2 - k_0^2}.$$

In the case of $^3S_1$-$^3D_1$ channel we can write:

$$h(s) = \sum_{L=S,D} h_{L}(s) = h_{S}(s) + h_{D}(s).$$

Below we will use index $S$ for $L = J - 1$ wave and $D$ for $L = J + 1$ wave.

On the other hand, $T$-matrix for coupled case can be written in the following form:

$$t(s) \equiv t(0, 0, 0, 0; s) = \frac{16\pi}{\sqrt{s}\sqrt{s - 4m^2}} \left( \begin{array}{cc} \cos 2\epsilon e^{2i\delta_S} - 1 & i\sin 2\epsilon e^{i(\delta_S + \delta_D)} \\ i\sin 2\epsilon e^{i(\delta_S + \delta_D)} & \cos 2\epsilon e^{2i\delta_D} - 1 \end{array} \right),$$

where we introduced $\delta_S, \delta_D$ and $\epsilon$ being phases shift in $^3S_1, ^3D_1$ channel and the mixing parameter $\epsilon$ and $\bar{p} = \sqrt{s/4 - m^2}$.

At the mass of the bound state $M_b$, $T$-matrix has a simple pole at the total momentum squared $s$, and the bound state condition can be written in the following form:

$$t(k_0', |k'|, k_0, |k|; s) = \frac{B(k_0', |k'|, k_0, |k|; s = M_b^2)}{s - M_b^2} + R(k_0', |k'|, k_0, |k|; s),$$

where functions $B$ and $R$ are regular at the point $s = M_b^2$ and corresponds to the Eq. (2.13). The bound state energy $E_b$ is connected to $M_b$ as: $M_b = 2m - E_b$. The bound state condition Eq. (4.41), with the help of Eq. (4.36) and Eq. (4.37), can be presented in a form:

Bound state, if it exists, is described by the simple pole in $T$ matrix, namely,

$$\lambda^{-1} = -h_{S}(s = M_b^2) - h_{D}(s = M_b^2).$$
Low-energy parameters

Now, we can introduce the low-energy parameters for $^3S_1$ wave, namely, $a_t$ and $r_t$ by the following equation:

$$\bar{p} \cot \delta_S(s) = -a_t^{-1} + \frac{r_t}{2} \bar{p}^2 + \mathcal{O}(\bar{p}^3).$$ (4.43)

$S$ matrix for the triplet case has the following representation:

$$S = 1 + i \alpha t(s) = \begin{pmatrix} \cos 2\epsilon e^{2i\delta_S} & i \sin 2\epsilon e^{i(\delta_S + \delta_D)} \\ i \sin 2\epsilon e^{i(\delta_S + \delta_D)} & \cos 2\epsilon e^{2i\delta_D} \end{pmatrix},$$ (4.44)

with $\alpha = -\sqrt{s}\sqrt{s - 4m^2}/16\pi$. From Eq. (4.36), we can write

$$t_S(s) \equiv t/ss(0, \bar{p}, 0, \bar{p}; s) = t_{Sr}(s) + it_{Si}(s),$$ (4.45)

$$t_D(s) \equiv t_{DD}(0, \bar{p}, 0, \bar{p}; s) = t_{Dr}(s) + it_{Di}(s),$$ (4.46)

$$t_{Sr} = \frac{n_SD_r}{|d|^2}, \quad t_{Si} = -\frac{n_SD_i}{|d|^2},$$ (4.47)

$$t_{Dr} = \frac{n_DD_r}{|d|^2}, \quad t_{Di} = -\frac{n_DD_i}{|d|^2},$$ (4.48)

$$n_S = [g^s](0, \bar{p})^2, \quad n_D = [g^D](0, \bar{p})^2,$$ (4.49)

$$d_r = \lambda^{-1} + h_{Sr} + h_{Dr}, \quad d_i = h_{Si} + h_{Di}.\quad (4.50)$$

From Eqs. (4.10) and (4.44)-(4.50), we see that

$$\cot 2\delta_S = \frac{S_{Sr}}{S_{Si}} = \frac{1 - \alpha t_{Si}}{\alpha t_{Sr}} = \frac{\alpha^{-1} - t_{Si}}{t_{Sr}} = \frac{\alpha^{-1}|d|^2 + n_SD_i}{n_SD_r} = \frac{2h_{Si}d_i - |d|^2}{2h_{Si}d_r}.\quad (4.51)$$

Since we are interesting in an elastic scattering only and taking into account only $h^e_{0i}$ and $h^e_{2i}$, it is easy to show that:

$$\alpha = -\frac{2h_{Si}}{n_S} = -\frac{2h_{Di}}{n_D} = -\frac{1}{16\pi} \sqrt{s}\sqrt{s - 4m^2}$$ (4.52)

Substituting Eq. (4.52) into Eq. (4.51) and taking into account Eq. (4.50) and the following decompositions into series on $\bar{p}$:

$$h_{Sr}(s) = h_{S0} + \bar{p}^2h_{S2} + \mathcal{O}(\bar{p}^3), \quad h_{Dr}(s) = h_{D0} + \bar{p}^2h_{D2} + \mathcal{O}(\bar{p}^3),$$ (4.53)

$$h_{Si}(s) = \bar{p}(h_{S1} + \bar{p}^2h_{S3} + \mathcal{O}(\bar{p}^3)), \quad h_{Di}(s) = \bar{p}(h_{D1} + \bar{p}^2h_{D3} + \mathcal{O}(\bar{p}^3)),$$ (4.54)

we can obtain

$$\bar{p} \cot 2\delta_S = -\frac{\lambda^{-1} + h_{S0} + h_{D0}}{2h_{S1}}$$

$$- \frac{1}{2h_{S1}} \left[ (h_{S1}(h_{S2} + h_{D2}) - h_{S3}(\lambda^{-1} + h_{S0} + h_{D0})) \frac{1}{h_{S1}} \right]$$

$$- (h_{S1}^2 - h_{D1}^2) \frac{1}{\lambda^{-1} + h_{S0} + h_{D0}} \bar{p}^2 + \mathcal{O}(\bar{p}^3).$$ (4.55)
On the other hand, we have:
\[
\bar{p} \cot 2\delta_S = \frac{\bar{p} \cot \delta_S^2 - 1}{2 \cot \delta_S} = -\frac{1}{2a_S} + \frac{1}{4}(2a_S + r_S) + \mathcal{O}(\bar{p}^3). \tag{4.56}
\]

Here we used the decomposition Eq. (4.43). Comparing Eqs. (4.55) and (4.56), we can obtain:
\[
a_s = \frac{h_{s1}}{\lambda^{-1} + h_{s0} + h_{D0}}, \tag{4.57}
\]
\[
r_s = 2\left[\left((\lambda^{-1} + h_{s0} + h_{D0})\frac{h_{s3}}{h_{s1}} - h_{s2} - h_{D2}\right)\frac{1}{h_{s1}} - \frac{h_{D1}^2}{h_{s1}(\lambda^{-1} + h_{s0} + h_{D0})}\right]. \tag{4.58}
\]

The Yamaguchi type of functions
We use a covariant generalization of the Yamaguchi type functions [18] for \(g^r(k_0, |\mathbf{k}|)\):
\[
g^S(k_0, |\mathbf{k}|) = \frac{1}{k_0^2 - |\mathbf{k}|^2 - \beta_0^2 + i\epsilon}, \tag{4.59}
\]
\[
g^P(k_0, |\mathbf{k}|) = \frac{\sqrt{-k_0^2 + |\mathbf{k}|^2}}{(k_0^2 - |\mathbf{k}|^2 - \beta_1^2 + i\epsilon)^2}, \tag{4.60}
\]
\[
g^D(k_0, |\mathbf{k}|) = \frac{C(k_0^2 - |\mathbf{k}|^2)}{(k_0^2 - |\mathbf{k}|^2 - \beta_2^2 + i\epsilon)^2}, \tag{4.61}
\]
where \(\beta_0, \beta_1, \beta_2\) and \(C\) are parameters. These equations are obtained by the covariantization \(|\mathbf{k}|^2 \rightarrow -k_0^2 + |\mathbf{k}|^2\) of the non-relativistic Yamaguchi type function,
\[
g^S(|\mathbf{k}|) = \frac{1}{|\mathbf{k}|^2 + \beta_0^2 + i\epsilon}, \tag{4.62}
\]
\[
g^P(|\mathbf{k}|) = \frac{|\mathbf{k}|}{(|\mathbf{k}|^2 + \beta_1^2 + i\epsilon)^2}, \tag{4.63}
\]
\[
g^D(|\mathbf{k}|) = \frac{C(|\mathbf{k}|^2)}{(|\mathbf{k}|^2 + \beta_2^2 + i\epsilon)^2}. \tag{4.64}
\]
The numerator of Eq. (4.63) express the \(p\)-wave character, that is, \(|\mathbf{k}| = |\mathbf{k}|^{l=1}\). The numerator of Eq. (4.64) express the \(d\)-wave character, that is, \(|\mathbf{k}| = |\mathbf{k}|^{l=2}\).

In this case, the functions \(h_0(s)\) and \(h_2(s)\) could be written in the following form:
\[
h_s(s, \beta_0) \equiv -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2 d|\mathbf{k}| \frac{1}{(\sqrt{s}/2 - E_k^1 + i\epsilon)^2 - k_0^2} \frac{1}{(k_0^2 - E_{\beta_0}^2 + i\epsilon)^2}, \tag{4.65}
\]
\[
h_p(s, \beta_1) \equiv -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2 d|\mathbf{k}| \frac{1}{(\sqrt{s}/2 - E_k^1 + i\epsilon)^2 - k_0^2} \frac{(-k_0^2 + |\mathbf{k}|^2)}{(k_0^2 - E_{\beta_1}^2 + i\epsilon)^4}. \tag{4.66}
\]
where \( E_{\beta_i} = \sqrt{|k|^2 + \beta_i^2} \) (\( i = 0, 1, 2 \)). We should note that Eq. (4.65) equals Eq. (4.19). We know the analytic properties of the function \( h_0(s) \). Therefore in order to calculate functions, \( h_P(s, \beta_1) \) and \( h_D(s, \beta_2) \) can be expressed as

\[
h_P(s, \beta_1) = -\frac{1}{2} \left[ \partial_{\beta_1}^2 + \frac{1}{3} \beta_1^2 \partial_{\beta_1}^2 \right] h_S(s, \beta_1), \tag{4.68}
\]

\[
h_D(s, \beta_2) = C^2 \left[ 1 + \beta_2^2 \partial_{\beta_2}^2 + \frac{1}{6} \beta_2^4 \partial_{\beta_2}^2 \right] h_S(s, \beta_2), \tag{4.69}
\]

because

\[
\begin{align*}
-k_0^2 + |k|^2 & \equiv -\frac{1}{(k_0^2 - |k|^2 - \beta_1^2)^4} - \frac{\beta_1^2}{(k_0^2 - |k|^2 - \beta_1^2)^4} \\
&= -\frac{1}{2} \left[ \partial_{\beta_1}^2 + \frac{1}{3} \beta_1^2 \partial_{\beta_1}^2 \right] \frac{1}{(k_0^2 - |k|^2 - \beta_1^2)^4}, \tag{4.70}
\end{align*}
\]

\[
\begin{align*}
\frac{(k_0^2 - |k|^2)^2}{(k_0^2 - |k|^2 - \beta_1^2)^4} & \equiv \frac{1}{(k_0^2 - |k|^2 - \beta_1^2)^4} + \frac{2\beta_1^2}{(k_0^2 - |k|^2 - \beta_1^2)^4} + \frac{\beta_1^4}{(k_0^2 - |k|^2 - \beta_1^2)^4} \\
&= \left[ 1 + \beta_2^2 \partial_{\beta_2}^2 + \frac{1}{6} \beta_2^4 \partial_{\beta_2}^2 \right] \frac{1}{(k_0^2 - |k|^2 - \beta_2^2)^2}. \tag{4.71}
\end{align*}
\]

**Normalization condition**

Normalization condition for \( ^3S_1 - ^3D_1 \) waves has the form:

\[
\int dk_0 \int |k|^2 d|k| (E_k - \frac{M}{2})(|\phi_S|^2 + |\phi_D|^2) = N, \tag{4.72}
\]

which is the special case when we consider the \( ^3S_1^+ \) and \( ^3D_1^+ \) state only in the discussion of the appendix D. Introducing the “pseudo-probabilities” \( p_S \) and \( p_D \) as following:

\[
\frac{1}{N} \int dk_0 \int |k|^2 d|k| (E_k - \frac{M}{2})|\phi_L|^2 = p_L, \tag{4.73}
\]

we can rewrite Eq. (4.72) in the form:

\[
 p_S + p_D = 1. \tag{4.74}
\]

Below we introduced the radial part of the BS amplitude \( \phi \) by the following equation:

\[
\phi_L(k_0, |k|) = g^{[L]}(k_0, |k|)/((M/2 - E_k)^2 - k_0^2), \tag{4.75}
\]

which corresponds to Eq. (2.32).

It is easy to connect Eq. (4.73) with functions Eq. (4.65) and Eq. (4.67), namely:

\[
p_L = \frac{1}{N} \int dk_0 \int |k|^2 d|k| (E_k - \frac{M}{2}) \frac{|g^{[L]}(k_0, |k|)|^2}{((M/2 - E_k)^2 - k_0^2)^2} \\
= \frac{1}{2N} \frac{d}{d\sqrt{s}} \int_{s=M^2} dk_0 \int |k|^2 d|k| \frac{|g^{[L]}(k_0, |k|)|^2}{(\sqrt{s/2} - E_k)^2 - k_0^2} \\
= i\frac{2\pi^3}{N} \frac{d}{d\sqrt{s}} \int_{s=M^2} h_L(s) \equiv h'_L(s) (L = S, D). \tag{4.76}
\]
Determination of the parameters

For a coupled channel problem, we have four parameters to determine, namely, $\lambda$, $\beta_0$, $\beta_2$ and $C$. In order to determine them, we use the following physical equations:

1. Eq. (4.42) determines the parameter $\lambda$ for the experimental value of a deuteron binding energy $M = (2m - E^{exp}_D)$:

   $$
   \lambda^{-1} = -h_S(s = M^2, \beta_0) - h_D(s = M^2, \beta_2) = 
   -h_S(s = M^2, \beta_0) - C^2\tilde{h}_D(s = M^2, \beta_2),
   $$

   (4.77)

   where $\tilde{h}_D(s, \beta_2) = \left[1 + \beta_2^2 \frac{\partial}{\partial \beta_2} + \frac{1}{6} \beta_2^4 \frac{\partial^2}{\partial \beta_2^2}\right] h_S(s, \beta_2)$ according with Eq. (4.69).

2. The “pseudo-probability” of $p_D$ can be expressed in the following form: $p_D = h'_D/(h'_S + h'_D)$. Taking into account Eq. (4.69), we can express parameter $C$ in the term of $p_D$:

   $$
   C^2 = \frac{p_D h'_S}{1 - p_D h'_D},
   $$

   (4.78)

3. In order to obtain the parameters $\beta_0$ and $\beta_2$, we use Eq. (4.58) with the experimental values for $a_t$ and $r_t$ and fixed $\lambda$ and $C^2$ from the previous two items and solve two nonlinear equations:

   $$
   a_t^{exp} = \frac{h_{S1}}{\lambda^{-1} + h_{S0} + C^2\tilde{h}_{D0}},
   $$

   (4.79)

   $$
   r_t^{exp} = 2 \left[ \left( (\lambda^{-1} + h_{S0} + C^2\tilde{h}_{D0}) \frac{h_{S3}}{h_{S1}} - h_{S2} - C^2\tilde{h}_{D2} \right) \frac{1}{h_{S1}} 
   - \frac{C^2\tilde{h}_{D1}^2}{h_{S1}(\lambda^{-1} + h_{S0} + C^2\tilde{h}_{D0})} \right].
   $$

   (4.80)

Now we can calculate the internal parameters of the $NN$ kernel by using the above equations to reproduce the experimental values for the phase shifts (data are from SAID program http://gwdac.phys.gwu.edu/), deuteron energy and quadrupole moment, low-energy parameters (data are from [28]).

### 4.1.3 Calculation

1. In order to find the parameters $\lambda$, $\beta_0$, $\beta_2$ and $C$ in $^3S_1$ coupled channel, we solve the nonlinear equations ($t$ stands for triplet):

   $$
   a_t^{exp} = a_t(\lambda, \beta_0, \beta_2, C), \quad E^{exp}_d = r_0(\lambda, \beta_0, \beta_2, C),
   $$

   (4.81)

   $$
   p_d = p_d(\lambda, \beta_0, \beta_2, C), \quad q_d^{exp} = q_d(\lambda, \beta_0, \beta_2, C).
   $$
Table 5: Parameters for $^1S_0$ and $^3S_1$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$^1S_0$</th>
<th>$^3S_1$ (pd = 4%)</th>
<th>$^3S_1$ (pd = 5%)</th>
<th>$^3S_1$ (pd = 6%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ (GeV$^2$)</td>
<td>-0.285549</td>
<td>-0.502690</td>
<td>-0.429637</td>
<td>-0.364905</td>
</tr>
<tr>
<td>$\beta_0$ (GeV)</td>
<td>0.221858</td>
<td>0.251241</td>
<td>0.246706</td>
<td>0.242285</td>
</tr>
<tr>
<td>$\beta_2$ (GeV)</td>
<td>0.293994</td>
<td>0.324341</td>
<td>0.350007</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1.6471</td>
<td>2.4071</td>
<td>3.2735</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Parameters for $^3P_0$, $^1P_1$ and $^3P_1$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$^3P_0$ (n=3)</th>
<th>$^3P_0$ (n=5)</th>
<th>$^1P_1$ (n=4)</th>
<th>$^1P_1$ (n=5)</th>
<th>$^3P_1$ (n=7)</th>
<th>$^3P_1$ (n=9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ (GeV$^2$)</td>
<td>-0.029420</td>
<td>-0.016123</td>
<td>0.091535</td>
<td>0.19513</td>
<td>0.31263</td>
<td>0.65701</td>
</tr>
<tr>
<td>$\beta_1$ (GeV)</td>
<td>0.23833</td>
<td>0.21861</td>
<td>0.27673</td>
<td>0.30891</td>
<td>0.33890</td>
<td>0.38191</td>
</tr>
</tbody>
</table>

2. In order to find the parameters $\lambda$ and $\beta$ for $^3P_0$, $^1P_1$ and $^3P_1$ channels, we use a procedure to minimize the $\chi^2$ function:

$$\chi^2 = \sum_{i=1}^{n} \frac{(\delta^{\exp}(s_i) - \delta(s_i))^2}{(\Delta\delta^{\exp}(s_i))^2},$$

(4.82)

where $n$ is the number of the experimental points we use. Note that $\Delta\delta^{\exp}$ is the data error. By introducing $\Delta\delta^{\exp}$, we can fit the experimental data depending on its accuracy.

The results of the calculations are given in Tables 5 and 6 and Figs. 7, 8, 9 and 10.
4.1.4 Discussion

- Fig. 7 show that $^1S_0$ and $^3S_1$ phase shifts calculated from the rank I separable potential and the experimental data as function of $T_{lab}$, which is the kinetic energy in the laboratory frame. The red line represents for the $^1S_0$ phase shift calculated from the rank I separable potential, and blue line is experimental data. As shown in the Fig. 7, the $^1S_0$ phase shift begins from the 0 degree and rise at first, then turns to decrease at $T_{lab} \sim 5$ MeV. Up to $T_{lab} \sim 100$ MeV, we can reproduce the $^1S_0$ experimental value. However, we can not reproduce the phase shift at $T_{lab} > 100$ MeV. This is due to the property of the rank I separable potential, namely, a rank I separable potential can not reproduce the contribution of the attraction and the repulsion at the same time. In Fig. 7, the green line, dark blue line and pink line express $^3S_1$ phase shift calculated from the rank I separable potential in the case of $p_D = 4, 5$ and 6%, where $p_D$ express the probability of the $D$-wave function. The yellow line is the experimental data. As shown in the Fig. 7, all the phase shifts look almost the same and begin from the 180 degree and decrease, Up to $T_{lab} \sim 300$ MeV, it has good agreement with the experimental value. Although we can not reproduce the phase shift at $T_{lab} > 300$ MeV due to the same reason as in the case of $^1S_0$

- Fig. 8 show that $^3P_0$ phase shifts calculated from the rank I separable potential and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 3$ for Eq. (4.82) and $n = 5$ for the green line. The blue line is the experimental data. As shown in the Fig. 8, both phase shifts begin from the 0 degree and rise at first, then turn to decrease at $T_{lab} \sim 50$ MeV. As in the case of $^1S_0$, we can not reproduce the phase shift $T_{lab} \sim 100$ MeV

- Fig. 9 show that $^1P_1$ phase shifts calculated from the rank I separable potential and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 4$ for Eq. (4.82) and $n = 5$ for the green line. The blue line is the experimental data. As shown in the Fig. 9, both phase shifts can reproduce the phase shift up to $T_{lab} \sim 100$ MeV. They starts from 0 degree and decrease as the energy increase at first, and then increase and approach to the 0 degree again.

- Fig. 10 show that $^3P_1$ phase shifts calculated from the rank I separable potential and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 7$ for Eq. (4.82) and $n = 9$ for the green line. The blue line is the experimental data. As shown in the Fig. 10, both phase shifts can reproduce the phase shift up to $T_{lab} \sim 200$ MeV. They starts from 0 degree and decrease as the energy increase at first, and then turns to increase and approach to the 0 degree again.

Using rank I potential, we could reproduce the phase shift up to certain energy. In order to reproduce the phase shifts for higher energy, we need to improve the separable potential as we will discuss in the next section.
Figure 7: $^1S_0$ and $^3S_1$ phase shift.

Figure 8: $^3P_0$ phase shift.

Figure 9: $^1P_1$ phase shift.

Figure 10: $^3P_1$ phase shift.
4.2 Extended rank I Yamaguchi type potential

In this section, we change the form of the functions \( g^I(k_0, |k|) \) of rank I separable potential which we use in the previous section. We add one more term as follows

\[
g^I_k(k_0, |k|) = \sqrt{|-k_0^2 + |k|^2|} + C_{12}\sqrt{|(-k_0^2 + |k|^2)^3|}. \tag{4.83}
\]

Using these functions, we perform the same calculation for \( p \)-wave phase shifts as in the previous section.

In order to find the parameters \( \lambda, C_{12}, \beta_{11} \) and \( \beta_{12} \) in uncoupled \( ^3P_0, ^1P_1 \) and \( ^3P_1 \) channels, we use procedure to minimize the \( \chi^2 \) value:

\[
\chi^2 = \sum_{i=1}^{n} \frac{(\delta_{\text{exp}}(s_i) - \delta(s_i))^2}{(\Delta \delta_{\text{exp}}(s_i))^2}, \tag{4.84}
\]

The results of the calculations are given in Tables 7 and Figs. 11-13. We also show the results of the rank I separable potential discussed in the section 4.1 for comparison, where \( n \) is the number of the experimental points.

<table>
<thead>
<tr>
<th>( \lambda ) (GeV)</th>
<th>( C_{12} ) (GeV)</th>
<th>( \beta_{11} ) (GeV)</th>
<th>( \beta_{12} ) (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ^3P_0 )</td>
<td>( ^3P_0 )</td>
<td>( ^1P_1 )</td>
<td>( ^1P_1 )</td>
</tr>
<tr>
<td>n=4</td>
<td>n=5</td>
<td>n=6</td>
<td>n=8</td>
</tr>
<tr>
<td>-2.09</td>
<td>-0.0108</td>
<td>0.00416</td>
<td>0.008323</td>
</tr>
<tr>
<td>0.91</td>
<td>9.74</td>
<td>-25.7</td>
<td>-31.549</td>
</tr>
<tr>
<td>0.401</td>
<td>0.209</td>
<td>0.172</td>
<td>0.18716</td>
</tr>
<tr>
<td>0.31</td>
<td>0.495</td>
<td>0.388</td>
<td>0.44545</td>
</tr>
</tbody>
</table>

| \( ^3P_1 \)       | \( ^3P_1 \)       | \( ^3P_1 \)       |
| n=10              | n=10              | n=15              | n=19              |
| 0.00706           | 0.006861          | 0.0342            | 0.0632            |
| -28.92            | -22.7             | -24.3             | -26.7             |
| 0.183             | 0.200             | 0.235             | 0.250             |
| 0.428             | 0.409             | 0.503             | 0.535             |

Table 7: Parameters for \( ^3P_0, ^1P_1 \) and \( ^3P_1 \) channels.
Figure 11: $^3P_0$ phase shift.

Figure 12: $^1P_1$ phase shift.

Figure 13: $^3P_1$ phase shift.
4.2.1 Discussion

- Fig. 11 show that the \(3P_0\) phase shifts calculated from the extended rank I separable potential and the experimental data as function of \(T_{\text{lab}}\). The red line is for the case when we take \(n = 4\) for Eq. (4.84) and the green line \(n = 5\). And we show the blue line, which is \(3P_0\) phase shifts calculated by the rank I separable potential for comparison. The pink line is experimental data. As shown in the Fig. 11, we can reproduce the phase shift up to \(T_{\text{lab}} \sim 150\) MeV. It is shown that we can improve the result as in the case of rank I separable potential.

- Fig. 12 show that \(1P_1\) phase shift calculated from the extended rank I separable potential and the experimental data as function of \(T_{\text{lab}}\). The red line is for the case when we take \(n = 6\) for Eq. (4.84), the green line \(n = 8\) and dark blue line \(n = 10\). And we also show the phase shift rank I calculated from the rank I separable potential by pink line. The blue line is experimental data. As shown in the Fig. 12, we can reproduce the phase shift by the extended rank I separable potential up to \(T_{\text{lab}} \sim 600\) MeV, while up to \(T_{\text{lab}} \sim 100\) MeV in the case of the rank I separable potential.

- Fig. 13 show that \(3P_1\) phase shift calculated from the extended rank I separable potential and the experimental data as function of \(T_{\text{lab}}\). The red line is for the case when we take \(n = 10\) for Eq. (4.84), the green line \(n = 15\), the dark blue line \(n = 19\). And we also show the phase shift calculated from the rank I separable potential by the pink line. The blue line is experimental data. As shown in the Fig. 12, we can reproduce the phase shift by the extended rank I separable potential up to \(T_{\text{lab}} \sim 700\) MeV, while up to \(T_{\text{lab}} \sim 100\) MeV in the case of the rank I separable potential.

As a result, it is shown that the extended rank I separable potential improve the fit for phase shift in the case of \(1P_1, 3P_1\). On the other hand, in the case of \(3P_0\), we can find the little difference. It imply that we can not reproduce the phase shift which has repulsive and attractive property like \(3P_0\) phase shift whatever the type of function we choose. Therefore, we try to increase the rank of the separable potential in the section 4.4.
Table 8: Parameters for $^3P_0$, $^1P_1$ and $^3P_1$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$^3P_0$</th>
<th>$^3P_0$</th>
<th>$^1P_1$</th>
<th>$^1P_1$</th>
<th>$^3P_1$</th>
<th>$^3P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>n=3</td>
<td>n=5</td>
<td>n=4</td>
<td>n=5</td>
<td>n=7</td>
<td>n=9</td>
</tr>
<tr>
<td>$\lambda$ (GeV$^2$)</td>
<td>-0.029880</td>
<td>-0.016259</td>
<td>0.089406</td>
<td>0.188387</td>
<td>0.301376</td>
<td>0.621283</td>
</tr>
<tr>
<td>$\beta_1$ (GeV)</td>
<td>0.238550</td>
<td>0.218680</td>
<td>0.276340</td>
<td>0.308175</td>
<td>0.338114</td>
<td>0.380215</td>
</tr>
</tbody>
</table>

4.3 The another type separable potential

As we discussed in the section 2.5, we take the form of the $g$-function, so that it keeps angular momentum algebra here in this section. And we compare the results with those of section 4.1.

4.3.1 Formalism

We follow the same formalism as in the section 4.1 except the form of $g$-function. Here we use the following functions for $g^{[k]}(k_0, k)$ for $P$-wave:

$$g^{[P]}_1(k_0, |k|) = \frac{|k|}{(k_0^2 - |k|^2 - \beta_1^2 + i\epsilon)^2} \quad (4.85)$$

4.3.2 Calculation and result

We obtain the parameter sets which reproduce the phase shifts by the rank I separable potential using Eq. (4.85). All calculation methods are the same as those in the section 4.1. As a result, we obtain the following results as shown in the Table. 8 and Figs. 14, 15 and 16.

4.3.3 Discussion

- Fig. 14 show that $^3P_0$ phase shifts calculated from the rank I separable potential given by Eq. (4.85) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 3$ for Eq. (4.82) and $n = 5$ for the dark blue line. We show the results of subsection 4.1 by the green and pink line for comparison. The blue line is the experimental data. As shown in the Fig. 14, all results are almost same as those of subsection 4.1.

- Fig. 15 show that $^1P_1$ phase shifts calculated from the rank I separable potential given by Eq. (4.85) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 4$ for Eq. (4.82) and $n = 5$ for the green line. We show the results of subsection 4.1 by the green and pink line for comparison. The blue line is the experimental data. As shown in the Fig. 15, all results are almost same as those of subsection 4.1.

- Fig. 16 show that $^3P_1$ phase shifts calculated from the rank I separable potential given by Eq. (4.85) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 7$ for Eq. (4.82) and $n = 9$ for the green line. We
show the results of subsection 4.1 by the green and pink line for comparison. The blue line is the experimental data. As shown in the Fig. 15, all results are almost same as those of subsection 4.1.

Seeing the Table. 8, we can see that all parameter sets are almost the same as those of Table. 6. And also from Figs. 14, 15 and 16, we can see that the lines of the two cases is almost the same. These result shows that the differences of the denominator of the $g$-function between Eq. (4.85) and Eq. (4.60) does not affect the results largely.
Figure 14: $^3P_0$ phase shift.

Figure 15: $^1P_1$ phase shift

Figure 16: $^3P_1$ phase shift.
4.4 Rank II Yamaguchi type potential

In the previous section, we showed that both the rank I separable potential and the extended rank I separable potential can not reproduce the phase shift which has the repulsion and attraction simultaneously. In this section we introduce the rank II separable potential expecting that we improve the reproduction to the all kinds of phase shift.

4.4.1 Formalism

The BS equation we should solve is

\[ T_{L'L'}(k'_0', |k'_1|, k_0, |k|; s) = V_{L'L'}(k'_0', |k'_1|, k_0, |k|; s) + \frac{i}{4\pi^3} \sum \int dk''_0' \int |k''|^2 d|k''| \frac{V_{L'L''}(k''_0', |k'_1|, k''_0, |k''|; s) T_{L'L''}(k''_0, |k''|, k_0, |k|; s)}{(\sqrt{s'/2} - E_{k''} + i\epsilon)^2 - k''_0^2}. \] (4.86)

Here \( T \) is the partial-wave decomposed \( T \)-matrix and \( V \) is the kernel of the \( NN \) interaction. There is only one term in the sum for the single channel problem \( (L = J) \) and there are two terms for the coupled channel problem \( (L = J \pm 1) \).

Assuming that the rank II separable form for the partial-wave decomposed kernels of \( NN \) interactions

\[ V_{L'L'}(k'_0', |k'_1|, k_0, |k|; s) = \sum_{i,j=1}^{2} \lambda_{ij} g_i^L(k'_0', |k'_1|) g_j^L(k_0, |k|), \] (4.87)

we can solve Eq. (4.87) and write for the \( T \) matrix:

\[ T_{L'L'}(k'_0', |k'_1|, k_0, |k|; s) = \sum_{i,j=1}^{2} \tau_{ij}(s) g_i^L(k'_0', |k'_1|) g_j^L(k_0, |k|), \] (4.88)

with the function \( \tau(s) \) being:

\[ \tau^{-1}(s)_{ij} = \lambda_{ij}^{-1} + H_{ij}(s). \] (4.89)

Function \( H_{ij}(s) \) has the following form:

\[ H_{ij}(s) = \sum_{L} H_{ij}^L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \sum_{L} \frac{[g_i^L(k_0, |k|)[g_j^L(k_0, |k|)]}{(\sqrt{s/2} - E_{k} + i\epsilon)^2 - k_0^2}. \] (4.90)

As in the previous section we use the following equation for the on-mass-shell \( T \) matrix for the singlet (uncoupled triplet) case:

\[ T_{LL}(s) \equiv T_{LL}(0, \bar{p}, 0, \bar{p}, s) = -\frac{16\pi}{\sqrt{s} \sqrt{s - 4m^2}} e^{i\delta} \sin \delta, \] (4.91)

and for the coupled triplet case:

\[ T_{LL}(s) = \frac{8\pi}{\sqrt{s} \sqrt{s - 4m^2}} \begin{pmatrix} \cos 2\epsilon e^{2i\delta} - 1 & i \sin 2\epsilon e^{i(\delta + \delta)} \\ i \sin 2\epsilon e^{i(\delta - \delta)} & \cos 2\epsilon e^{2i\delta} - 1 \end{pmatrix}, \] (4.92)

with \( \bar{p} = \sqrt{s/4 - m^2} = \sqrt{mT_{lab}/2} \). Here, we introduced the phase shifts \( \delta \equiv \delta_{L=J}, \delta_\perp \equiv \delta_{L=J\mp 1} \) and mixing parameter \( \epsilon \).
4.4.2 $P$-wave case

For the $P$-wave channel, $T$-matrix can be expressed in the following way.

$$T_{LL}(k'_0, |k'|, k_0, |k|; s) = \sum_{i,j=1}^{2} t_{ij} = \sum_{i,j=1}^{2} \tau_{ij} g_i^P(k'_0, |k'|) g_j^P(k_0, |k|).$$  \hspace{1cm} (4.93)

And the on-mass-shell expression is

$$T_{LL}(s) = \frac{n(s)}{d(s)} = \sum_{i,j=1}^{2} n_{ij}(s) d_{ij}(s) = \sum_{i,j=1}^{2} \tau_{ij} [g_i^P(0, \bar{p}) g_j^P(0, \bar{p})].$$  \hspace{1cm} (4.94)

Here we assume the following condition

$$\text{Im}(n_{ij}(s)) = 0.$$  \hspace{1cm} (4.95)

Then the phase shifts can be expressed in the following way:

$$\cot \delta(s) = \frac{\text{Re}(T_{LL})}{\text{Im}(T_{LL})} = \frac{\sum_{i,j=1}^{2} \text{Re}(t_{ij})}{\sum_{i,j=1}^{2} \text{Im}(t_{ij})},$$  \hspace{1cm} (4.96)

where the real and imaginary part of $t_{ij}(s)$ can be expressed in the following way respectively,

$$\text{Re}(t_{ij}) = \text{Re}\left[\tau_{ij} g_i^L(0, \bar{p}) g_j^L(0, \bar{p})\right]$$
$$= \left[g_i^{[L]}(0, \bar{p}) g_j^{[L]}(0, \bar{p})\right] \text{Re}(\tau_{ij})$$
$$= \left[g_i^{[L]}(0, \bar{p}) g_j^{[L]}(0, \bar{p})\right]$$
$$\times \left((\lambda^{-1})_{ij} + \text{Re}(H_{ij})\right) + \text{Im}(H_{ij}) \left((\lambda^{-1})_{ij} + \text{Re}(H_{ij})\right)^{-1} \text{Im}(H_{ij})^{-1},$$

$$\text{Im}(t_{ij}) = \text{Im}\left[\tau_{ij} g_i^L(0, \bar{p}) g_j^L(0, \bar{p})\right]$$
$$= \left[g_i^{[L]}(0, \bar{p}) g_j^{[L]}(0, \bar{p})\right] \text{Im}(\tau_{ij})$$
$$= \left[g_i^{[L]}(0, \bar{p}) g_j^{[L]}(0, \bar{p})\right]$$
$$\times \left((\text{Im}(H_{ij}) + (\lambda^{-1})_{ij} + \text{Re}(H_{ij})\right) \left(\text{Im}(H_{ij})\right)^{-1} \left((\lambda^{-1})_{ij} + \text{Re}(H_{ij})\right)^{-1}.$$

We use the simple covariantization of the Yamaguchi type functions for $g^P(k_0, |k|)$ as in the previous section,

$$g_1^P(k_0, |k|) = \frac{\sqrt{|-k_0^2 + |k|^2|}}{(k_0^2 - |k|^2 - \beta^2_1 + i\epsilon)^2}$$  \hspace{1cm} (4.99)

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\[ g^P_2(k_0, |k|) = \frac{C \sqrt{-k_0^2 + |k|^2} |k|^3}{(k_0^2 - |k|^2 \beta_2^2 + i\epsilon)^3}. \] (4.100)

Let us define the functions \( h_0(s) \) and \( h_{00}(s) \) in the following way:

\[
h_0(s, \beta) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k} + i\epsilon)^2 - k_0^2} \frac{1}{(k_0^2 - E_\beta^2 + i\epsilon)^2},
\]

here \( E_\beta = \sqrt{|k|^2 + \beta^2} \).

\[
h_{00}(s, \beta_1, \beta_2) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k} + i\epsilon)^2 - k_0^2} \frac{1}{k_0^2 - |k|^2 - \beta_2^2 + i\epsilon} \times \frac{1}{k_0^2 - |k|^2 - \beta_2^2 + i\epsilon}.
\]

Using \( h_0(s, \beta) \) and \( h_{00}(s, \beta_1, \beta_2) \), we can express the following equations for the function \( H_{ij} \) which was defined in the Eq. (4.90):

\[
H^{[P]}_{11}(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \left[ h_0(k_0, |k|) \right]^2
\]

\[
= -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \frac{k_0^2 - |k|^2}{(k_0^2 - |k|^2 - \beta_2^2)^3}.
\]

\[
= -\frac{1}{2\beta_1^2} \left[ D^1 + \frac{1}{3} D^2 \right] h_0(s, \beta_1)
\]

\[
H^{[P]}_{12}(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \left[ g_1^{[P]}(k_0, |k|) g_2^{[P]}(k_0, |k|) \right]
\]

\[
= -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \frac{\sqrt{k_0^2 - |k|^2}}{C(k_0^2 - |k|^2)^3}
\]

\[
= \frac{C}{\beta_2^2} \left[ D^{01} + \frac{1}{2} D^{02} + D^{11} + \frac{1}{2} D^{12} \right] h_{00}(s, \beta_1, \beta_2)
\]

\[
H^{[P]}_{22}(s) = H^{[P]}_{12}(s)
\]

\[
H^{[P]}_{22}(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \left[ g_1^{[P]}(k_0, |k|) \right]^2
\]

\[
= -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2 - E_k}^2 - k_0^2)} \frac{C \sqrt{k_0^2 - |k|^2}}{(k_0^2 - |k|^2 - \beta_2^2)^3}
\]

\[
= -\frac{C^2}{2\beta_2^2} \left[ D^1 + D^2 + \frac{1}{4} D^3 + \frac{1}{60} D^4 \right] h_0(s, \beta_2)
\]

where notations for differentiation were introduced:

\[
D^n f(\beta) \equiv \beta^{2n} \partial^{(n)}_{\beta^2} f(\beta),
\]

\[
D^{mn} f(\beta_1, \beta_2) \equiv \beta_1^{2n} \beta_2^{2m} \partial^{(n)}_{\beta_1^2} \partial^{(m)}_{\beta_2^2} f(\beta_1, \beta_2),
\]

\[
D^0 f(\beta) \equiv f(\beta).
\]
4.4.3 \(^3S_1, \, ^1S_0\) channel

In this case, \(T\)-matrix can be expressed in the following way.

\[
T_{00}(k'_0, |k'|, k_0, |k|; s) = \sum_{i,j=1}^{2} t_{ij} = \sum_{i,j=1}^{2} \tau_{ij} g_{i}^{S}(k'_0, |k'|) g_{j}^{S}(k_0, |k|),
\]

(4.109)

and the on-mass-shell expression is:

\[
T_{00}(s) = \frac{n(s)}{d(s)} = \sum_{i,j=1}^{2} \frac{n_{ij}(s)}{d_{ij}(s)} = \sum_{i,j=1}^{2} \tau_{ij} g_{i}^{S}(0, \bar{p}) g_{j}^{S}(0, \bar{p})
\]

(4.110)

Here we assume the following condition,

\[
I m(n_{ij}(s)) = 0.
\]

(4.111)

Then, the phase shift can be expressed in the following way

\[
cot\delta(s) = -\sum_{i,j=1}^{2} \left[ g_{i}^{S}(0, \bar{p}) g_{j}^{S}(0, \bar{p}) \right] \times \left( (\lambda^{-1})_{ij} + Re(H_{ij}) \right) + I m(H_{ij}) \left( (\lambda^{-1})_{ij} + Re(H_{ij}) \right)^{-1} I m(H_{ij})^{-1}
\]

\[
\times \left( I m(H_{ij}) + \frac{\sum_{i,j=1}^{2} \left[ g_{i}^{S}(0, \bar{p}) g_{j}^{S}(0, \bar{p}) \right]}{\sum_{i,j=1}^{2} \left( (\lambda^{-1})_{ij} + Re(H_{ij}) \right) \left( I m(H_{ij}) \right)^{-1} \left( (\lambda^{-1})_{ij} + Re(H_{ij}) \right)^{-1}} \right)
\]

(4.112)

We use the general covariantization of the Yamaguchi type functions for \(g^{[L]}(k_0, |k|)\) as in the previous section:

\[
g_{1}^{S}(k_0, |k|) = \frac{1}{(k_0^2 - |k|^2 - \beta_1^2 + i\epsilon)}
\]

(4.113)

\[
g_{2}^{S}(k_0, |k|) = \frac{C(k_0^2 - |k|^2)}{(k_0^2 - |k|^2 - \beta_2^2 + i\epsilon)^2},
\]

(4.114)

Using \(h_0(s, \beta)\) and \(h_{00}(s, \beta_1, \beta_2)\) which was define Eq. (4.101), we can obtain the following equations for function \(H_{ij}\) which was defined by Eq. (4.90),

\[
H_{11}^{[S]}(s) = -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \left[ \frac{1}{(\sqrt{s/2 - E_k} k_0^2 - k_0^2)} \right] [g_{1}^{S}(k_0, |k|)]^2
\]

\[
= -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \left[ \frac{1}{(\sqrt{s/2 - E_k} k_0^2 - k_0^2)} \frac{1}{(k_0^2 - |k|^2)^2} \right]
\]

\[
= h_0(s, \beta_1),
\]

(4.115)

\[
H_{12}^{[S]}(s) = -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \left[ \frac{1}{(\sqrt{s/2 - E_k} k_0^2 - k_0^2)} \right] g_{1}^{S}(k_0, |k|) g_{2}^{S}(k_0, |k|)
\]

\[
= -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \left[ \frac{1}{(\sqrt{s/2 - E_k} k_0^2 - k_0^2)} \frac{1}{(k_0^2 - |k|^2)^2} \frac{C(k_0^2 - |k|^2)}{(k_0^2 - |k|^2 - \beta_2^2)^2} \right]
\]

\[
= C [1 + D^{01}] h_{00}(s, \beta_1 \beta_2),
\]

(4.116)
\[ H_{21}^{[S]}(s) = H_{12}^{[S]}(s), \quad (4.117) \]

\[ H_{22}^{[S]}(s) = -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2-E}k_p^2-k_0^2)} [g_2^{[S]}(k_0, |k|)]^2 \]

\[ = -\frac{i}{4\pi} \int dk_0 \int |k|^2 d|k| \frac{1}{(\sqrt{s/2-E}k_p^2-k_0^2)} \frac{C^2(k_0^2-|k|^2)^2}{(k_0^2-|k|^2)^2} \]

\[ = C^2 \left[ 1 + D^1 + \frac{1}{6} D^2 \right] h_0(s, \beta_2), \quad (4.118) \]

where the notations for differentiation were introduced:

\[ D^n f(\beta) \equiv \beta^n \partial_{\beta^2} f(\beta), \quad (4.119) \]

\[ D^m f(\beta_1, \beta_2) \equiv \beta_1^n \partial_{\beta_1^2} \beta_2^m \partial_{\beta_2^2} f(\beta_1, \beta_2), \quad (4.120) \]

\[ D^0 f(\beta) \equiv f(\beta). \quad (4.121) \]

### Low energy parameter

At first, we try to expand g-functions by \( \bar{p} \).

\[ g_1^{[S]}(0, \bar{p}) = \frac{1}{(-\bar{p}^2 - \beta_1^2)} \approx -\frac{1}{\beta_1^2} (1 - \frac{\bar{p}^2}{\beta_1^2}) + O(\bar{p}^3) \quad (4.122) \]

\[ g_2^{[S]}(0, \bar{p}) = \frac{(-\bar{p}^2)}{(-\bar{p}^2 - \beta_2^2)^2} = -\frac{\bar{p}}{\beta_2^2} \frac{1}{1 + \frac{\bar{p}^2}{\beta_2^2}} \approx -\frac{\bar{p}^2}{\beta_2^2} + O(\bar{p}^4) \quad (4.123) \]

Therefore, T-matrix can be expressed in the following way,

\[ T_{00}(s) = \tau_{11}(s) g_1^{[S]}(0, \bar{p}) g_1^{[S]}(0, \bar{p}) + \tau_{12}(s) g_1^{[S]}(0, \bar{p}) g_2^{[S]}(0, \bar{p}) + \tau_{21}(s) g_2^{[S]}(0, \bar{p}) g_1^{[S]}(0, \bar{p}) + \tau_{22}(s) g_2^{[S]}(0, \bar{p}) g_2^{[S]}(0, \bar{p}) \]

\[ \approx \tau_{11}(s) \left( -\frac{1}{\beta_1^2} (1 - \frac{\bar{p}^2}{\beta_1^2}) \right) + \tau_{12}(s) \left( \frac{\bar{p}^2}{\beta_2^2 \beta_1^2} \right) + \tau_{21}(s) \left( \frac{\bar{p}^2}{\beta_2^2 \beta_1^2} \right) + O(\bar{p}^4) \]

\[ = -\frac{\tau_{11}(s)}{\beta_1^4} + \beta_2^2 \tau_{12}(s) + \beta_1^2 \tau_{21}(s) + O(\bar{p}^4)) \quad (4.124) \]

Next we expand \( \tau \) matrix in the following way.

\[ \tau_{11}(s) \approx (\tau_{11})_0 + i(\tau_{11})_1 \bar{p} + (\tau_{11})_2 \bar{p}^2 + O(\bar{p}^3). \quad (4.125) \]

Then

\[ \cot(\delta(s)) = \frac{Re(T)}{Im(T)} \approx \frac{Re(-\frac{\tau_{11}(s)}{\beta_1^2})}{Im(-\frac{\tau_{11}(s)}{\beta_1^2})} = \frac{Re(\tau_{11})}{Im(\tau_{11})} \approx \frac{(\tau_{11})_0 + (\tau_{11})_2 \bar{p}^2}{(\tau_{11})_1 \bar{p}} \quad (4.126) \]

\[ \bar{p} \cot(\delta(s)) = \frac{(\tau_{11})_0}{(\tau_{11})_1} + O(\bar{p}^3) \quad (4.127) \]

Therefore, the low energy parameter \( a_S \) can be expressed by

\[ a_S = -\frac{(\tau_{11})_1}{(\tau_{11})_0}. \quad (4.128) \]
On the other hand, $\tau$ is defined by

$$
\tau^{-1}(s)_{ij} = \lambda_{ij}^{-1} + H_{ij}(s).
$$

(4.129)

We can express $\tau(s)_{ij}$ elements by $\tau^{-1}(s)_{ij}$

$$
\tau_{ij}(s) = \frac{1}{\tau^{-1}(s)_{11}\tau^{-1}(s)_{22} - \tau^{-1}(s)_{21}\tau^{-1}(s)_{12}} \begin{pmatrix}
\tau^{-1}(s)_{22} & -\tau^{-1}(s)_{12} \\
-\tau^{-1}(s)_{21} & \tau^{-1}(s)_{11}
\end{pmatrix}.
$$

(4.130)

Thus $\tau_{11}$ can be expressed as

$$
\tau_{11}(s) = \frac{\tau^{-1}(s)_{22}}{\tau^{-1}(s)_{11}\tau^{-1}(s)_{22} - \tau^{-1}(s)_{21}\tau^{-1}(s)_{12}} \\
\approx \frac{(\tau_{22})_0 + i(\tau_{22})_1\bar{p} + (\tau_{22})_2p^2 + \mathcal{O}(p^3)}{\alpha_0 + ip\alpha_1 + \bar{p}^2} \\
\approx \frac{1}{\alpha_0}(\tau_{22}^{-1})_0 + ip\frac{1}{\alpha_0}(-\frac{\alpha_1}{\alpha_0}(\tau_{22}^{-1})_0 + (\tau_{22}^{-1})_1),
$$

(4.131)

where

$$
\alpha_0 \equiv (\tau^{-1}(s)_{11})_0(\tau^{-1}(s)_{22})_0 - (\tau^{-1}(s)_{21})_0(\tau^{-1}(s)_{12})_0 \\
= (\lambda_{11}^{-1} + (H_{11})_0)(\lambda_{22}^{-1} + (H_{22})_0) - (\lambda_{12}^{-1} + (H_{12})_0)^2,
$$

(4.132)

$$
\alpha_1 \equiv (\tau^{-1}(s)_{11})_1(\tau^{-1}(s)_{22})_0 + (\tau^{-1}(s)_{11})_0(\tau^{-1}(s)_{22})_0 \\
- (\tau^{-1}(s)_{21})_1(\tau^{-1}(s)_{12})_0 - (\tau^{-1}(s)_{21})_0(\tau^{-1}(s)_{12})_0 \\
= (H_{11})(\lambda_{22}^{-1} + (H_{22})_0) + (H_{22})(\lambda_{11}^{-1} + (H_{11})_0) - 2(\lambda_{12}^{-1} + (H_{12})_0)(H_{12})_1.
$$

(4.133)

Finally $a_s$ can be expressed

$$
a_s = \frac{\alpha_1}{\alpha_0} - \frac{(H_{22})_1}{\lambda_{22}^{-1} + (H_{22})_0}.
$$

(4.134)

### 4.4.4 $^3S_1$ state without D-wave

We consider $^3S_1$ state without coupling with $^3D_1$ state by imposing the bound condition. It is not usual way, because we need a $^3D_1$ state for bound state. However, it is possible to have a bound state in Bethe-Salpeter approach without $^3D_1$ state by considering $^3S_1$ bound state by using the $T$-matrix property as follows.

$$
\text{det}|\tau^{-1}(s)| = 0
$$

(4.135)

$$
\iff
$$

$$
(\tau^{-1}(s = M_6^2)_{11})(\tau^{-1}(s = M_6^2)_{22}) - (\tau^{-1}(s = M_6^2)_{21})(\tau^{-1}(s = M_6^2)_{12}) \\
= (\lambda_{11}^{-1} + H_{11}(s = M_6^2))(\lambda_{22}^{-1} + H_{22}(s = M_6^2)) - (\lambda_{12}^{-1} + H_{12}(s = M_6^2))^2 \\
= 0
$$

(4.136)
Table 9: The binding energy and low-energy parameters for $^3S_1$ and $^1S_0$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$a_{0s}$ (Fm)</th>
<th></th>
<th>$a_{0t}$ (Fm)</th>
<th>$E_d$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1S_0$</td>
<td></td>
<td>$^3S_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Calculated (5p)</td>
<td>-23.745</td>
<td>Calculated (5p)</td>
<td>5.419</td>
<td>2.224606</td>
</tr>
<tr>
<td>Calculated (4p)</td>
<td>-23.747</td>
<td>Calculated (4p)</td>
<td>5.411</td>
<td>2.224503</td>
</tr>
<tr>
<td>Experiment</td>
<td>-23.748±0.010</td>
<td>Experiment</td>
<td>5.424±0.004</td>
<td>2.224644±0.000046</td>
</tr>
</tbody>
</table>

4.4.5 Calculation

We calculate in the following ways:

1. In order to find the parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$ and $\lambda_{22}$ which reproduce the $^1S_0$ phase shift, we use the procedure to minimize the function,

$$
\chi^2 = \sum_{i=1}^{n} (\delta_{\text{exp}}(s_i) - \delta(s_i))^2/\Delta\delta_{\text{exp}}(s_i)^2 + (a_{0s}^{\text{exp}} - a_{0s}^{\text{cal}})^2/\Delta a_{0s}^{\text{exp}}^2 .
$$

(4.137)

2. In order to find the parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$ and $\lambda_{22}$ which reproduce the $^3S_1$ phase shift, we use the procedure to minimize the function,

$$
\chi^2 = \sum_{i=1}^{n} (\delta_{\text{exp}}(s_i) - \delta(s_i))^2/\Delta\delta_{\text{exp}}(s_i)^2 + (a_{0t}^{\text{exp}} - a_{0t}^{\text{cal}})^2/\Delta a_{0t}^{\text{exp}}^2

+ (E_d^{\text{exp}} - E_d^{\text{cal}})^2/\Delta E_d^{\text{exp}}^2 .
$$

(4.138)

3. In order to find the parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$ and $\lambda_{22}$ which reproduce the $^3P_0$, $^1P_1$ and $^3P_1$ phase shifts, we use procedure to minimize function,

$$
\chi^2 = \sum_{i=1}^{n} (\delta_{\text{exp}}(s_i) - \delta(s_i))^2/\Delta\delta_{\text{exp}}(s_i)^2 .
$$

(4.139)

Here $\delta_{\text{exp}}(s_i)$, $\delta(s_i)$ and $\Delta\delta_{\text{exp}}(s_i)$ is the experimental data, the theoretical value and the experimental error of the phase shift at the energy of $s_i$ respectively. $a_{0s}^{\text{exp}}$, $a_{0s}^{\text{cal}}$ and $\Delta a_{0s}^{\text{exp}}$ is the experimental data, the theoretical value and the experimental error of the low energy parameter for the energy $s_i$ respectively. $E_d^{\text{exp}}$, $E_d^{\text{cal}}$ and $\Delta E_d^{\text{exp}}$ is the experimental data, the theoretical value and the experimental error of the binding energy of the deuteron respectively.

4.4.6 Result

The results of the calculations are given in Tables 9, 10, 11 and Figs. 17, 18, 19, 20.
Table 10: Parameters for $^3S_1$, $^1S_0$, $^3P_0$, $^1P_1$ and $^3P_1$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$^3S_0$</th>
<th>$^3S_1$</th>
<th>$^3P_0$</th>
<th>$^1P_1$</th>
<th>$^3P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$ (GeV)</td>
<td>0.3849</td>
<td>0.3095</td>
<td>0.21425</td>
<td>0.37036</td>
<td>0.3741</td>
</tr>
<tr>
<td>$\beta_2$ (GeV)</td>
<td>0.6877</td>
<td>0.8376</td>
<td>0.51963</td>
<td>0.363609</td>
<td>0.40618</td>
</tr>
<tr>
<td>$\lambda_{11}$ (GeV$^2$)</td>
<td>0.866</td>
<td>-0.394</td>
<td>-0.0116239</td>
<td>0.9343</td>
<td>0.7125</td>
</tr>
<tr>
<td>$\lambda_{12}$ (GeV$^2$)</td>
<td>21.89</td>
<td>17.654</td>
<td>0.36589</td>
<td>1.8371</td>
<td>1.57199</td>
</tr>
<tr>
<td>$\lambda_{22}$ (GeV$^2$)</td>
<td>-7.5</td>
<td>21.982</td>
<td>43.887</td>
<td>6.9663</td>
<td>11.523</td>
</tr>
</tbody>
</table>

Table 11: Parameters for $^3S_1$, $^1S_0$, $^3P_0$, $^1P_1$ and $^3P_1$ channels.

<table>
<thead>
<tr>
<th></th>
<th>$^3S_0$</th>
<th>$^3S_1$</th>
<th>$^3P_0$</th>
<th>$^1P_1$</th>
<th>$^3P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$ (GeV)</td>
<td>0.2999</td>
<td>0.3217</td>
<td>0.231617</td>
<td>0.24083</td>
<td>0.28638</td>
</tr>
<tr>
<td>$\beta_2$ (GeV)</td>
<td>0.5066</td>
<td>0.418</td>
<td>0.574154</td>
<td>0.39692</td>
<td>0.43468</td>
</tr>
<tr>
<td>$\lambda_{11}$ (GeV$^2$)</td>
<td>-0.8371</td>
<td>-1.48</td>
<td>-0.0241143</td>
<td>0.033093</td>
<td>0.08919</td>
</tr>
<tr>
<td>$\lambda_{12}$ (GeV$^2$)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\lambda_{22}$ (GeV$^2$)</td>
<td>-40.8</td>
<td>8.229</td>
<td>108.465</td>
<td>5.0481</td>
<td>10.8589</td>
</tr>
</tbody>
</table>

4.4.7 Discussion

- Fig. 17 show that $^1S_0$ and $^3S_1$ phase shifts calculated from the rank II separable potential using the parameter sets determined by the Eq. (4.137) and the experimental data as function of $T_{lab}$, which is the kinetic energy in the laboratory frame. The red line and green line represents for the $^1S_0$ phase shift calculated from the rank II separable potential using the parameter sets shown in the Table. 10 and 11 respectively and dark blue is experimental data. As shown in the Fig. 17, the $^1S_0$ phase shift begins from the 0 degree and rise at first, then turns to decrease at $T_{lab} \sim 5$ MeV. Up to $T_{lab} \sim 600$ MeV, we can reproduce the $^1S_0$ phase shift experimental value, while we can not reproduce the phase shift which is more than $T_{lab} > 100$ MeV in the case of the rank I separable potential,

Similarly, the pink line and blue line represents for the $^3S_1$ phase shift calculated from the rank II separable potential using the parameter sets shown in the Table. 10 and 11 respectively and yellow line is experimental data. Up to $T_{lab} \sim 600$ MeV, we can reproduce the $^3S_1$ phase shift experimental value, while we can not reproduce the phase shift which is more than $T_{lab} > 100$ MeV in the case of the rank I separable potential,

These results show that the rank II separable potential can reproduce the contribution of the attraction and repulsion at the same time.

- Fig. 18 show that the $^3P_0$ phase shift calculated from the rank II separable potential using the parameter sets determined by the Eq. (4.139) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 15$ for Eq. (4.139) with 4 parameters and the green line for 5 parameters. The blue line is experimental
Figure 17: $^1S_0$ and $^3S_1$ phase shifts.

Figure 18: $^3P_0$ phase shifts.

Figure 19: $^1P_1$ phase shifts.

Figure 20: $^3P_1$ phase shifts.
data. As shown in the Fig. 18, we can reproduce the phase shift up to $T_{lab} \sim 600$ MeV.

- Fig. 19 show that $^1P_1$ phase shift calculated from the rank II separable potential using the parameter sets determined by the Eq. (4.139) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 12$ for Eq. (4.139) with 4 parameters and the green line for 5 parameters. The blue line is experimental data. As shown in the Fig. 19, we can reproduce the phase shift up to $T_{lab} \sim 600$ MeV.

- Fig. 20 show that $^1P_1$ phase shift calculated from the extended rank II separable potential using the parameter sets determined by the Eq. (4.139) and the experimental data as function of $T_{lab}$. The red line is for the case when we take $n = 15$ for Eq. (4.139) with 4 parameters and the green line 5 parameters. The blue line is experimental data. As shown in the Fig. 20, we can reproduce the phase shift up to $T_{lab} \sim 600$ MeV.

### 4.4.8 Conclusion

In this section, we have constructed the covariant rank II separable potentials for the nucleon-nucleon interaction in $^3S_1$, $^1S_0$, $^3P_1$, $^1P_1$, $^3P_0$ channels. As a result, we have found that the rank II Yamaguchi potentials are able to reproduce the deuteron static properties and the phase shifts up to $T_{lab} = 600 MeV$ while the rank I separable potential can reproduce up to the lower energy. This is because the rank II separable can reproduce the contribution of the attraction and repulsion at the same time. We can interpret this fact intuitively in the following way. If we set $\lambda_{12} = 0$, the potential $V$ can be expressed the following way.

$$ V_{L'L}(k'_0, |k'|, k_0, |k|; s) = \lambda_{11} g_{11}^{[L']}(k'_0, |k'|) g_{11}^{[L]}(k_0, |k|) $$

$$ + \lambda_{22} g_{22}^{[L']}(k'_0, |k'|) g_{22}^{[L]}(k_0, |k|), \quad (4.140) $$

Now we have to remember that $\lambda$ corresponds to a coupling constant and $\beta$ a mass parameter. We can easily understand that the first term play a role of repulsive force, while the second term attractive force from the sign of $\lambda$, when $\lambda > 0$ and $\lambda < 0$. In this way, a rank II separable potential can have the property of the attraction and the repulsion at the same time.
5 The effect of the negative component of two nucleon system.

5.1 Motivation

In previous sections, we investigated mostly properties of the $NN$-scattering by using the relativistic BS equation. In the BS approach, the effect of the LS-force and negative energy components are naturally included. So far, however, we have ignored the effect of negative energy components in the calculations for simplicity. Here in this section we would like to investigate the effect of the negative energy components on the two nucleon system.

Traditionally, the $NN$ interaction has been studied based on the Schrödinger equation with one boson exchange potential. In the non-relativistic framework, electron-deuteron scattering is expressed as shown in Fig. 21, where (a) is the impulse process and (b) process with exchange currents. On the other hand, if we consider the deuteron in a relativistic framework, electron-deuteron scattering can be expressed as shown in Fig. 22. In the relativistic impulse approximation (RIA), each nucleon line has both negative and positive energy states, in contrast with the non-relativistic approximation where each nucleon line contains only positive energy states. The right hand side of Fig. 22 shows the decomposition of the RIA diagram into several diagrams in the non-relativistic reduction, where the diagrams are depicted in the chronological order. One of those diagrams expresses the Z-graph whose nucleon line goes back as corresponding to the negative energy states. Therefore, we can interpret that Z-graph part corresponds to the exchange current as shown in Fig. 22 in the non-relativistic approach. It implies that a part of the dynamical effect of the inclusion of the negative energy components can be interpreted by the exchange currents.

For the relativistic description for the deuteron, we prepare the complete two-nucleon set including negative energy states of $P$-wave in addition to the $^3S_1$ and $^3D_1$ states of positive energy. We solve the Bethe-Salpeter equation by using the separable ansatz including the negative energy states, and determine the parameters to reproduce deuteron form factors.
Figure 21: Non-relativistic deuteron

Figure 22: Relativistic deuteron
5.2 Formalism

Let us start again with the Bethe-Salpeter Equation (BSE) for the NN $T$-matrix, Eq. (2.13):

$$T_{\alpha\beta,\delta\gamma}(P, p', p) = V_{\alpha\beta,\delta\gamma}(P, p', p) + i \int \frac{d^4k}{(2\pi)^4} V_{\alpha\beta,\delta\gamma}(P, p', k) S_{\epsilon\eta}(P/2 + k) S_{\lambda\rho}(P/2 - k) T_{\eta\rho,\delta\gamma}(P, k, p),$$  \hspace{1cm} (5.1)

where the Greek letters express the spinor indices. The bound state corresponds to a pole in the $T$-matrix at $P^2 = M_B^2$ ($M_B$ is the mass of the bound state):

$$T(P, p', p) = \frac{\Gamma(P, p') \Gamma(P, p)}{P^2 - M_B^2} + R(P, p', p),$$  \hspace{1cm} (5.2)

where $\Gamma_{\alpha\beta}$ is the vertex function of BSE, and $R_{\alpha\beta,\delta\gamma}$ is regular at $P^2 = M_B^2$. We can express the BS amplitude by the vertex function as:

$$\Phi(P, p) = S\left(\frac{P}{2} + p\right) S\left(\frac{P}{2} - p\right) \Gamma(P, p),$$  \hspace{1cm} (5.3)

and we obtain the equation for the BS amplitude from Eq. (5.1), (5.2) and (5.3):

$$\Phi(P, p) = i S\left(\frac{P}{2} + p\right) S\left(\frac{P}{2} - p\right) \int \frac{d^4k}{(2\pi)^4} V(P, p, k) \Phi(P, k)$$  \hspace{1cm} (5.4)

5.3 Solutions to the BSE

After partial wave decompositions as explained in section 2.2, the BS equation for the $T$-matrix is given by

$$T_{\alpha\beta}(p_0', |p'|, p_0, |p|; s) = V_{\alpha\beta}(p_0', |p'|, p_0, |p|; s) + \frac{i}{2\pi^2} \int d{k_0} k^2 d|k| \sum_{\gamma\delta} V_{\alpha\gamma}(p_0', |p'|, k_0, |k|; s) S_{\gamma\delta}(k_0, |k|; s) T_{\delta\beta}(k_0', |k|, p_0, |p|; s).$$

Here the indices of Greek character represent quantum numbers ($JLS_\rho$), and the summation takes over all partial waves. When we include both positive and negative energy states, we have eight states, namely, $^3S_1^+, ^3D_1^+, ^3S_1^-, ^3D_1^-, ^1P_e^1, ^1P_o^1, ^3P_e^1$ and $^3P_o^1$, which are numbered as $1, \ldots, 8$ as discussed in section 2.3. $T$, $V$ and $S$ are $8 \times 8$ matrices where $n$-th column and row corresponds to the $n$-th state. In this eight dimensional basis, the propagator $S_{\alpha\beta}$ is expressed by

$$S = \begin{pmatrix}
S_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S_+ & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_- & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & S_e & 0 & S_o & 0 \\
0 & 0 & 0 & 0 & 0 & S_e & 0 & S_o \\
0 & 0 & 0 & 0 & S_o & 0 & S_e & 0 \\
0 & 0 & 0 & 0 & S_o & 0 & S_e & 0
\end{pmatrix},$$  \hspace{1cm} (5.5)
where two nucleon propagators $S_\alpha$ are presented as

\[ S_+ = \frac{1}{(\sqrt{s}/2 - e_k + i\epsilon)^2 - k_0^2}, \]

\[ S_- = \frac{1}{(\sqrt{s}/2 + e_k - i\epsilon)^2 - k_0^2}, \]

\[ S_{+-} = \frac{1}{(\sqrt{s}/2 - e_k + k_0 + i\epsilon) (\sqrt{s}/2 + e_k - k_0 - i\epsilon)}, \]

\[ S_{-+} = \frac{1}{(\sqrt{s}/2 + e_k + k_0 - i\epsilon) (\sqrt{s}/2 - e_k - k_0 + i\epsilon)}, \]

\[ S_e = S_{ee} = S_{oo} = \frac{S_{+-} + S_{-+}}{2}, \]

\[ S_o = S_{eo} = S_{oe} = \frac{S_{+-} - S_{-+}}{2}. \]  

(5.6)

Now let us introduce the separable ansatz in the following manner:

\[ V_{\alpha\beta}(p_0', |p|, p_0, |p|; s) = \sum_{i,j=1}^{N} \lambda_{ij} g_i^{(a)}(p_0', |p'|) g_j^{(b)}(p_0, |p|), \quad \lambda_{ij} = \lambda_{ji}, \]  

(5.7)

\[ T_{\alpha\beta}(p_0', |p'|, p_0, |p|; s) = \sum_{i,j=1}^{N} \tau_{ij} g_i^{(a)}(p_0', |p'|) g_j^{(b)}(p_0, |p|), \]  

(5.8)

where we assume that $\lambda_{ij}$ is symmetric under the interchange of $i, j$ and

\[ (\tau^{-1}(s))_{ij} = (\lambda^{-1})_{ij} - H_{ij}(s). \]  

(5.9)

Then, the solution to the radial part of the BS equation can be written as

\[ \phi_{jLS\rho}(p_0, |p|) = \sum_{\rho'} S_{\rho\rho'}(p_0, |p|) g_{jLS\rho}(p_0, |p|) \]

\[ = \sum_{\rho'} \sum_{i,j=1}^{N} S_{\rho\rho'}(p_0, |p|; s) \lambda_{ij} g_i^{(jLS\rho)}(p_0, |p|) c_j(s), \]  

(5.10)

where $c_i(s)$ satisfy the following equation:

\[ c_i(s) - \sum_{k,j=1}^{N} H_{ik}(s) \lambda_{kj} c_j(s) = 0, \]  

(5.11)

\[ H_{ij}(s) = \frac{i}{2\pi^2} \sum_{LS\rho\rho'} \int d|k| d|k| S_{\rho\rho'}(k_0, |k|; s) g_i^{(jLS\rho)}(k_0, |k|) g_j^{(LS\rho')} (k_0, |k|). \]  

(5.12)

Because $^3P_1$ spin-angular parts has complex form (See Table. 3 of section 2.3), we include $^3P_1$ and $^1P_1$ but ignore $^3P_1$ and $^3P_1$ among negative energy components for simplicity in actual calculation. We also ignore $^3S_1$, $^3D_1$ states, because we expect that the contribution from those two states is second order in the non-relativistic expansion (in powers of velocity) and is small.
5.4 Extension of the covariant Graz-II interaction with $P$-waves

For a realistic description of the deuteron when including both the positive and negative energy components in the BS amplitude, we extend the the covariant Graz-II interaction with including $P$-wave components. For this purpose, we adopt a rank IV separable ansatz. The Graz-II interaction is a rank III separable potential suggested by Matthiasch [10]. Rupp made a covariant Graz-II potential by simple covariantization [25].

In the covariant Graz-II interaction, they considered only positive energy states, $^3S_1^+$ and $^3D_1^+$. Here in this work we build a rank IV separable potential by improving the rank III covariant Graz-II interaction with including the negative energy states. The functions $g_i^\alpha$, which we use take the following forms

\[
g_1^{4S_0}(p_0,|p|) = \frac{1 - \gamma_1(p_0^2 - p^2)}{(p_0^2 - p^2 - \beta_{11}^2)^2},
\]

\[
g_2^{4S_1}(p_0,|p|) = -\frac{(p_0^2 - p^2)}{(p_0^2 - p^2 - \beta_{12}^2)^2},
\]

\[
g_3^{4D_1^+}(p_0,|p|) = \frac{(p_0^2 - p^2)(1 - \gamma_2(p_0^2 - p^2))}{(p_0^2 - p^2 - \beta_{21}^2)(p_0^2 - p^2 - \beta_{22}^2)^2},
\]

\[
g_4^{4P_0}(p_0,|p|) = \frac{|p|}{(p_0^2 - p^2 - \beta_0^2)^2},
\]

\[
g_4^{4P_1}(p_0,|p|) = \frac{|p|}{m(p_0^2 - p^2 - \beta_0^2)^2},
\]

\[
g_3^{3S_1^+}(p_0,|p|) = g_{1,2,4}(p_0,|p|) = g_{1,2,3}(p_0,|p|) = 0,
\]

\[
g_3^{3S_0^+}(p_0,|p|) = g_{1,2,3,4}(p_0,|p|) = 0.
\]

For $^3S_1^+$ and $^3D_1^+$, we use the same type of function as the covariant Graz-II Interaction. What is new here is the inclusion of functions for the $^1P_1^{\alpha-o}$-waves. The last line reflects our assumption that we ignore $^3P_0^\alpha$, $^3P_1^\alpha$, $^3S_0^-$ and $^3D_0^-$. Having the ansatz Eq. (5.13), the solution Eq. (5.10) to the BSE can be written as

\[
\phi_{3S_1^+}(p_0,|p|) = (c_1\lambda_{11} + c_2\lambda_{12} + c_3\lambda_{13} + c_4\lambda_{14})S_+g_1^{4S_0}(p_0,|p|) + (c_1\lambda_{12} + c_2\lambda_{22} + c_3\lambda_{23} + c_4\lambda_{24})S_+g_2^{4S_1}(p_0,|p|),
\]

\[
\phi_{3D_1^+}(p_0,|p|) = (c_1\lambda_{13} + c_2\lambda_{23} + c_3\lambda_{33} + c_4\lambda_{34})S_+g_3^{4D_1^+}(p_0,|p|),
\]

\[
\phi_{1P_0}(p_0,|p|) = (c_1\lambda_{14} + c_2\lambda_{24} + c_3\lambda_{34} + c_4\lambda_{44})(S_0 g_4^{1P_0^\alpha}(p_0,|p|) + S_0 g_4^{1P_0^\alpha}(p_0,|p|)),
\]

\[
\phi_{1P_1}(p_0,|p|) = (c_1\lambda_{14} + c_2\lambda_{24} + c_3\lambda_{34} + c_4\lambda_{44})(S_0 g_4^{1P_0^\alpha}(p_0,|p|) + S_0 g_4^{1P_0^\alpha}(p_0,|p|)),
\]

\[
\phi_{3S_1^+}(p_0,|p|) = \phi_{3D_1^-}(p_0,|p|) = \phi_{3P_0}(p_0,|p|) = \phi_{3P_1}(p_0,|p|) = 0,
\]

where we have used the facts that $\lambda_{ij} = \lambda_{ji}$, that $\phi_{1P_1}$ is even and that $\phi_{1P_1}$ is odd under $p_0 \rightarrow -p_0$.  

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5.5 Elastic Electron-Deuteron Scattering

5.5.1 Relativistic Kinematics

The differential cross section for unpolarized elastic electron-deuteron scattering in the one-photon-exchange approximation is given by

\[ \frac{d\sigma}{d\Omega_e} = \left( \frac{d\sigma}{d\Omega_e} \right)_\text{Mott} \left[ A(q^2) + B(q^2) \tan^2 \theta_e \right]. \]  

(5.15)

Here \( \left( \frac{d\sigma}{d\Omega_e} \right)_\text{Mott} \) is the Mott cross section which is given by

\[ \left( \frac{d\sigma}{d\Omega_e} \right)_\text{Mott} = \frac{\alpha^2 \cos^2 \theta_e/2}{4E_e^2(1 + 2E_e/M \sin^2 \theta_e/2)}, \]

(5.16)

where \( \theta_e \) is the electron scattering angle, \( M \) the deuteron mass and \( E_e \) the incident electron energy. The functions \( A(q^2) \) and \( B(q^2) \) are the deuteron structure functions which can be related to the form factors by

\[ A(q^2) = F_C^2(q^2) + \frac{8}{9} \eta^2 F_Q^2(q^2) + \frac{2}{3} \eta F_M^2(q^2), \]

\[ B(q^2) = \frac{4}{3} \eta(1 + \eta) F_M^2(q^2), \]

(5.17)

where \( \eta = -q^2/4M^2 = Q^2/4M^2 \). The electric \( F_C(q^2) \), the magnetic \( F_M(q^2) \) and the quadrupole \( F_Q(q^2) \) form factors are normalized as

\[ F_C(0) = 1, \quad F_M(0) = \frac{\mu_D}{m}, \quad F_Q(0) = M^2 Q_D, \]

(5.18)

where \( m \) is the nucleon mass, \( \mu_D \) the magnetic moment and \( Q_D \) the quadrupole moment of the deuteron. The tensor polarization components of the final deuteron can be written through the deuteron form factors as follows:

\[ t_{20} [A + B \tan^2 \theta_e/2] = -\frac{1}{\sqrt{2}} \left[ \frac{8}{9} \eta F_C F_Q + \frac{8}{9} \eta^2 F_Q + \frac{1}{3} \eta(1 + 2(1 + \eta) \tan^2 \theta_e/2) F_M^2 \right], \]

\[ t_{21} [A + B \tan^2 \theta_e/2] = \frac{2}{\sqrt{3}} \eta(\eta + \eta^2 \sin^2 \theta_e/2)^{1/2} F_M F_Q \sec \theta_e/2, \]

(5.19)

\[ t_{22} [A + B \tan^2 \theta_e/2] = -\frac{1}{2\sqrt{3}} \eta F_M^2. \]

They are defined by the following equation.

\[ t_{kq} = \frac{\text{Tr}(M_{fi} M_{fi}^\dagger \tau_{kq})}{\text{Tr}(M_{fi} M_{fi}^\dagger)} \]  

(5.20)

Here \( M_{fi} \) is the amplitude with \( \tau_{kq} \) being the irreducible tensor operator [47].

We can obtain Eq. (5.15) from the following amplitude

\[ M_{fi} = i e^2 \bar{u}_{m'}(l') \gamma^\mu u_m(l) \frac{1}{q^2} \langle D'M'|J_\mu|D'M \rangle, \]

(5.21)
where \( u_m(l) \) is the free electron spinor with 4-momentum \( l \) and the spin projection \( m \), \( q = l - l' = P' - P \) the 4-momentum transfer and \( P(P') \) the initial (final) deuteron momentum. \( |D\mathcal{M}\rangle \) is the deuteron state with total angular momenta projection \( \mathcal{M} \) and \( J_\mu \) is the electromagnetic current operator.

The deuteron current matrix element is usually parameterized in the following way

\[
\langle D'\mathcal{M}'|J_\mu|D\mathcal{M}\rangle = -ie \xi^*_{\alpha\mathcal{M}'}(P') \xi_\beta\mathcal{M}(P) \left[(P' + P)_\mu \left(g^{\alpha\beta} F_1(q^2) - \frac{q^\alpha q^\beta}{2M^2} F_2(q^2)\right)\right] - (q^\alpha g^\beta_\mu - q^\beta g^\alpha_\mu) G_1(q^2),
\]

where \( \xi_\mathcal{M}(P) \) and \( \xi^*_{\mathcal{M}'}(P') \) are the polarization 4-vectors of the initial and final deuteron. Form factors \( F_{1,2}(q^2), G_1(q^2) \) are related to \( F_C(q^2), F_Q(q^2) \) and \( F_M(q^2) \) by the equations

\[
F_C = F_1 + 2\frac{q^2}{3} \eta [F_1 + (1 + \eta) F_2 - G_1], F_Q = F_1 + (1 + \eta) F_2 - G_1, F_M = G_1.
\]

The normalization condition for the deuteron current matrix element is given by

\[
\lim_{q^2 \to 0} \langle D'\mathcal{M}'|J_\mu|D\mathcal{M}\rangle = 2eP_\mu \delta_{\mathcal{M}\mathcal{M}'}.
\]

The normalization of the BS amplitude is discussed in detail in section 2.1.

In order to calculate the deuteron form factors, let us choose the laboratory frame (deuteron at rest). In the laboratory frame the relevant momentum variables take the following form (the \( z \)-axis is along the photon momentum):

\[
P = (M, 0), \quad P' = (M(1 + 2\eta), 0, 0, 2M\sqrt{\eta}\sqrt{1 + \eta}),
\]

\[
q = (2M\eta, 0, 0, 2M\sqrt{\eta}\sqrt{1 + \eta}),
\]

\[
\xi_{\mathcal{M}=+1}(P) = \xi_{\mathcal{M}=+1}(P') = -\frac{1}{\sqrt{2}}(0, 1, i, 0),
\]

\[
\xi_{\mathcal{M}=-1}(P) = \xi_{\mathcal{M}=-1}(P') = \frac{1}{\sqrt{2}}(0, 1, -i, 0),
\]

\[
\xi_{\mathcal{M}=0}(P) = (0, 0, 0, 1), \quad \xi_{\mathcal{M}=0}(P') = (2\sqrt{\eta}\sqrt{1 + \eta}, 0, 0, 1 + 2\eta).
\]

From Eq. (5.24), (5.25) and Eq. (5.22), we obtain:

\[
\langle \mathcal{M}'|J_0|\mathcal{M}\rangle = 2Me(1 + \eta) \left\{ F_1 \delta_{\mathcal{M}\mathcal{M}'} + 2\eta [F_1 + (1 + \eta) F_2 - G_1] \delta_{\mathcal{M}'0}\delta_{\mathcal{M}0}\right\},
\]

\[
\langle \mathcal{M}'|J_x|\mathcal{M}\rangle = \frac{2Me}{\sqrt{2}} \sqrt{\eta} \sqrt{1 + \eta} G_1 \left\{ \delta_{\mathcal{M}'\mathcal{M}+1} - \delta_{\mathcal{M}'\mathcal{M}-1}\right\}.
\]

5.5.2 Deuteron current

In the relativistic impulse approximation, the deuteron current matrix element can be written as

\[
\langle D'\mathcal{M}'|J_\mu|D\mathcal{M}\rangle = ie \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \Phi_{\mathcal{M}'}(P', p') \Gamma^{(p+n)}(q) \Phi_\mathcal{M}(P, p) (S^{(2)}_{\text{IR}}(q^2))^{-1} \right],
\]

\[
\Gamma^{(S)}_\mu(q) = \gamma_\mu F_1^{(S)}(q^2) - \frac{\gamma_\mu \hat{q} - \hat{q} \gamma_\mu}{4m} F_2^{(S)}(q^2),
\]
where $\Phi_M(P, p)$ is the BS amplitude of the deuteron, $P' = P + q$ and $p' = p + q/2$. $q$ is the momentum transfer and $\eta = -q^2/4M^2 = Q^2/4M^2$ where $M$ is the deuteron mass. $\Gamma^{(S)}_\mu(q)$ is the vertex of $\gamma NN$ interaction. The form factors of the nucleon $F^{(S)}_{1,2}$ is the summation of two nucleons.

The resulting expressions for the deuteron current matrix element can be written as

$$
\langle D' M' | J^{RI A}_\mu | D M \rangle = \mathcal{I}_{1,2}^{M', M}(q^2) F^{(S)}_1(q^2) + \mathcal{I}_{2}^{M', M}(q^2) F^{(S)}_2(q^2),
$$

where the function $\mathcal{I}_{1,2}^{M', M}(q^2)$ is obtained by the taking the trace in the $\gamma$ matrix space of Eq. (5.27) and the substitution of the scalar products into Eq. (5.27).

### 5.5.3 Lorentz transformation

In Eq. (5.28), the radial part of the BS amplitude for the final state deuteron $\phi_{1' L'S' L'}(k'_0, |k'|)$ depends on the momentum variable $k'$ in laboratory frame. The momenta of the initial deuteron ($P$) and of the final deuteron ($P'$) in the laboratory frame are related by the Lorentz transformation:

$$
P' = \mathcal{L} P = \mathcal{L}(M, 0), \quad k' = \mathcal{L} k,
$$

where the Lorentz transformation matrix $\mathcal{L}$ is of the form:

$$
\mathcal{L} = \begin{pmatrix}
1 + 2\eta & 0 & 0 & 2\sqrt{\eta} \sqrt{1 + \eta} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2\sqrt{\eta} \sqrt{1 + \eta} & 0 & 0 & 1 + 2\eta
\end{pmatrix}
$$

The components of the 4-vector of the final state $k'$ are expressed by $k' = (k'_0, k'_x, k'_y, k'_z)$, and $|k'| = \sqrt{k'_x^2 + k'_y^2 + k'_z^2}$. Using Eq. (5.24), (5.29) and (5.30) we obtain

$$
k'_0 = (1 + 2\eta)k_0 - 2\sqrt{\eta} \sqrt{1 + \eta k_z - M \eta},
$$
\[ k'_x = k_x, \quad k'_y = k_y, \]
\[ k'_z = (1 + 2\eta)k_z - 2\sqrt{\eta(1 + \eta)k_0 + M\sqrt{1 + \eta}}, \quad (5.31) \]

where \(k_0, k_x, k_y, k_z\) are the components of the 4-vector \(k\) of the initial state.

In order to calculate the deuteron form factors, we need to know three matrix elements with different total angular momentum projections and current component, for example, \(\langle 0|J_0|0\rangle\), \(\langle 1|J_0|1\rangle\) and \(\langle 1|J_x|0\rangle\). Finally, we obtain the following equations.

\[ F_C = \frac{1}{2M} \left\{ \frac{1}{3}(1 + \eta) \right\} \frac{\langle P', M' = 0|J_0|P, M = 0 \rangle + 2 \langle P', M' = +1|J_0|P, M = +1 \rangle}{\sqrt{1 + \eta}} \]
\[ F_M = \frac{1}{M\sqrt{2\eta}} \frac{\langle P', M' = +1|J_x|P, M = 0 \rangle}{\sqrt{1 + \eta}} \]
\[ F_Q = \frac{1}{2M} \left\{ \frac{1}{\eta(1 + \eta)} \right\} \frac{\langle P', M' = 0|J_0|P, M = 0 \rangle - \langle P', M' = +1|J_0|P, M = +1 \rangle}{\sqrt{1 + \eta}} \quad (5.32) \]

We can obtain tensor polarization components by inserting these equations into Eqs. (5.20).

### 5.5.4 Calculation with fewer parameters

When we use the rank IV separable potential, Eq. (5.11) takes the following form

\[ c_1 - \sum_{k,j=1}^{3} H_{1k}\lambda_{kj}c_j = c_4(H_{11}\lambda_{14} + H_{12}\lambda_{24}), \]
\[ c_2 - \sum_{k,j=1}^{3} H_{2k}\lambda_{kj}c_j = c_4(H_{21}\lambda_{14} + H_{22}\lambda_{24}), \]
\[ c_3 - \sum_{k,j=1}^{3} H_{3k}\lambda_{kj}c_j = c_4(H_{33}\lambda_{34}), \]
\[ c_4 - \sum_{k,j=1}^{3} H_{4k}\lambda_{kj}c_j = c_4(H_{44}\lambda_{44}). \quad (5.33) \]

\(\lambda_{j4}, \ H_{4j}(j = 1, 2, 3, 4)\) and \(c_4\) appear due to the inclusion of \(1P^o_1\) and \(1P^o_2\)-states. Since the fourth component in the rank IV ansatz is the new point in this thesis, we would like to discuss in some detail the quantities carrying the fourth index. First, we note that \(H_{4j}(j = 1, 2, 3) = 0\) due to the choice of the \(g\)-functions as given in Eqs. (5.13). On the other hand, \(H_{44}\) can take a finite value. We would like to discuss \(H_{44}\) in detail.

The evaluation of \(H_{44}\) contains a \(k_0\)-integral, and the result is affected by the location of poles of the \(g\)-functions and propagator \(S_n\) as defined by the Eq. (5.6). For deuteron problems, we take the location of poles of \(g^{1P^o_1}, g^{1P^o_2}\) in the same side of the complex \(k_0\)-plane as \(g^{3S^1}\). As we will refer in detail later, \(P\)-wave part influence on the form factors and tensor polarizations through \(H_{44}, c_j(j = 1, 2, 3, 4)\) and \(\lambda_{j4}(j = 1, 2, 3, 4)\). This is an important contribution in order to improve the agreement with the experimental data by including the negative energy components in bound state problems.
However, if we adopt the same prescription for scattering problems, a finite $H_{44}$ causes a serious problem, which change the phase shift drastically, even if the rate of $P$-wave is very small. Therefore, we try to solve this problem in the following way. For $H_{44}$, we have the following expression:

$$H_{44} = \frac{i}{2\pi^2} \int dk_0 k^2 d|k| \left( S_e \left( (g_4^{1P^e})^2 + (g_4^{1P^o})^2 \right) + 2S_o \left( g_4^{1P^e} g_4^{1P^o} \right) \right)$$

where

$$g^{+-} \equiv g_4^{1P^e} + g_4^{1P^o},$$

$$g^{-+} \equiv g_4^{1P^e} - g_4^{1P^o}.$$

At first sight, Eq. (5.34) takes a finite value. However, by locating the poles of $g^{+-}$ in the upper side of the complex $k_0$-plane as $S^{+-}$, and the poles of $g^{-+}$ lower as $S^{-+}$, Eq. (5.34) becomes zero. (See Fig. 24.) This prescription to make the vanishing contribution for $H_{44}$ turns out to be crucially important in order to reproduce the experimental phase shifts. By assuming different pole location of the $g$-function for the bound state and scattering problems, we are able to reproduce reasonable results. In this way, we will be able to obtain reasonable results for both scattering and bound state (deuteron) problems simultaneously.

In actual calculations, we set the special treatment as follows to reduce the number of parameters:

$$\lambda_{14} = -\sqrt{\lambda_{11}} u_4, \ \lambda_{24} = \sqrt{\lambda_{22}} u_4, \ \lambda_{34} = \sqrt{\lambda_{33}} u_4, \ \lambda_{44} = u_4^2.$$ (5.35)

We try to find the some parameter sets to reproduce the form factors and tensor polarizations, and phase shifts.
5.6 Results

We calculated the deuteron form factors and tensor polarizations for the three purposes. First is to investigate how the deuteron form factors and tensor polarizations depend on the nucleon form factors. Second is to investigate how the deuteron form factors and tensor polarizations depend on the ratio of the $D$-wave. Third is to investigate how the deuteron form factors and tensor polarizations depend on the rate of the $P$-wave components.

5.6.1 Dependence on the intrinsic nucleon form factor

In this subsection, we investigate how the deuteron form factors and tensor polarizations depend on the intrinsic form factor of the nucleon. To do this, we shall briefly summarize the electromagnetic form factors. The electric $G_E(q^2)$ and the magnetic $G_M(q^2)$ form factors of nucleons are related to the Dirac $F_1(q^2)$ and Pauli and $F_2(q^2)$ form factors as follows:

\[
G_E(q^2) = F_1(q^2) + \frac{q^2}{4m^2} F_2(q^2),
\]
\[
G_M(q^2) = F_1(q^2) + F_2(q^2).
\]

We use three kinds of the nucleon form factors in the calculations, dipole fit, vector meson dominance model and relativistic harmonic oscillator model. They have the following properties.

- **Dipole fit**
  \[
  G_M^p(q^2) = (1 + \kappa_p) G_E^p(q^2),
  \]
  \[
  G_M^n(q^2) = \kappa_n G_E^p(q^2),
  \]
  \[
  G_E^p(q^2) = 0,
  \]
  \[
  G_E^n(q^2) = 1/(1 - q^2/0.71 \text{(GeV)}^2)^2,
  \]

where $\kappa_p = 1.7928$ and $\kappa_n = -1.9130$ are the anomalous magnetic moments of the nucleons.

- **Vector meson dominance model (VMDM) [44]**

\[
F_1^{(S)}(t) = \left[ \frac{m_\omega^2}{m_\omega^2 - q^2} \gamma_\omega + (1 - \gamma_\omega) \right] F_{1L},
\]
\[
F_2^{(S)}(t) = \left[ \frac{m_\omega^2}{m_\omega^2 - q^2} \kappa_\omega \gamma_\omega + (1 + \kappa_p + \kappa_n - \kappa_\omega \gamma_\omega) \right] F_{2L},
\]
\[
F_{1L} = \frac{\lambda_1^2}{\lambda_1^2 + \tilde{q}^2} \frac{\lambda_2^2}{\lambda_3^2 + \tilde{q}^2},
\]
\[
F_{2L} = \frac{\lambda_1^2}{\lambda_1^2 + \tilde{q}^2} \left( \frac{\lambda_2^2}{\lambda_3^2 + \tilde{q}^2} \right)^2,
\]
\[
\tilde{q}^2 = -q^2 \frac{\ln (\lambda_3^2 - \tilde{q}^2)/\lambda_3^2}{\ln \lambda_2^2/\lambda_3^2},
\]

where $\lambda_1 = 0.795 \text{(GeV)}$, $\lambda_2 = 2.27 \text{(GeV)}$, $\lambda_3 = 0.29 \text{(GeV)}$, $\kappa_\omega = 0.163$, $\gamma_\omega = 0.411$, $m_\omega = 0.784 \text{(GeV)}$. 

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- Relativistic harmonic oscillator model (RHOM) [45]

\[
G_E^p = I^{(3)}(q^2),
\]
\[
G_E^n = -\frac{q^2}{2m^2} I^{(3)}(q^2),
\]
\[
\frac{G_M^p(q^2)}{1 + \kappa_p} = \frac{G_M^n(q^2)}{\kappa_n} = I^{(3)}(q^2),
\]
\[
I^{(3)}(q^2) = \frac{1}{(1 - q^2/2m^2)^2} \exp \left( \frac{1}{2\alpha_3} \frac{q^2}{1 - q^2/2m^2} \right),
\]
\[
\alpha_3 = 0.42 \text{(GeV)}^2.
\]

The first model assumes that the neutron electric form factor is equal to zero. Two other models lead to a nonzero \(G_E^n\). The form factors of proton and neutron by each models will be shown in Figs. 25. Figs. 25-(a), (b), (c) and (d) represents the magnetic form factor of a proton, the electric form factor of a proton, the magnetic form factor of a neutron and the electric form factor of a neutron respectively. The red, green and blue line represents the results calculated by the Dipole fit, Vector meson dominance model and Relativistic oscillator model respectively. The pink line represents the experimental data. Seeing Figs. 25, we can reproduce the proton and neutron form factors by all models.
Using these nucleon form factors, we compute various electromagnetic form factors of the deuteron. Fig. 26-(a), (b), (c) shows the charge, magnetic and quadrupole form factors respectively. Fig. 26-(d), (e), (f) show the \( t_{20} \), \( t_{21} \) and \( t_{22} \) tensor polarizations respectively. All results are calculated including only positive energy states, \( ^3S_1^+ \) and \( ^3D_1^+ \), namely, without negative energy components. The ratio of the \( D \)-wave is fixed to 4%. We use the same parameter set shown as the result of Rupp and Tjon [25] of \( P_d = 4\% \) except \( \lambda_{14} = \lambda_{24} = \lambda_{34} = \lambda_{44} = \gamma_3 = 0 \). The red line is calculated with the nucleon form factor of the Dipole fit, the green line with the vector meson dominance model and the blue line with the relativistic harmonic oscillator model, which are compared with experimental data (the pink line). The data of the charge and quadrupole form factor are taken from [40], those of the magnetic form factor are from [43], and those of the tensor polarizations are from [40, 41]. From this comparison, we verify that the deuteron form factors are not very sensitive to nucleon form factor. This is natural since all three nucleon form factors agree with experimental data qualitatively.

In the following discussions, we discuss one of them, the dipole fit model.

### 5.6.2 The role of the \( D \)-wave

Next we investigate the role of the \( D \)-wave in the form factors and tensor polarizations. Fig. 27-(a), (b), (c) show the charge, magnetic and quadrupole form factors respectively. Fig. 27-(d), (e), (f) show the \( t_{20} \), \( t_{21} \) and \( t_{22} \) tensor polarizations, respectively. All quantities are calculated from the BS approach, considering only \( ^3S_1^+ \) and \( ^3D_1^+ \) states. The red line is the result obtained with the \( D \)-wave probability 4\%, the green line 5\%, the blue line 6\%. The corresponding parameter sets are the as the results of Rupp and Tjon [25] of \( P_d = 4\%\), \( P_d = 4\% \) and \( P_d = 6\%\) except that \( \lambda_{14} = \lambda_{24} = \lambda_{34} = \lambda_{44} = \gamma_3 = 0 \). The pink line shows the experimental data. The data of the charge and quadrupole form factors are taken from [40], that of the magnetic form factor are taken from [43], and the data of the tensor polarizations are from [40] and [41]. In this calculation, we verify that it is not easy to make a significant improvement as to reproduce the experimental data whatever we change the ratio of the \( D \)-wave. These results imply that in order to reproduce the experimental data in the BS approach, it is not sufficient to consider the positive energy components, namely, \( ^3S_1^+ \) and \( ^3D_1^+ \) states.

### 5.6.3 The role of negative energy \( P \)-wave

Finally, we investigate the effect of the negative energy \( P \)-wave on the form factors and tensor polarizations. The contents of this subsection is the new result obtained in this thesis. We calculate the form factors and tensor polarizations from BS approach including positive energy components, \( ^3S_1^+ \) and \( ^3D_1^+ \) and negative energy components, \( ^1P_1^e \) and \( ^1P_1^o \). We change the \( P \)-wave probability from zero to a few percent in order to see the effect of \( P \)-wave, while we fix the \( ^3D_1^+ \)-wave probability to 4\%. Fig. 28-(a), (b), (c) shows charge, magnetic and quadrupole form factors respectively. Fig. 28-(d), (e), (f) the \( t_{20} \), \( t_{21} \) and \( t_{22} \) tensor polarizations respectively.

The red lines represent the result when the case of the ratio of the \( P \)-wave is 0\%, the green lines 1\%, the blue lines 3\%, except for Fig. 28-(b). For Fig. 28-(b), the red, green and blue lines represent the results with \( P \)-wave ratio is 0\%, 0.5\% and 1\%, respectively.
Figure 25: Proton and neutron form factors.
The parameter set used for each calculation is the same as that of Rupp and Tjon [25] of $P_d = 4\%$ except for the followings:

- $u_4 = -3.5, \beta_3 = 0.481$ GeV, $\gamma_3 = -15.0$ for 0.5%.
- $u_4 = -5.0, \beta_3 = 0.481$ GeV, $\gamma_3 = -15.0$ for 1%.
- $u_4 = -8.5, \beta_3 = 0.481$ GeV, $\gamma_3 = -15.0$ for 3%.

In all figures, the pink lines represent the experimental data. The data of the charge and quadrupole form factors are taken from [40], that of the magnetic form factor are taken from [43], and the data of the tensor polarizations are taken from [40, 41].

We find a tendency that better agreement with experimental data is achieved if we include the finite amount of $P$-wave amplitudes of the negative energy components. The agreement of the charge form factor is improved drastically. As for magnetic form factors, the dip structure (the change in the sign of the form factor) around $Q^2 \sim 2$ GeV/c$^2$ is reproduced by including the negative energy components. About $t_{20}$ and $t_{21}$, the agreement in the high momentum transfer region $Q^2 > 1$ GeV/c$^2$ is improved as in the cases of the charge form factors. The agreement of $t_{22}$ is also improved.

### 5.7 Conclusion

We have seen that the inclusion of negative energy states ($^1P^e_1$, $^1P^o_1$) has improved form factors and tensor polarizations, while the results of section 5.6.1 and 5.6.2 imply that the dependence of the intrinsic nucleon form factor and the ratio of $D$-wave component do not affect very much on calculational results.

We found that 3% $P$-wave ratio is the most appropriate in order to reproduce the charge form factors and tensor polarizations. For magnetic form factors we can reproduce the experimental data by including the $P$-wave components by 0.5%. We expect that this inconsistency of the rates of the $P$-wave would be improved by including the other negative components, namely $^3P^e_0$, $^3P^o_0$, $^3S^-_1$ and $^3D^-_1$ states. This is one of interesting future works.

On the other hand, using the same parameter set as in section 5.6.3 we can nicely reproduce the $^3S^-_1$ and $^3D^-_1$ phase shifts as shown in Figs. 29, where we have seen that the negative energy components affect very little on scattering properties. Therefore, the inclusion of the negative energy components can explain simultaneously the two-nucleon scattering problems and electromagnetic properties of the deuteron.

Finally, we would like to compare our results with the results of the no-relativistic framework, in order to relate the negative energy components with the exchange currents. Figs. 30-(a), (c) and (e) show the charge form factor, the deuteron structure function $B$ defined by Eq. (5.17), and the quadrupole form factor calculated in the non-relativistic framework, respectively. The solid lines represent the results in the non-relativistic impulse approximation with the inclusion of exchange currents, and the dotted lines are for the results in the non-relativistic impulse approximation. Figs. 30-(a) and (e) are the results from the work of Gari and Hyuga [7], while Fig. 30-(c) are from the work of Blunden and Riska work [12].

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Figs. 30-(b), (d) and (f) are the same plots as those of (a), (c) and (e) in the relativistic framework.

We can see that the exchange currents shift the location of the dip structure of the charge form factor to the left side, while a similar effect is seen in the relativistic method by including negative energy components as shown in Figs. 30-(a) and (b). Similarly, the same tendency is observed for the structure function $B$ as in Figs. 30-(c) and (d). Quantitatively, the effect of the negative energy components shift the location of the dip structure to the left side in the relativistic case more strongly than exchange currents in the non-relativistic case. As for the quadrupole form factor, the exchange currents move the results to the upper side, which is the same as the effect of the negative energy components as shown in Figs. 30-(e) and (f).

We find that in the comparison with the experimental data the results of the non-relativistic calculation are sensitive to the model of exchange currents, and a fine tuning of the parameters is needed. However, the relativistic calculation gives us a good agreement with experimental data just by including the negative energy components. Therefore, we conclude that the relativistic calculation provides a more consistent description for the deuteron properties.
Figure 26: Form factors and Tensor polarization. D=4% Dipole, VMDM, RHOM.
Figure 27: Form factor and Tensor polarization. D=4,5,6 % P=0%.
Figure 28: Tensor polarization. D=4%, P=0,1,3%
Figure 29: $^3S_1$ and $^3D_1$ phase shifts. D=4%, P=0,1,3%
6 Summary

In this thesis, we investigated the properties of two-nucleon systems using the relativistic Bethe-Salpeter approach.

First of all, we have studied the general formalism of the Bethe-Salpeter approach for two-nucleon systems. We derived the Bethe-Salpeter equation for the $T$-matrix for scattering problems and the integral equation for the Bethe-Salpeter amplitude for deuteron problems in the momentum space. Then, we showed the method for the partial-wave decomposition, introducing the concept of the $\rho$-spin which was essential for the treatment of a complete set of the two-nucleon system including negative energy states in the relativistic formalism. Consequently, we have eight basis states for $J^P = 1^+$ channels for the deuteron, while these are only two states in the non-relativistic framework. Then, we showed the general solution to the Bethe-Salpeter equation by using the separable ansatz which makes it easy to solve the Bethe-Salpeter equation.

Since it is not easy to get physics insight of the separable ansatz, secondly, we have studied the relation between the separable and one-boson-exchange potential for the relativistic Bethe-Salpeter equation. We expected that the meaning of the separable potential could be understood with physics ground, not just a mathematically convenient tool. It turned out that it was not easy to reproduce properties of a single term of the one-boson-exchange potential, separately by the rank I separable potential. Such a property was shown analytically in the non-relativistic framework, especially for a very strong interaction of sigma meson exchange potential. However, it was shown that the separable potential is still useful to parameterize the total nucleon-nucleon interaction, where the very strong attractive potential is largely canceled by the strong repulsion due to omega meson exchange. The resulting strength of the nuclear force is just suited to being expressed by a separable ansatz.

Third, we solved the Bethe-Salpeter equation to the scattering problem for $J = 0, 1$ channels. There, although keeping the relativistic framework, we ignored the negative energy components in the intermediate states. The ignorance of the negative energy components can be justified by considering the location of poles of $g$-functions. We have then found the several parameter sets to reproduce the phase shifts and low energy parameters. As a result, it was shown that at least Rank II was necessary in order to reproduce the phase shifts up to few hundreds MeV.

Finally, we investigated the effect of the negative energy components on the two-nucleon system. We have solved the Bethe-Salpeter equation, including $1P^e_1, 1P^o_1$ states, both for scattering states and bound states. We have fixed the parameter sets such that they reproduce electromagnetic properties of the deuteron and phase shifts. In general, the inclusion of the negative energy components improved systematically the form factors and tensor polarizations. As for charge form factor, the position of the dips and the agreement at high momentum transfer region ($Q^2 > 1 \text{ (GeV/c)}^2$) were improved. The dip structure of magnetic form factor could be reproduced. No significant change was found as for the quadrupole form factor. As for $t_{20}, t_{21}$ and $t_{22}$, the agreement was improved at high momentum transfer region ($Q^2 > 1 \text{ (GeV/c)}^2$). It turned out that the mixture of about 3% of $P$-wave components was needed to reproduce the electromagnetic properties
Figure 30: Comparison with the results of non-relativistic calculations. NRIA stands for
the non-relativistic impulse approximation, and RIA for the relativistic impulse approxi-
mation. +EXC includes the exchange currents.
and tensor polarizations in the relativistic impulse approximation. As for a magnetic form factor, about 0.5% of $P$-wave components was appropriate. The different rates of the mixture of negative energy components might be related to our incomplete treatment of negative energy components. In fact, in the present work we have ignored negative energy components of $^3P_e$, $^3P_o$, $^3S_1^-$ and $^3D_1^-$. The inclusion of the full set of the negative energy components will be a future work.

In this thesis, we have discussed that a relativistic approach provides a straightforward and systematic method to compute two-nucleon properties, at least conceptionally. However, we have found that actual calculations are very complicated due to the increase in the number of the degrees of freedom (antiparticle) and due to the necessity of dealing with energy variables in the complex plane. Therefore, in this work, ignoring mathematical rigor, we have just included only two negative energy components of $^1P_e^1$ and $^1P_o^1$ as a part of complete set for the deuteron wave function. Although not complete set an explicit inclusion of the negative energy components was performed correctly for the first time in the relativistic Bethe-Salpter approach.

As one of important works which have not been explored much is to understand the physical meaning of the negative energy components, although we have attempted to do so to some extent by making comparison with non-relativistic calculations with the inclusion of the exchange currents. A better understanding of the negative energy components will make a firm ground on the use of the relativistic method.
Appendix

A Derivation of the normalization condition Eq. (2.20)

Let us start with the BS equation in the following form:

$$T = V + VST.$$  \hspace{1cm} (A.1)

Formally the $T$-matrix can be obtained as

$$T = \frac{1}{V^{-1} - S}. \hspace{1cm} (A.2)$$

In this simplified notation, we can write Eq. (2.13) as

$$T = |\Gamma\rangle \langle \Gamma| + \sum_n c_n |\chi_n\rangle \langle \chi_n|, \hspace{1cm} (A.3)$$

where $|\Gamma\rangle$ is the vertex function for the bound state, and $|\chi_n\rangle$ is the continuum states except bound state. Thus we have the following orthogonal condition,

$$\langle \chi_n| \Gamma \rangle = 0 \hspace{1cm} (A.4)$$

If we substitute Eq. (A.2) into Eq. (A.3), we obtain

$$(V^{-1} - S)\left(\frac{|\Gamma\rangle \langle \Gamma|}{P^2 - M^2} + \sum_n c_n |\chi_n\rangle \langle \chi_n|\right) = 1. \hspace{1cm} (A.5)$$

Multiplying $P^2 - M^2$, we obtain

$$(V^{-1} - S)|\Gamma\rangle \langle \Gamma| + \sum_n (V^{-1} - S)c_n (P^2 - M^2)|\chi_n\rangle \langle \chi_n| = P^2 - M^2. \hspace{1cm} (A.6)$$

When we differentiate the both side by $P^\mu$, we obtain

$$\frac{\partial}{\partial P^\mu}(V^{-1} - S)|\Gamma\rangle \langle \Gamma| + (V^{-1} - S)|\Gamma\rangle \frac{\partial}{\partial P^\mu} \langle \Gamma|$$

$$+ 2P^\mu \sum_n (V^{-1} - S)c_n |\chi_n\rangle \langle \chi_n| = 2P^\mu. \hspace{1cm} (A.7)$$

On the other hand, from Eq. (A.6) we can immediately obtain

$$(V^{-1} - S)|\Gamma\rangle = 0 \hspace{1cm} (A.8)$$

Thus

$$\frac{\partial}{\partial P^\mu}\left((V^{-1} - S)|\Gamma\rangle\right) \langle \Gamma| + 2P^\mu \sum_n (V^{-1} - S)c_n |\chi_n\rangle \langle \chi_n| = 2P^\mu \hspace{1cm} (A.9)$$

If we multiply $\langle \Gamma|$ from left and $|\Gamma\rangle$ from right for both sides and use condition Eq. (A.4), we obtain

$$\langle \Gamma| \frac{\partial}{\partial P^\mu}(V^{-1} - S)|\Gamma\rangle = 2P^\mu. \hspace{1cm} (A.10)$$

If the interaction kernel doesn’t depend on the total momentum $P$ we obtain Eq. (2.20).
B The property of the rank I separable potential

In order to get some insight of the property of the rank I separable potential, we consider the BSE in the non-relativistic case, namely the Lippmann-Schwinger equation. For simplicity, we consider only the $S$-wave amplitude. Thus, we have the LS equation

\[
\begin{align*}
t(|k'|, |k|; E) &= v(|k'|, |k|) \\
&+ \frac{1}{2\pi^2} \int |k'|^2 d|k''| \frac{v(|k''|, |k|; E)}{(E - \frac{|k''|^2}{m} + i\epsilon)},
\end{align*}
\]

(B.1)

where $m$ is the mass of the nucleon and is twice of the reduced mass. Using a rank I separable ansatz, we adopt an interaction kernel $V$ as follows

\[
v(|k'|, |k|) = \lambda g(|k|) g(|k'|),
\]

(B.2)

which corresponds to Eq. (2.55) with $N = 1$, and $\lambda$ is the parameter which expresses the strength of the potential. Here we assume that $g(|k|) \to 0$ ($|k| \to \infty$). Then the $T$-matrix can be expressed as

\[
t(|k'|, |k|; E) = \tau(E) g(|k|) g(|k'|)
\]

(B.3)

\[
\tau(E) = \frac{1}{\lambda^{-1} + h(E)},
\]

(B.4)

and the function $h(E)$ is given by

\[
h(E) = -\frac{1}{2\pi^2} \int |k|^2 d|k| \frac{[g(|k|)]^2}{(E - \frac{k^2}{m} + i\epsilon)}.
\]

(B.5)

Now let us investigate bound state properties. We would like to see how many bound states are allowed. To do that, we look for the pole of the denominator of the Eq. (B.4);

\[
\Delta(E) = 1 + \lambda h(E),
\]

(B.6)

in the bound state region $E \leq 0$. When $E \to -\infty$,

\[
\Delta(-\infty) = 1
\]

(B.7)

and when $E = 0$,

\[
\Delta(0) = 1 + \lambda \frac{m}{2\pi^2} \int d|k| [g(|k|)]^2.
\]

(B.8)

On the other hand

\[
\frac{\partial \Delta(E)}{\partial E} = \lambda \frac{1}{2\pi^2} \int |k|^2 d|k| \frac{[g(|k|)]^2}{(E - \frac{k^2}{m} + i\epsilon)^2},
\]

(B.9)

therefore $\Delta(E)$ is monotonically decreasing from 1 to $\Delta(0)$, for $\lambda^{-1} \leq 0$ for an attractive interaction. Therefore, we have only one bound state, if

\[
1 + \lambda \frac{m}{2\pi^2} \int d|k| [g(|k|)]^2 \leq 0
\]

(B.10)
and no bound state otherwise. The rank I separable ansatz allows only one bound state no matter how strong the attractive coupling constants $\lambda$ is.

Finally we show the derivation of the Eq. (B.5) from the relativistic framework. We start from the rank I separable potential solution Eq. (4.5)

$$h(s) = -\frac{i}{4\pi^3} \int dk_0 \int |k|^2 d|k| \frac{[g(k_0, |k|)]^2}{(\sqrt{s}/2 - E_k + i\epsilon)^2 - k_0^2}.$$  \hspace{1cm} \text{(B.11)}

By assuming that the function $g$ is independent of $p_0$, namely $g(k_0, |k|) = g(|k|)$, we can perform the $k_0$-integration by considering the pole at $k_0 = \sqrt{s}/2 - E_k + i\epsilon$. Thus we obtain

$$h(s) = -\frac{1}{2\pi^2} \int |k|^2 d|k| \frac{[g(|k|)]^2}{\sqrt{s} - 2E_k + i\epsilon}.$$  \hspace{1cm} \text{(B.12)}

Using the relation $\sqrt{s} = M = 2m + E \ (E \leq 0)$

$$\sqrt{s} - 2E_k = (2m + E) - 2(m^2 + k^2)^{(1/2)}$$

$$\approx (2m + E) - 2m(1 + \frac{k^2}{2m})$$

$$= E - \frac{k^2}{m}.$$  \hspace{1cm} \text{(B.13)}
C The probability of the BSE

In this appendix we discuss how probability is defined from the BS amplitude. From the normalization condition Eq. (2.20) when \( \mu = 0 \), we can obtain the normalization condition for the radial part as follows,

\[
\int dk_0 \int d|k| |k|^2 \Gamma^\dagger(k_0, |k|) \left[ \frac{\partial}{\partial \sqrt{s}} S(k_0, |k|; s) \right]_{\sqrt{s} = M} \Gamma(k_0, |k|) = -2M , \tag{C.1}
\]

where the integration for the angular part is already done and \( \Gamma(k_0, |k|) \) is the radial part of the vertex function. For the deuteron of \( J = 1 \), \( S \) is the two nucleon propagator which is given as a matrix form for eight channels \( (3S^+_1, 3S^-_1, 3D^+_1, 3D^-_1, 3P^e_1, 3P^o_1, 1P^e_1, 1P^o_1) \),

\[
S = \begin{pmatrix}
S_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S_+ & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_e & 0 & S_o & 0 & 0 \\
0 & 0 & 0 & 0 & S_e & 0 & S_o & 0 \\
0 & 0 & 0 & 0 & 0 & S_o & 0 & S_e \\
0 & 0 & 0 & 0 & 0 & 0 & S_e & S_o
\end{pmatrix} , \tag{C.2}
\]

where components \( S_\alpha \) are given as

\[
S_+ = \frac{1}{(\sqrt{s}/2 - E_k + i\epsilon)^2 - k_0^2} , \tag{C.3}
\]

\[
S_- = \frac{1}{(\sqrt{s}/2 + E_k - i\epsilon)^2 - k_0^2} , \tag{C.4}
\]

\[
S_{+-} = \frac{1}{(\sqrt{s}/2 - E_k + k_0 + i\epsilon) (\sqrt{s}/2 + E_k - k_0 - i\epsilon)} , \tag{C.5}
\]

\[
S_{-+} = \frac{1}{(\sqrt{s}/2 + E_k + k_0 - i\epsilon) (\sqrt{s}/2 - E_k + k_0 + i\epsilon)} , \tag{C.6}
\]

\[
S_e = S_{ee} = S_{oo} = \frac{S_{+-} + S_{-+}}{2} , \tag{C.7}
\]

\[
S_o = S_{eo} = S_{oe} = \frac{S_{+-} - S_{-+}}{2} . \tag{C.8}
\]

Substituting the definition of the radial part of the BS amplitude

\[
\Gamma(k_0, |k|) = S^{-1}\phi(k_0, |k|) \tag{C.9}
\]
obtained from Eq. (2.17) into Eq. (C.1), we find

\[
\int dk_0 \frac{d |k| |k|^2 \phi^\dagger (k_0, |k|) \frac{\sqrt{S}}{\sqrt{\pi}} [S^{-1}] \frac{\partial}{\partial \sqrt{S}} S \frac{\sqrt{\pi}}{S \sqrt{\pi} M} \phi (k_0, |k|)}{d |k| |k|^2 \phi^\dagger (k_0, |k|) \omega_{\alpha \beta} \phi_\beta (k_0, |k|)} = -2M.
\]

(C.10)

In the second line we have used eight components explicitly,

\[
\omega_{\alpha \beta} = \left[ S^{-1} \frac{\partial}{\partial \sqrt{S}} S \frac{\sqrt{\pi}}{S \sqrt{\pi} M} S \frac{\sqrt{\pi}}{\sqrt{\pi}} M \right]_{\alpha \beta}
\]

\[
\begin{pmatrix}
-\frac{M}{2} + E_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{M}{2} + E_k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{M}{2} - E_k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{M}{2} - E_k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{M}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{M}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{M}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{M}{2}
\end{pmatrix},
\]

\[
\phi (k_0, |k|) = \begin{pmatrix}
\phi_{3S^+_1} (k_0, |k|) \\
\phi_{3D^+_1} (k_0, |k|) \\
\phi_{3S^-_1} (k_0, |k|) \\
\phi_{3D^-_1} (k_0, |k|) \\
\phi_{1P^+_1} (k_0, |k|) \\
\phi_{3P^+_1} (k_0, |k|) \\
\phi_{1P^-_1} (k_0, |k|) \\
\phi_{3P^-_1} (k_0, |k|)
\end{pmatrix},
\]

\[
\phi^\dagger (k_0, |k|) = (\phi_{3S^+_1}^* (k_0, |k|), \phi_{3D^+_1}^* (k_0, |k|), \phi_{3S^-_1}^* (k_0, |k|), \phi_{3D^-_1}^* (k_0, |k|), \\
\phi_{1P^+_1}^* (k_0, |k|), \phi_{3P^+_1}^* (k_0, |k|), \phi_{1P^-_1}^* (k_0, |k|), \phi_{3P^-_1}^* (k_0, |k|)).
\]

From these equations, we can define the probability of finding the \( \alpha \)-state the total deuteron state by the following expression,

\[
P_\alpha = \frac{1}{N} \int dk_0 \int d |k| |k|^2 \omega_{\alpha \beta} |\phi_\alpha (k_0, |k|)|^2,
\]

(C.11)

where \( N \) is determined by

\[
\sum_\alpha P_\alpha = 1 \quad (\alpha = 3 \quad S^+_1, \quad 3 \quad D^+_1, \quad 3 \quad S^-_1, \quad 3 \quad D^-_1, \quad 1 \quad P^+_1, \quad 3 \quad P^+_1, \quad 3 \quad P^-_1, \quad 3 \quad P^-_1)
\]

(C.12)

\( P_\alpha \) is often called by as "pseudo probability" which is normalized by the baryon charge.
For example in the case of $^{3}S_{1}^{+}$

\[
P_{^{3}S_{1}^{+}} = \frac{1}{N} \int d\vec{k} \int d|\vec{k}|^2 \omega_\alpha |\phi_\alpha(k_0, |\vec{k}|)|^2
\]

\[
= \frac{1}{N} \int d\vec{k}_0 \int d|\vec{k}|^2 (-\frac{M}{2} + E_k) \frac{1}{\left((M/2 - E_k + i\epsilon)^2 - k_0^2\right)^2} \Gamma_{^{3}S_{1}^{+}}(k_0, |\vec{k}|)^2
\]

If we assume that $\Gamma_{^{3}S_{1}^{+}}(k_0, |\vec{k}|)$ has no pole in the $k_0$-plane (when there is a pole, we need careful treatment of the location of the pole of $\Gamma$), we have the only one pole from the propagator $S_+$, namely at $k_0 = \frac{M}{2} - E_k + i\epsilon$. As a result we can obtain

\[
P_{^{3}S_{1}^{+}} = \frac{1}{N'} \int d|\vec{k}|^2 \left(\frac{\Gamma_{^{3}S_{1}^{+}}(\vec{k}_0, |\vec{k}|)}{(M/2 - E_k)}\right)^2
\]  \hspace{1cm} (C.13)

where $\vec{k}_0 = M/2 - E_k$. Here we have used

\[
\int d\vec{k}_0 \left(\frac{1}{k_0^2 - k_0^2}\right)^2 \Gamma_{^{3}S_{1}^{+}}(k_0, |\vec{k}|)^2 = \int d\vec{k}_0 \frac{\partial}{\partial k_0^2} \left(\frac{1}{k_0^2 - k_0^2}\right) \Gamma_{^{3}S_{1}^{+}}(k_0, |\vec{k}|)^2
\]

\[
= \frac{\partial}{\partial k_0^2} \left(\frac{2\pi i}{4k_0} \Gamma_{^{3}S_{1}^{+}}(\vec{k}_0, |\vec{k}|)^2\right)
\]

\[
= -\frac{2\pi i}{4k_0} \Gamma_{^{3}S_{1}^{+}}(\vec{k}_0, |\vec{k}|)^2
\]  \hspace{1cm} (C.14)

From Eq. (C.13) we can consider that

\[
\frac{\Gamma_{^{3}S_{1}^{+}}(\vec{k}_0, |\vec{k}|)}{(M/2 - E_k)}
\]  \hspace{1cm} (C.15)

corresponds to the momentum space wave function in the non-relativistic framework. Formally the Schrodinger equation can be expressed as

\[
(H_0 + V)\psi = E\psi, \hspace{1cm} (C.16)
\]

where $\psi$ is a wave function, $H_0$ the kinetic energy and $V$ a potential energy. Then the solution can be expressed as

\[
\psi = \frac{1}{E - H_0} V\psi \equiv \frac{1}{E - H_0} \Gamma
\]  \hspace{1cm} (C.17)

Here we used that $V\psi = \Gamma$, which is obtained from Eq. (2.16) and Eq. (2.17). Comparing with Eq. (C.15) and Eq. (C.17), we can immediately understand that Eq. (C.15) corresponds to the Schrodinger wave function.
D Bonn potential

In this appendix, we summarize the Bonn potential. The Bonn potential is given as a superposition of various meson exchanges. In a one meson exchange scheme, the potential can be expressed conveniently in momentum space. Let $q$ and $q'$ be the relative momentum of the initial and final two-nucleon momentum. Also we define their sum and differences

$$k = q' - q, \quad p = \frac{1}{2}(q' + q) \tag{D.1}$$

The meson exchange potential is given in a fully relativistic way. However, in the non-relativistic limit in a small momentum region, they are given as functions of spatial components of momentum variables. For instance,

$$V_{\text{sc}}(k, p) = -\frac{g_{\text{sc}}^2}{k^2 + m_{\text{sc}}^2} \left( 1 - \frac{p^2}{2m^2} + \frac{k^2}{8m^2} - \frac{i}{2m^2} S \cdot (k \times p) \right), \tag{D.2}$$

$$V_{\text{ps}}(k, p) = -\frac{g_{\text{ps}}^2}{4m^2} \frac{(\sigma_1 \cdot k)(\sigma_2 \cdot k)}{k^2 + m_{\text{ps}}^2} \tag{D.3}$$

$$V_{\text{v}}(k, p) = \frac{1}{k^2 + m^2} \left( g_{\text{vc}}^2 \left( 1 + \frac{3p^2}{2m^2} - \frac{3i}{2m^2} S \cdot (k \times p) - \sigma_1 \cdot \sigma_2 \frac{k^2}{4m^2} + \frac{1}{4m^2} (\sigma_1 \cdot k)(\sigma_2 \cdot k) \right) + \frac{g_{\text{vc} f_{\text{vc}}}}{2m} \left( \frac{k^2}{m} + \frac{4i}{m} S \cdot (k \times p) - \sigma_1 \cdot \sigma_2 \frac{k^2}{m^2} + \frac{1}{m} (\sigma_1 \cdot k)(\sigma_2 \cdot k) \right) + \frac{f_{\text{vc}}^2}{4m^2} (-\sigma_1 \cdot \sigma_2 k^2 + (\sigma_1 \cdot k)(\sigma_2 \cdot k)) \right). \tag{D.4}$$

where sc denotes scalar meson, ps denotes pseudoscalar meson, v denotes vector meson exchanges which contribute dominantly to the $NN$ interaction. The momentum space, interaction can be transformed into those in $r$-space. The results are

$$V_{\text{ps}}(m_{\text{ps}}, r) = \frac{1}{12} \frac{g_{\text{ps}}^2}{4\pi} m_{\text{ps}} \left( \left( \frac{m_{\text{ps}}}{m} \right)^2 Y(m_{\text{ps}} r) \sigma_1 \cdot \sigma_2 + Z(m_{\text{ps}} r) S_{12} \right), \tag{D.5}$$

$$V_{\text{s}}(m_{\text{s}}, r) = -\frac{g_{\text{s}}^2}{4\pi} m_{\text{s}} \left( 1 - \frac{1}{4} \left( \frac{m_{\text{s}}}{m} \right)^2 \right) + \frac{1}{4m^2} (\nabla^2 Y(m_{\text{s}} r) + Y(m_{\text{s}} r) \nabla^2) + \frac{1}{2} Z_1(m_{\text{s}} r) L \cdot S \right), \tag{D.6}$$
\[ V_v(m_v, r) = \frac{g_v^2}{4\pi} m_v \left( 1 + \frac{1}{2} \left( \frac{m_v}{m} \right)^2 \right) Y(m_v r) - \frac{3}{4m^2} Z(m_v r) (\nabla^2 Y(m_v r) + Y(m_v r) \nabla^2) \]
\[ + \frac{1}{6} \left( \frac{m_v}{m} \right)^2 Y(m_v r) \sigma_1 \cdot \sigma_2 - \frac{3}{2} Z_1(m_v r) \mathbf{L} \cdot \mathbf{S} - \frac{1}{12} Z(m_v r) S_{12} \]
\[ + \frac{1}{2} g_v f_v m_v \left( \left( \frac{m_v}{m} \right)^2 Y(m_v r) + \frac{2}{3} \left( \frac{m_v}{m} \right)^2 Y(m_v r) \sigma_1 \cdot \sigma_2 \right) \]
\[ - 4Z_1(m_v r) \mathbf{L} \cdot \mathbf{S} - \frac{1}{3} Z(m_v r) S_{12} \]
\[ + \frac{f_v^2}{4\pi} m_v \left( \frac{1}{6} \left( \frac{m_v}{m} \right)^2 Y(m_v r) \sigma_1 \cdot \sigma_2 - \frac{1}{12} Z(m_v r) S_{12} \right) , \]  
(D.7)

where

\[ Y(x) = \frac{e^{-x}}{x}, \quad Z(x) = \left( \frac{m_\alpha}{m} \right)^2 (1 + \frac{3}{x} + \frac{3}{x^2}) Y(x) , \]  
(D.8)

\[ Z_1(x) = \left( \frac{m_\alpha}{m} \right)^2 (1 + \frac{1}{x^2}) Y(x), \quad S_{12} = 3 \frac{(\sigma_1 \cdot \mathbf{r}) (\sigma_2 \cdot \mathbf{r})}{r^2} - \sigma_1 \cdot \sigma_2 ; \]  
(D.9)

\[ \nabla^2 = \frac{1}{r} \frac{d^2}{dr^2} r - \frac{\mathbf{L}^2}{r^2} . \]  
(D.10)

For actual calculations, we introduce the form factor. The following prescription has been conveniently introduced:

\[ V_\alpha(r) = V_\alpha(m_\alpha, r) - \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,1}, r) + \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,2}, r) \]  
(D.11)

\[ \Lambda_{\alpha,1} \equiv \Lambda_\alpha + \epsilon, \quad \Lambda_{\alpha,2} \equiv \Lambda_\alpha - \epsilon, \quad \epsilon = 10 \text{MeV} \]  
(D.12)

In momentum space, we find the interaction with the form factor as

\[ V_\alpha(p) = V_\alpha(m_\alpha, p) - \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,1}, p) + \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2} V_\alpha(\Lambda_{\alpha,2}, p) . \]  
(D.13)

For small \( \epsilon \), we can verify that the form factor is equivalent to the monopole (at each vertex) type form factor. Let us check it for

\[ V_\alpha(m, p) = \frac{g_v^2}{p^2 + m_\alpha^2} . \]  
(D.14)

The result is

\[ V_\alpha(p) = \frac{1}{p^2 + m_\alpha^2} - \frac{\Lambda_\alpha^2 - m_\alpha^2}{\Lambda_\alpha^2 - \Lambda_\alpha^2} \frac{1}{p^2 + \Lambda_\alpha^2} + \frac{\Lambda_\alpha^2 - m_\alpha^2}{\Lambda_\alpha^2 - \Lambda_\alpha^2} \frac{1}{p^2 + \Lambda_\alpha^2} \]
\[ = \frac{1}{(p^2 + m_\alpha^2)(\Lambda_\alpha^2 - \Lambda_\alpha^2)(p^2 + \Lambda_\alpha^2)(p^2 + \Lambda_\alpha^2)} ((\Lambda_\alpha^2 - \Lambda_\alpha^2)(\Lambda_\alpha^2 + p^2)(\Lambda_\alpha^2 + p^2) \]
\[ - (p^2 + m_\alpha^2)(\Lambda_\alpha^2 - \Lambda_\alpha^2)(p^2 + m_\alpha^2 + \Lambda_\alpha^2 + \Lambda_\alpha^2)) \]
\[
\frac{\omega^2_{\pi\pi}}{m_\omega^2} \quad m_\alpha (MeV) \quad \Lambda (GeV)
\]

<table>
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<th></th>
<th>(\frac{\omega^2_{\pi\pi}}{m_\omega^2})</th>
<th>(m_\alpha) (MeV)</th>
<th>(\Lambda) (GeV)</th>
</tr>
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<td>550</td>
<td>2.0</td>
</tr>
<tr>
<td>(\rho)</td>
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<td>769</td>
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<td>(\omega)</td>
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</tr>
<tr>
<td>(\pi)</td>
<td>14.9</td>
<td>138.03</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 12: Mass and coupling constants.

\[
\begin{align*}
&= \frac{1}{(p^2 + m_\alpha^2)(\Lambda_2^2 - \Lambda_1^2)(p^2 + \Lambda_1^2)(p^2 + \Lambda_2^2)}(\Lambda_2^2 - \Lambda_1^2)(m_\alpha^2 - \Lambda_2^2)(m_\alpha^2 - \Lambda_1^2) \\
&= \frac{1}{(m_\alpha^2 - \Lambda_1^2)(m_\alpha^2 - \Lambda_2^2)} \\
&= \frac{1}{(p^2 + m_\alpha^2)(p^2 + \Lambda_1^2)(p^2 + \Lambda_2^2)}(\Lambda_2^2 - m_\alpha^2)^2
\end{align*}
\]

(D.15)

In the last step we used \(\epsilon \ll m_\alpha\).

We show the resulting potential in \(r\)-space in the Figs. 31. The left and right panel is for the results with and without the form factor. For each cases, separate contributions of one-meson-exchange for \(\sigma\), \(\omega\), \(\rho\) and \(\pi\) exchanges are shown as well as the total potential. For the parameters of coupling constants and mass for each meson, we have used the values shown in Table. 12.
Figure 31: Potential with (left panel) and without (right panel) the form factors.
E  Magnetic moment

The magnetic moment is defined as

$$\mu = \langle J M = J | \mu_z | J M = J \rangle$$

(E.1)

where \( \mu_z \) is the z component of the magnetic operator \( \mu \). The definition of the operator of the magnetic moment is

$$\mu = \mu_p + \mu_n,$$

$$= \ell_p + \mu_p \sigma_p + \mu_n \sigma_n,$$

(E.2)

$$= \frac{1}{2} l + (\mu_p + \mu_n) s + \frac{1}{2}(\mu_p - \mu_n)(\sigma_p - \sigma_n),$$

(E.3)

where

$$s = \frac{\sigma_p + \sigma_n}{2}, \mu_p = 2.79279, \mu_n = -1.91315.$$  

(E.4)

Now the deuteron state is expressed by eight basis states in the relativistic framework:

$$|D\rangle = C_{3S^+_1}|3S^+_1\rangle + C_{3D^+_1}|3D^+_1\rangle + C_{3S^-_1}|3S^-_1\rangle + C_{3D^-_1}|3D^-_1\rangle + C_{1S^+_1}|1S^+_1\rangle + C_{3P^+_1}|3P^+_1\rangle + C_{1P^+_1}|1P^+_1\rangle + C_{3P^-_1}|3P^-_1\rangle,$$

(E.5)

where \( |\alpha\rangle \) are normalized spin-angular part of the basis states defined by

$$|^{2S+1}L^p, M\rangle = |L, S\rangle_{JM} \otimes |\rho_1 \rho_2\rangle.$$  

(E.6)

Each state can be expressed

$$|^{3S^+_1}\rangle = (0 0 1 S_2 | 1 M \rangle | 0 0 \rangle \otimes | 1 S_2 \rangle \otimes | + + \rangle,$$

$$|^{3D^+_1}\rangle = (2 L 1 S_2 | 1 M \rangle | 2 L 2 \rangle \otimes | 1 S_2 \rangle \otimes | + + \rangle,$$

$$|^{3S^-_1}\rangle = (0 0 1 S_2 | 1 M \rangle | 0 0 \rangle \otimes | 1 S_2 \rangle \otimes | - - \rangle,$$

$$|^{3D^-_1}\rangle = (2 L 1 S_2 | 1 M \rangle | 2 L 2 \rangle \otimes | 1 S_2 \rangle \otimes | - - \rangle,$$

$$|^{1P^+_1}\rangle = (1 L 0 0 | 1 M \rangle | 1 L 2 \rangle \otimes | 0 0 \rangle \otimes | + - \rangle,$$

$$|^{1P^-_1}\rangle = (1 L 0 0 | 1 M \rangle | 1 L 2 \rangle \otimes | 0 0 \rangle \otimes | - + \rangle,$$

$$|^{3P^+_1}\rangle = (1 L 1 S_2 | 1 M \rangle | 1 L 2 \rangle \otimes | 1 S_2 \rangle \otimes | + - \rangle,$$

$$|^{3P^-_1}\rangle = (1 L 1 S_2 | 1 M \rangle | 1 L 2 \rangle \otimes | 1 S_2 \rangle \otimes | - + \rangle.$$  

(E.7)

If we ignore \(^3S^-_1, \, ^3D^-_1\) states for simplicity,

$$\langle D | \mu | D \rangle = \left( \langle ^3S^+_1 | C^*_{3S^+_1} + \langle ^3D^+_1 | C^*_{3D^+_1} + \langle ^1P^+_1 | C^*_{1P^+_1} \right.$$

$$+ \langle ^3P^+_1 | C^*_{3P^+_1} + \langle ^1P^-_1 | C^*_{1P^-_1} + \langle ^3P^-_1 | C^*_{3P^-_1} \right)$$

$$\mu \left( C_{3S^+_1} | ^3S^+_1 \rangle + C_{3D^+_1} | ^3D^+_1 \rangle + C_{1P^+_1} | ^1P^+_1 \rangle + C_{3P^+_1} | ^3P^+_1 \rangle \right)$$

(E.8)
In the last step we have used $J_{E.1}$. For further calculations, we will use these relations,

\[
\langle J M | \mu_2 | J M \rangle = \langle J M | \mu | J M \rangle = \langle J M | \mu_1 | J M \rangle = \frac{1}{\sqrt{3}} \langle J M | 1 0 | J M \rangle \langle J M | \mu_1 | 1 \rangle = \frac{1}{\sqrt{3}} \langle 1 1 1 0 | 1 1 \rangle \langle 1 1 | \mu_1 | 1 \rangle = \frac{1}{\sqrt{6}} \langle 1 | \mu_1 | 1 \rangle . \tag{E.9}
\]

In the last step we have used $J = 1$ and $M = 1$. And we use one more relation,

\[
\langle 1 | \mu_1 | 1 \rangle = 3 \sqrt{3} \left\{ l_1 \ s_1 \ 1 \right\} \langle l_2 | T_1 | l_1 \rangle \langle s_2 | \sigma_1 | s_1 \rangle . \tag{E.10}
\]

At first, we calculate the diagonal element.

### E.1 Diagonal elements

#### E.1.1 $\langle ^3S_1^+ | \mu | ^3S_1^+ \rangle$

We can express the magnetic moment as a sum of the orbital angular momentum and the spin part,

\[
\langle ^3S_1^+ | \mu | ^3S_1^+ \rangle = |c_{^3S_1^+}|^2 \left\{ \frac{1}{2} \langle ^3S_1^+ | \mu | ^3S_1^+ \rangle + (\mu_p + \mu_n) \langle ^3S_1^+ | \mu_s | ^3S_1^+ \rangle \right\} . \tag{E.11}
\]
For the orbital momentum part,
\[
\langle 3S^+ || |l| 3S^+ \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\} \langle 0||L||0\rangle \langle 1||1\rangle = 0. \tag{E.12}
\]

In the last step, we have used
\[
\langle 0||L||0\rangle = 0. \tag{E.13}
\]

For the spin part,
\[
\langle 3S^+ || |s| 3S^+ \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\} \langle 0||1||0\rangle \langle 1||s||1\rangle = 0.
\tag{E.14}
\]

Finally we obtain
\[
\langle 3S^+ |\mu| 3S^+ \rangle = (\mu_p + \mu_n)|C_{3S^+}|^2. \tag{E.15}
\]

\textbf{E.1.2} \quad \langle 3D^+ |\mu| 3D^+ \rangle

Similarly, we write the matrix element for the $D$-state as
\[
\langle 3D^+ || |\mu| 3D^+ \rangle = |C_{3D^+}|^2 \left\{ \begin{array}{cc} \frac{1}{2} \langle 3D^+ || |l| 3D^+ \rangle \\ + (\mu_p + \mu_n)\langle 3D^+ || |s| 3D^+ \rangle \end{array} \right\}. \tag{E.16}
\]

For each terms,
\[
\langle 3D^+ || |l| 3D^+ \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right\} \langle 2||L||2\rangle \langle 1||1||1\rangle
\]
\[
= 3\sqrt{3} \frac{1}{6\sqrt{5}} \sqrt{2 \cdot 3 \cdot 5 \sqrt{3}}
\]
\[
= \frac{3\sqrt{6}}{2}, \tag{E.17}
\]
\[
\langle 3D^+ || |s| 3D^+ \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\} \langle 2||1||2\rangle \langle 1||s||1\rangle
\]
\[
= 3\sqrt{3} \frac{-1}{6\sqrt{15}} \sqrt{5\sqrt{6}}
\]
\[
= -\frac{\sqrt{6}}{2}. \tag{E.18}
\]
Finally we obtain
\[ \langle 3D^+_1|\mu|3D^+_1 \rangle = |C_{3D^+_1}|^2 \left\{ \frac{1}{\sqrt{6}} \frac{3\sqrt{6}1}{2} + (\mu_p + \mu_n) \frac{1}{\sqrt{6}} \left( -\frac{\sqrt{6}}{2} \right) \right\} \]
\[ = |C_{3D^+_1}|^2 \left\{ \frac{3}{4} - (\mu_p + \mu_n) \frac{1}{2} \right\} . \]  
(E.19)

Below we summarize the results of other matrix elements

E.1.3  \[ \langle 1P^+|\mu|1P^+ \rangle \]
\[ \langle 1P^-||\mu||1P^- \rangle = |C_{1P^-}|^2 \left\{ \frac{1}{2} \langle 1P^-||l||1P^- \rangle + (\mu_p + \mu_n) \langle 1P^-||s||1P^- \rangle \right\} \]  
(E.20)

\[ \langle 1P^-||l||1P^- \rangle = 3\sqrt{3} \left\{ \begin{array}{ll} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{array} \right\} \langle 1||L||1 \rangle \langle 0||1||0 \rangle \]
\[ = 3\sqrt{3} \times \frac{1}{3\sqrt{3}} \times 1 \]
\[ = \sqrt{6} \]  
(E.21)

\[ \langle 1P^+||s||1P^+ \rangle = 3\sqrt{3} \left\{ \begin{array}{ll} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{array} \right\} \langle 1||1||1 \rangle \langle 0||s||0 \rangle \]
\[ = 3\sqrt{3} \times \frac{1}{3\sqrt{3}} \sqrt{3} \times 0 \]
\[ = 0 \]  
(E.22)

\[ \langle 1P^-|\mu|1P^- \rangle = |C_{1P^-}|^2 \left\{ \frac{1}{2} \times \frac{1}{\sqrt{6}} \times \sqrt{6} \right\} \]
\[ = |C_{1P^-}|^2 \frac{1}{2} \]  
(E.23)

E.1.4  \[ \langle 1P^-|\mu|1P^- \rangle \]
\[ \langle 1P^-|\mu|1P^- \rangle = |C_{1P^-}|^2 \left\{ \frac{1}{2} \times \frac{1}{\sqrt{6}} \times \sqrt{6} \right\} \]
\[ = |C_{1P^-}|^2 \frac{1}{2} \]  
(E.24)
\[ \langle 3P^+^-|\mu|3P^+^- \rangle = \langle 3P^+^-| \mu |3P^+^- \rangle = |c_{3P^+^-}|^2 \left\{ \frac{1}{2} \langle 3P^+^-| |\mu| - 1 \rangle \langle 3P^+^-| |\mu| - 1 \rangle + (\mu_p + \mu_n) \langle 3P^+^-| |s| - 1 \rangle \langle 3P^+^-| |s| - 1 \rangle \right\} \] (E.25)

\[ \langle 3P^+^-| |\mu| |3P^+^- \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right\} \langle 1||L||1 \rangle \langle 1||1||1 \rangle \]
\[ = 3\sqrt{3} \times \frac{1}{18} \times \sqrt{3} \]
\[ = \frac{\sqrt{6}}{2} \] (E.26)

\[ \langle 3P^+^-| |s| |3P^+^- \rangle = 3\sqrt{3} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\} \langle 1||1||1 \rangle \langle 0||s||0 \rangle \]
\[ = 3\sqrt{3} \times \frac{1}{18} \sqrt{1 \cdot 2 \cdot 3} \]
\[ = \frac{\sqrt{6}}{2} \] (E.27)

\[ \langle 3P^+^-| |\mu| |3P^+^- \rangle = |C_{3P^+^-}|^2 \left\{ \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{2} \times \frac{1}{2} + (\mu_p + \mu_n) \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{2} \right\} \]
\[ = |C_{3P^+^-}|^2 \left\{ \frac{1}{4} + (\mu_p + \mu_n) \frac{1}{2} \right\} \] (E.28)

E.1.6 \[ \langle 3P^+^-| |\mu| |3P^+^- \rangle \]

\[ \langle 3P^+^-| |\mu| |3P^+^- \rangle = |C_{3P^+^-}|^2 \left\{ \frac{1}{4} + (\mu_p + \mu_n) \frac{1}{2} \right\} \] (E.29)

E.2 Off-diagonal elements

We have no off-diagonal term because of the orthogonal.
Finally we reached the expression of the magnetic form factor.

\[
\mu = (\mu_p + \mu_n)|C_3s_1^+|^2 + \left(\frac{3}{4} - \frac{1}{2}(\mu_p + \mu_n)\right)|C_3d_1^+|^2 + \frac{1}{2}(|C_1p_1^+|^2 + |C_1p_1^-|^2) \\
+ \frac{1}{2}(\mu_p + \mu_n)(|C_3p_1^-|^2 + |C_3p_1^+|^2) + \frac{1}{4}(|C_3p_1^-|^2 + |C_3p_1^+|^2) \\
= (\mu_p + \mu_n)|C_3s_1^+|^2 + \left(\frac{3}{4} - \frac{1}{2}(\mu_p + \mu_n)\right)|C_3d_1^+|^2 + \frac{1}{2}(|C_1p_1^+|^2 + |C_1p_1^-|^2) \\
+ \frac{1}{2}(\mu_p + \mu_n)(|C_3p_1^-|^2 + |C_3p_1^+|^2) + \frac{1}{4}(|C_3p_1^-|^2 + |C_3p_1^+|^2) \\
= (\mu_p + \mu_n)p_3s_1^+ + \left(\frac{3}{4} - \frac{1}{2}(\mu_p + \mu_n)\right)p_3d_1^+ + \frac{1}{2}(p_1p_1^+ + p_1p_1^-) \\
+ \frac{1}{2}(\mu_p + \mu_n)(p_3p_1^- + p_3p_1^+ + \frac{1}{4}(p_3p_1^- + p_3p_1^+)) \tag{E.30}
\]

In the last line we use these relations,

\[
|C_3s_1^+|^2 = P_3s_1^+, \\
|C_3d_1^+|^2 = P_3d_1^+, \\
|C_1p_1^+|^2 = P_1p_1^+, \\
|C_3p_1^-|^2 = P_3p_1^- + \frac{1}{2}(P_1p_1^+ + P_1p_1^-) \\
|C_3p_1^+|^2 = P_3p_1^+ + \frac{1}{2}(P_1p_1^+ + P_1p_1^-) \tag{E.30}
\]

where \( P_\alpha \) is defined by Eq. (C.11).
References


