Basic and useful formulas of Angular momentum algebra

Kazuhito Mizuyama[1*]
[1] Institute of Research and Development, Duy Tan University, Da Nang 550000, Viet Nam
[2] Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Viet Nam
(Dated: November 3, 2021)

I. BASIC PROPERTIES OF 3J-SYMBOL

The relation between the Clebsh-Gordan's coefficient is

$$C^{j_3m_3}_{j_1m_1,j_2m_2} = \langle j_1m_1 : j_2m_2|j_3m_3\rangle = (-)^{-j_1+j_2-m_3}\widehat{j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \tag{1}$$

where $\widehat{j} = \sqrt{2j+1}$
3j-symbol has a property as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{-j_1-j_2-j_3}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \tag{3}$$

$$= (-)^{-j_1-j_2-j_3}\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} \tag{4}$$

$$= (-)^{-j_1-j_2-j_3}\begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} \tag{5}$$

Note that this realtion is correct for both the integer and harf-integer j's.
3j-symbol has the orthogonality shown as below

$$\sum_{j_3m_3}\widehat{j_3}^2\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1m_1'}\delta_{m_2m_2'} \tag{6}$$

$$\sum_{m_1m_2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \widehat{j_3}^{-2}\delta_{j_3j_3'}\delta_{m_3m_3'}\delta(j_1,j_2,j_3) \tag{7}$$

where $\delta(j_1,j_2,j_3) = 1$ if j_1, j_2, j_3 satisfy the triangular condition.

*Electronic address: mizukazu147@gmail.com

A. 3j-symbol and 6j-symbol

$$\sum_{\mu_1\mu_2\mu_3}(-1)^{l_1+l_2+l_3+\mu_1+\mu_2+\mu_3}\begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & \mu_2 & -\mu_3 \end{pmatrix}\begin{pmatrix} l_1 & j_2 & l_3 \\ -\mu_1 & m_2 & \mu_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & j_3 \\ \mu_1 & -\mu_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{8}$$

$$\sum_{\mu_3}(-1)^{l_1+l_2+l_3+\mu_1+\mu_2+\mu_3}\begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & \mu_2 & -\mu_3 \end{pmatrix}\begin{pmatrix} l_1 & j_2 & l_3 \\ -\mu_1 & m_2 & \mu_3 \end{pmatrix} = \sum_{j_3 m_3}\widehat{j_3}^{\,2}\begin{pmatrix} l_1 & l_2 & j_3 \\ \mu_1 & -\mu_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{9}$$

1. Useful formula

$$\begin{aligned}\begin{pmatrix} L & l' & l \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} j & j' & L \\ l' & l & \frac{1}{2} \end{Bmatrix} &= (-)^{j+j'}\left\{1+(-)^{L+l+l'}\right\}\begin{pmatrix} j & j' & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} l' & \frac{1}{2} & j' \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} l & \frac{1}{2} & j \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \frac{-1}{\widehat{ll'}}\left\{\frac{1+(-)^{L+l+l'}}{2}\right\}\begin{pmatrix} j & j' & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\ &= \frac{(-)^{j+j'}}{\widehat{ll'}\widehat{L}}\left\{\frac{1+(-)^{L+l+l'}}{2}\right\}\langle j1/2; j'-1/2|L0\rangle \end{aligned} \tag{10}$$

$$\begin{pmatrix} \lambda & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} \lambda & l' & l'' \\ l & 1 & L \end{Bmatrix} = \sum_{\mu}(-)^{l+1+L+\mu}\begin{pmatrix} \lambda & 1 & L \\ 0 & \mu & -\mu \end{pmatrix}\begin{pmatrix} l & l' & L \\ -\mu & 0 & \mu \end{pmatrix}\begin{pmatrix} l & 1 & l'' \\ \mu & -\mu & 0 \end{pmatrix} \tag{11}$$

$$\begin{aligned} &= (-)^{L+l+1}\begin{pmatrix} \lambda & 1 & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & 1 & l'' \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad +(-)^{L+l}\left\{1+(-)^{\lambda+l'+l''}\right\}\begin{pmatrix} \lambda & 1 & L \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} l & l' & L \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} l & 1 & l'' \\ 1 & -1 & 0 \end{pmatrix} \end{aligned} \tag{12}$$

B. 3j-symbol and 9j-symbol

$$\sum_{JM} \widehat{J}^2 \begin{pmatrix} J & j_{32} & j_{33} \\ M & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} j_{11} & j_{21} & J \\ m_{11} & m_{21} & M \end{pmatrix} \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ J & j_{32} & j_{33} \end{Bmatrix}$$
$$= \sum_{m_{12}m_{13}m_{22}m_{23}} \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ m_{11} & \boldsymbol{m_{12}} & \boldsymbol{m_{13}} \end{pmatrix} \begin{pmatrix} j_{21} & j_{22} & j_{23} \\ m_{21} & \boldsymbol{m_{22}} & \boldsymbol{m_{23}} \end{pmatrix} \begin{pmatrix} j_{12} & j_{22} & j_{32} \\ \boldsymbol{m_{12}} & \boldsymbol{m_{22}} & m_{32} \end{pmatrix} \begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \boldsymbol{m_{13}} & \boldsymbol{m_{23}} & m_{33} \end{pmatrix} \tag{13}$$

$$\sum_{JM,J'M'} \widehat{J}^2 \widehat{J'}^2 \begin{pmatrix} J & J' & j_{33} \\ M & M' & m_{33} \end{pmatrix} \begin{pmatrix} j_{11} & j_{21} & J \\ m_{11} & m_{21} & M \end{pmatrix} \begin{pmatrix} j_{12} & j_{22} & J' \\ m_{12} & m_{22} & M' \end{pmatrix} \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ J & J' & j_{33} \end{Bmatrix}$$
$$= \sum_{m_{13}m_{23}} \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ m_{11} & m_{12} & \boldsymbol{m_{13}} \end{pmatrix} \begin{pmatrix} j_{21} & j_{22} & j_{23} \\ m_{21} & m_{22} & \boldsymbol{m_{23}} \end{pmatrix} \begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \boldsymbol{m_{13}} & \boldsymbol{m_{23}} & m_{33} \end{pmatrix} \tag{14}$$

C. Special properties of 3j-symbol

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \frac{(-)^{j-m}}{\widehat{j}} \tag{15}$$

$$\sum_m (-)^m \begin{pmatrix} j & j & L \\ m & -m & M \end{pmatrix} = (-)^j \widehat{j} \delta_{L0} \delta_{M0} \tag{16}$$

$$\frac{1+(-)^{l+l'+L}}{2} \sqrt{l(l+1)} \begin{pmatrix} l' & l & L \\ 0 & 1 & -1 \end{pmatrix} = \frac{l'(l'+1) - l(l+1) - L(L+1)}{2\sqrt{L(L+1)}} \begin{pmatrix} l' & l & L \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{If } L \neq 0. \tag{17}$$

$$\begin{pmatrix} 1 & l & l'' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'',l+1} (-)^{l+1} \widehat{l''}^{-1} \sqrt{\frac{l+1}{2l+1}} + \delta_{l'',l-1} (-)^{l} \widehat{l''}^{-1} \sqrt{\frac{l}{2l+1}} \tag{18}$$

$$(-)^L \widehat{L} \begin{pmatrix} L+1 & 1 & L \\ -q & q & 0 \end{pmatrix} = (-)^L \widehat{L} \begin{pmatrix} L+1 & L & 1 \\ -q & 0 & q \end{pmatrix} = \begin{cases} (q=\pm 1) \quad (-)^L \widehat{L} \begin{pmatrix} L+1 & L & 1 \\ -1 & 0 & 1 \end{pmatrix} = \sqrt{\frac{L+2}{2(2L+3)}} \\ (q=0) \quad (-)^L \widehat{L} \begin{pmatrix} L+1 & L & 1 \\ 0 & 0 & 0 \end{pmatrix} = -\sqrt{\frac{L+1}{2L+3}} \end{cases} \tag{19}$$

$$(-)^L \widehat{L} \begin{pmatrix} L-1 & 1 & L \\ -q & q & 0 \end{pmatrix} = (-)^L \widehat{L} \begin{pmatrix} L & L-1 & 1 \\ 0 & -q & q \end{pmatrix} = \begin{cases} (q=\pm 1) \quad (-)^L \widehat{L} \begin{pmatrix} L & L-1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \sqrt{\frac{L-1}{2(2L-1)}} \\ (q=0) \quad (-)^L \widehat{L} \begin{pmatrix} L & L-1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{L}{2L-1}} \end{cases} \tag{20}$$

D. Special properties of 6j-symbol

$$\left\{\begin{array}{ccc} 1 & l'' & l \\ l & 1 & 1 \end{array}\right\} = \delta_{l'',l+1}\hat{l}^{-1}\frac{l}{\sqrt{6l(l+1)}} - \delta_{l'',l-1}\hat{l}^{-1}\frac{l+1}{\sqrt{6l(l+1)}} \tag{21}$$

$$\left\{\begin{array}{ccc} l & l & 1 \\ 1/2 & 1/2 & j \end{array}\right\} = (-)^{j+l+1/2}\frac{j(j+1)-l(l+1)-3/4}{\hat{l}\sqrt{6l(l+1)}} \tag{22}$$

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & 0 \end{array}\right\} = \frac{(-)^{j_1+j_2+j_3}}{\hat{j}_1\hat{j}_2}\delta_{j_1j_5}\delta_{j_2j_4} \tag{23}$$

$$\sum_j (-)^{j+j'+j''}\hat{j}^2 \left\{\begin{array}{ccc} j_1 & j_2 & j' \\ j_3 & j_4 & j \end{array}\right\}\left\{\begin{array}{ccc} j_1 & j_3 & j'' \\ j_2 & j_4 & j \end{array}\right\} = \left\{\begin{array}{ccc} j_1 & j_2 & j' \\ j_4 & j_3 & j'' \end{array}\right\} \tag{24}$$

$$\sum_j \hat{j}^2 \left\{\begin{array}{ccc} j_1 & j_2 & j' \\ j_3 & j_4 & j \end{array}\right\}\left\{\begin{array}{ccc} j_1 & j_2 & j'' \\ j_3 & j_4 & j \end{array}\right\} = \frac{\delta_{j'j''}}{\hat{j'}^2}\delta(j_1,j_2,j')\delta(j_3,j_4,j') \tag{25}$$

1. 6j-symbol and 9j-symbol in special case

$$\left\{\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & 0 \end{array}\right\} = \delta_{cf}\delta_{gh}\frac{(-)^{b+c+d+g}}{\sqrt{(2c+1)(2g+1)}}\left\{\begin{array}{ccc} a & b & c \\ e & d & g \end{array}\right\} \tag{26}$$

$$\left\{\begin{array}{ccc} j & 1 & \lambda \\ j' & 1 & \lambda' \\ J & 0 & L \end{array}\right\} = -\frac{(-)^{j+\lambda'+L}}{\sqrt{3}}\frac{\delta_{J,L}}{\hat{L}}\left\{\begin{array}{ccc} j & \lambda & 1 \\ \lambda' & j' & L \end{array}\right\} \tag{27}$$

$$\left\{\begin{array}{ccc} j & \frac{1}{2} & l \\ j & \frac{1}{2} & l \\ J & 1 & 0 \end{array}\right\} = \delta_{J,1}\left\{\begin{array}{ccc} j & \frac{1}{2} & l \\ j & \frac{1}{2} & l \\ 1 & 1 & 0 \end{array}\right\} \tag{28}$$

$$\left\{\begin{array}{ccc} a & b & c \\ d & e & c \\ g & g & 1 \end{array}\right\} = \frac{a(a+1)+e(e+1)-d(d+1)-b(b+1)}{2\sqrt{c(c+1)g(g+1)}}\left\{\begin{array}{ccc} a & b & c \\ d & e & c \\ g & g & 0 \end{array}\right\} \tag{29}$$

$$= (-)^{b+c+d+g}\frac{a(a+1)+e(e+1)-d(d+1)-b(b+1)}{2\sqrt{c(c+1)(2c+1)g(g+1)(2g+1)}}\left\{\begin{array}{ccc} a & b & c \\ e & d & g \end{array}\right\} \tag{30}$$

II. SPHERICAL HARMONICS

The most important and often used formula of the spherical harmonics is

$$Y_{lm_l}(\hat{\boldsymbol{r}})Y^*_{l'm'_l}(\hat{\boldsymbol{r}}) = \sum_{LM}(-)^{m'_l}\frac{\widehat{l}\widehat{l'}\widehat{L}}{\sqrt{4\pi}}\begin{pmatrix} l & l' & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l' & L & l \\ -m'_l & M & m_l \end{pmatrix}Y^*_{LM}(\hat{\boldsymbol{r}}) \tag{31}$$

$$= \sum_{LM}\langle l'||Y_L||l\rangle(-)^{l'-m'_l}\begin{pmatrix} l' & L & l \\ -m'_l & M & m_l \end{pmatrix}Y^*_{LM}(\hat{\boldsymbol{r}}) \tag{32}$$

$$\sum_{m_l}Y_{lm_l}(\hat{\boldsymbol{r}})Y^*_{lm_l}(\hat{\boldsymbol{r}}') = \frac{2l+1}{4\pi}P_l(\cos\theta) \qquad \text{where } \theta \text{ is the relative angle between } \hat{\boldsymbol{r}} \text{ and } \hat{\boldsymbol{r}}'. \tag{33}$$

In case $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}'$,

$$\sum_{m_l}Y_{lm_l}(\hat{\boldsymbol{r}})Y^*_{lm_l}(\hat{\boldsymbol{r}}) = \frac{\widehat{l}^2}{4\pi}P_l(1) = \frac{2l+1}{4\pi} \tag{34}$$

A. Spin-function: $S = \frac{1}{2}$

The spin function $\chi_{\frac{1}{2}m_s}(\sigma)$ satisfies the following properties,

$$\sum_{m_s}\chi_{\frac{1}{2}m_s}(\sigma')\chi^\dagger_{\frac{1}{2}m_s}(\sigma) = \delta_{\sigma\sigma'}, \qquad \sum_{\sigma}\chi^\dagger_{\frac{1}{2}m_s}(\sigma)\chi_{\frac{1}{2}m'_s}(\sigma) = \delta_{m_s m'_s} \tag{35}$$

$$\chi_{\frac{1}{2}m_s}(\sigma)\chi^\dagger_{\frac{1}{2}m'_s}(\sigma') = \frac{1}{2}\delta_{m_s m'_s}\langle\sigma|\sigma'\rangle + \frac{1}{2}(-)^{\frac{1}{2}-m'_s}\left\langle\frac{1}{2}||\sigma||\frac{1}{2}\right\rangle\sum_{\mu}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m'_s & \mu & m_s \end{pmatrix}(-)^{\mu}\langle\sigma|\sigma_{-\mu}|\sigma'\rangle \tag{36}$$

$$\text{where} \qquad \left\langle\frac{1}{2}||\sigma||\frac{1}{2}\right\rangle = \sqrt{6} \tag{37}$$

B. Spin-spherical harmonics

Coupling with the spin wave function, the so-called spin-spherical harmonics is defined by

$$\mathcal{Y}_{ljm}(\hat{\boldsymbol{r}}\sigma) = \sum_{m_l,m_s}\langle lm_l : \frac{1}{2}m_s|jm\rangle\chi_{\frac{1}{2}m_s}(\sigma)Y_{lm_l}(\hat{\boldsymbol{r}}) \tag{38}$$

$$\mathcal{Y}^*_{ljm}(\hat{\boldsymbol{r}}\sigma) = (-)^{l+1/2+j+m}\mathcal{Y}_{lj-m}(\hat{\boldsymbol{r}}\sigma) \tag{39}$$

$$\begin{aligned}\mathcal{Y}_{ljm}(\hat{\boldsymbol{r}}\sigma)\mathcal{Y}^*_{l'j'm'}(\hat{\boldsymbol{r}}\sigma') &= \frac{1}{2}\langle\sigma|\sigma'\rangle\sum_{LM}\langle l'j'||Y_L||lj\rangle(-)^{j'-m'}\begin{pmatrix} j' & L & j \\ -m' & M & m \end{pmatrix}Y^*_{LM}(\hat{\boldsymbol{r}}) \\ &+\frac{1}{2}\langle\sigma|\boldsymbol{\sigma}|\sigma'\rangle\cdot\sum_{L,JM_J}\langle l'j'||\boldsymbol{Y}_{JL}\cdot\boldsymbol{\sigma}||lj\rangle(-)^{j'-m'}\begin{pmatrix} j' & J & j \\ -m' & M_J & m \end{pmatrix}\boldsymbol{Y}^*_{JLM_J}(\hat{\boldsymbol{r}})\end{aligned} \tag{40}$$

$$\begin{aligned}&= \frac{1}{2}\langle\sigma|\sigma'\rangle\sum_{LM}\langle l'j'||Y_L||lj\rangle(-)^{j'-m'}\begin{pmatrix} j' & L & j \\ -m' & M & m \end{pmatrix}Y^*_{LM}(\hat{\boldsymbol{r}}) \\ &+\frac{1}{2}\sum_{L,JM_J,\mu}\langle l'j'||\boldsymbol{Y}_{JL}\cdot\boldsymbol{\sigma}||lj\rangle(-)^{j'-m'}\begin{pmatrix} j' & J & j \\ -m' & M_J & m \end{pmatrix} \\ &\times\widehat{J}(-)^{J-M_J}\begin{pmatrix} J & L & 1 \\ -M_J & M & \mu \end{pmatrix}(-)^{\mu}\langle\sigma|\sigma_{-\mu}|\sigma'\rangle Y^*_{LM}\end{aligned} \tag{41}$$

C. Special application of the sphrical harmonics

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi\sum_l i^l j_l(kr)\sum_m Y^*_{lm}(\widehat{\boldsymbol{k}})Y_{lm}(\widehat{\boldsymbol{r}}) \tag{42}$$

$$\delta(\boldsymbol{r}_1-\boldsymbol{r}_2) = \frac{\delta(r_1-r_2)}{r_1^2}\sum_m Y^*_{lm}(\widehat{\boldsymbol{r}}_1)Y_{lm}(\widehat{\boldsymbol{r}}_2) \tag{43}$$

$$\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \sum_l \frac{r^l_<}{r^{l+1}_>}P_l(\cos\theta) \tag{44}$$

$$= \sum_l \frac{r^l_<}{r^{l+1}_>}\frac{4\pi}{2l+1}\sum_m Y^*_{lm}(\widehat{\boldsymbol{r}}_1)Y_{lm}(\widehat{\boldsymbol{r}}_2) \tag{45}$$

$$\frac{e^{-\alpha|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{\alpha|\boldsymbol{r}_1-\boldsymbol{r}_2|} = -4\pi\sum_l j_l(i\alpha r_<)h^{(1)}_l(i\alpha r_>)\sum_m Y^*_{lm}(\widehat{\boldsymbol{r}}_1)Y_{lm}(\widehat{\boldsymbol{r}}_2) \tag{46}$$

D. Matrix element of the angular momentum operator

$$\langle j'm'|J_\mu|jm\rangle = \delta_{j'j}(-)^{-j+1-m'}\widehat{j}\sqrt{j(j+1)}\begin{pmatrix} j' & 1 & j \\ m' & -\mu & -m\end{pmatrix} \tag{47}$$

$$\leftrightarrow\ J_\mu Y_{jm} = \delta_{j'j}(-)^{-j+1-m'}\widehat{j}\sqrt{j(j+1)}\begin{pmatrix} j' & 1 & j \\ m' & -\mu & -m\end{pmatrix} Y_{j'm'} \tag{48}$$

$$\langle\frac{1}{2}m'_s|\sigma_\mu|\frac{1}{2}m_s\rangle = (-)^{\frac{1}{2}-m'_s}\sqrt{6}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m'_s & -\mu & -m_s\end{pmatrix} \tag{49}$$

$$\langle\frac{1}{2}m'_s|S_\mu|\frac{1}{2}m_s\rangle = (-)^{\frac{1}{2}-m'_s}\sqrt{\frac{3}{2}}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m'_s & -\mu & -m_s\end{pmatrix} \tag{50}$$

where σ_μ is the Pauli's matrix, $S_\mu = \frac{1}{2}\sigma_\mu$.

III. VECTOR SPHERICAL HARMONICS

A. Spherical cordinate system

First of all, the spherical unit vector is defined as

$$\boldsymbol{e}_0 = \boldsymbol{e}_z \tag{51}$$

$$\boldsymbol{e}_{\pm 1} = \mp\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_x \pm i\boldsymbol{e}_y\right) \tag{52}$$

$$\tag{53}$$

These vectors are associated a radial unit vector $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$ as

$$\begin{aligned}\hat{\boldsymbol{r}} &= \frac{\boldsymbol{r}}{r} \\ &= \sum_{\mu=0,\pm1} \boldsymbol{e}_\mu^* \sqrt{\frac{4\pi}{3}} Y_{1\mu}\end{aligned} \tag{54}$$

The spherical unit vector has some properties

$$\boldsymbol{e}_\mu^* = (-)^\mu \boldsymbol{e}_{-\mu}, \qquad \boldsymbol{e}_\mu^* \cdot \boldsymbol{e}_\nu = \delta_{\mu\nu} \tag{55}$$

$$\boldsymbol{e}_\mu \times \boldsymbol{e}_\nu = i\sqrt{2}\langle 1\mu : 1\nu | 1\mu+\nu\rangle \boldsymbol{e}_{\mu+\nu} \tag{56}$$

$$= i\sqrt{6}\begin{pmatrix} 1 & 1 & 1 \\ \mu & \nu & -\mu-\nu \end{pmatrix} \boldsymbol{e}_{-\mu-\nu}^* \tag{57}$$

A vector $\boldsymbol{A}$ is represented by using the spherical unit vector $\boldsymbol{e}_\mu$ as

$$\boldsymbol{A} = \sum_{i=x,y,z} A_i \boldsymbol{e}_i = \sum_{\mu=0,\pm1} A_\mu \boldsymbol{e}_\mu^*, \qquad A_\mu = \boldsymbol{e}_\mu \cdot \boldsymbol{A} \tag{58}$$

Inner product and vector product are described as

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_\mu (-)^\mu A_\mu B_{-\mu} \tag{59}$$

$$\boldsymbol{A}\times\boldsymbol{B} = \sum_{\mu\nu} (-)^{\mu+\nu} i\sqrt{6}\begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\nu & \mu+\nu \end{pmatrix} A_\mu B_\nu \boldsymbol{e}_{\mu+\nu}^* \tag{60}$$

$$(\boldsymbol{A}\times\boldsymbol{B})_\lambda = \boldsymbol{e}_\lambda \cdot (\boldsymbol{A}\times\boldsymbol{B}) \tag{61}$$

$$= \sum_{\mu\nu} (-)^\lambda i\sqrt{6}\begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\nu & \lambda \end{pmatrix} A_\mu B_\nu \tag{62}$$

B. Vector spherical harmonics

The vector spherical harmonics is a kind of the spherical harmonics coupled with the spherical unit vector as

$$\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}}) = \sum_{m_\lambda,q} \langle \lambda m_\lambda : 1q | LM \rangle \boldsymbol{e}_q Y_{\lambda m_\lambda}(\hat{\boldsymbol{r}}) \tag{63}$$

$$= \sum_{m_\lambda,q} (-)^{-\lambda+1-M} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ m_\lambda & q & -M \end{pmatrix} \boldsymbol{e}_q Y_{\lambda m_\lambda}(\hat{\boldsymbol{r}}) \tag{64}$$

$$\boldsymbol{Y}^*_{L\lambda M}(\hat{\boldsymbol{r}}) = (-)^{L+\lambda+1+M} \boldsymbol{Y}_{L\lambda -M}(\hat{\boldsymbol{r}}) \tag{65}$$

In case $M = 0, \lambda = L \pm 1$, the vector spherical harmonics is represented as

$$\boldsymbol{Y}_{L\lambda 0}(\hat{\boldsymbol{r}}) = \sum_{q} (-)^{L} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ -q & q & 0 \end{pmatrix} \boldsymbol{e}_q Y_{\lambda -q}(\hat{\boldsymbol{r}}) \tag{66}$$

$$= (-)^{L} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ -1 & 1 & 0 \end{pmatrix} \left(\boldsymbol{e}_{+1} Y_{\lambda -1}(\hat{\boldsymbol{r}}) + \boldsymbol{e}_{-1} Y_{\lambda +1}(\hat{\boldsymbol{r}})\right) + (-)^{L} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{e}_0 Y_{\lambda 0}(\hat{\boldsymbol{r}}) \tag{67}$$

Therefore

$$\boldsymbol{e}_x \cdot \boldsymbol{Y}_{L\lambda 0}(\hat{\boldsymbol{r}}) = \frac{1}{\sqrt{2}} \left(\boldsymbol{e}^*_{-1} - \boldsymbol{e}^*_{+1}\right) \cdot \boldsymbol{Y}_{L\lambda 0}(\hat{\boldsymbol{r}}) \tag{68}$$

$$= \frac{1}{\sqrt{2}} (-)^{L} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ -1 & 1 & 0 \end{pmatrix} \left(Y_{\lambda +1}(\hat{\boldsymbol{r}}) - Y_{\lambda -1}(\hat{\boldsymbol{r}})\right) \tag{69}$$

$$= \begin{cases} (\lambda = L+1) & \frac{1}{2}\sqrt{\frac{L+2}{2L+3}} \left(Y_{\lambda +1}(\hat{\boldsymbol{r}}) - Y_{\lambda -1}(\hat{\boldsymbol{r}})\right) \\ (\lambda = L-1) & \frac{1}{2}\sqrt{\frac{L-1}{2L-1}} \left(Y_{\lambda +1}(\hat{\boldsymbol{r}}) - Y_{\lambda -1}(\hat{\boldsymbol{r}})\right) \end{cases} \tag{70}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{Y}_{L\lambda 0}(\hat{\boldsymbol{r}}) = \boldsymbol{e}_0 \cdot \boldsymbol{Y}_{L\lambda 0}(\hat{\boldsymbol{r}}) = (-)^{L} \widehat{L} \begin{pmatrix} \lambda & 1 & L \\ 0 & 0 & 0 \end{pmatrix} Y_{\lambda 0}(\hat{\boldsymbol{r}}) \tag{71}$$

$$= \begin{cases} (\lambda = L+1) & -\sqrt{\frac{L+1}{2L+3}} Y_{\lambda 0}(\hat{\boldsymbol{r}}) \\ (\lambda = L-1) & \sqrt{\frac{L}{2L-1}} Y_{\lambda 0}(\hat{\boldsymbol{r}}) \end{cases} \tag{72}$$

$$\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{Y}_{L'\lambda' M'}(\hat{\boldsymbol{r}}) = \sum_{L''M''} (-)^{L+\lambda} \frac{\widehat{L}\widehat{L'}\widehat{\lambda}\widehat{\lambda'}\widehat{L''}}{\sqrt{4\pi}} \begin{pmatrix} \lambda & \lambda' & L'' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & L' & L'' \\ \lambda' & \lambda & 1 \end{Bmatrix} \begin{pmatrix} L & L' & L'' \\ M & M' & M'' \end{pmatrix} Y^*_{L''M''}(\hat{\boldsymbol{r}}) \tag{73}$$

$$Y^*_{l'm'_l}(\hat{\boldsymbol{r}}) \boldsymbol{Y}_{ll''m_l}(\hat{\boldsymbol{r}}) = \sum_{LJM_J} (-)^{J+L} \frac{\widehat{l}\widehat{l'}\widehat{l''}\widehat{L}\widehat{J}}{\sqrt{4\pi}} \begin{pmatrix} l'' & l' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 1 & L & J \\ l' & l & l'' \end{Bmatrix} (-)^{m'_l} \begin{pmatrix} l' & J & l \\ -m'_l & M_J & m_l \end{pmatrix} \boldsymbol{Y}^*_{JLM_J}(\hat{\boldsymbol{r}}) \tag{74}$$

$$\boldsymbol{J} Y_{jm} = \sum_{\mu} \boldsymbol{e}^*_\mu J_\mu Y_{jm} = \sum_{\mu,m'} (-)^{-j+1-m} \widehat{j} \sqrt{j(j+1)} \begin{pmatrix} j & 1 & j \\ m' & \mu & -m \end{pmatrix} \boldsymbol{e}_\mu Y_{jm'} = \sqrt{j(j+1)} \boldsymbol{Y}_{jjm} \tag{75}$$

C. Gradient formula

$$\nabla Y_{lm} f_l(r) = \left[\hat{\boldsymbol{r}}\frac{\partial}{\partial r} - i\frac{\hat{\boldsymbol{r}}\times\boldsymbol{l}}{r}\right] Y_{lm} f_l(r) \tag{76}$$

$$= \sum_{\eta=l\pm1} (-)^l \widehat{\eta} \begin{pmatrix} 1 & l & \eta \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{Y}_{l\eta m} \left[\frac{\partial}{\partial r} - \widehat{l}\sqrt{6l(l+1)} \begin{Bmatrix} 1 & \eta & l \\ l & 1 & 1 \end{Bmatrix} \frac{1}{r}\right] f_l(r) \tag{77}$$

Note that

$$(-)^l \widehat{\eta} \begin{pmatrix} 1 & l & \eta \\ 0 & 0 & 0 \end{pmatrix} = -\delta_{\eta,l+1}\sqrt{\frac{l+1}{2l+1}} + \delta_{\eta,l-1}\sqrt{\frac{l}{2l+1}} = \frac{(-)^{\frac{1}{2}(\eta-l+1)}}{\widehat{l}}\sqrt{\frac{\eta+l+1}{2}} \tag{78}$$

$$\widehat{l}\sqrt{6l(l+1)} \begin{Bmatrix} 1 & \eta & l \\ l & 1 & 1 \end{Bmatrix} = \delta_{\eta,l+1} l - \delta_{\eta,l-1}(l+1) = (-)^{\frac{1}{2}(l+1-\eta)}\frac{3l+1-\eta}{2} \tag{79}$$

Therefore finally one can get

$$\overrightarrow{\nabla} Y_{lm} f_l(r) = \sum_{\eta=l\pm1} \frac{(-)^{\frac{1}{2}(\eta-l+1)}}{\widehat{l}}\sqrt{\frac{\eta+l+1}{2}} \boldsymbol{Y}_{l\eta m} \left[\frac{\overrightarrow{\partial}}{\partial r} - (-)^{\frac{1}{2}(l+1-\eta)}\left(\frac{3l+1-\eta}{2}\right)\frac{1}{r}\right] f_l(r) \tag{80}$$

1. *special case of the gradient formula*

As an application of the gradient formula...

$$\nabla e^{+i\boldsymbol{kr}} = 4\pi \sum_{lm} i^l Y^*_{lm}(\widehat{\boldsymbol{k}}) \nabla Y_{lm}(\widehat{\boldsymbol{r}}) j_l(kr)$$

$$= 4\pi \sum_{lm} i^l Y^*_{lm}(\widehat{\boldsymbol{k}}) \left[-\sqrt{\frac{l+1}{2l+1}} \boldsymbol{Y}_{ll+1m}(\widehat{\boldsymbol{r}}) \left(\frac{\partial j_l(kr)}{\partial r} - \frac{l}{r} j_l(kr)\right) + \sqrt{\frac{l}{2l+1}} \boldsymbol{Y}_{ll-1m}(\widehat{\boldsymbol{r}}) \left(\frac{\partial j_l(kr)}{\partial r} + \frac{l+1}{r} j_l(kr)\right)\right] \tag{81}$$

(Using the recursion formula of the spherical Bessel function,)

$$= 4\pi k \sum_{lm} i^l Y^*_{lm}(\widehat{\boldsymbol{k}}) \left[\sqrt{\frac{l+1}{2l+1}} \boldsymbol{Y}_{ll+1m}(\widehat{\boldsymbol{r}}) j_{l+1}(kr) + \sqrt{\frac{l}{2l+1}} \boldsymbol{Y}_{ll-1m}(\widehat{\boldsymbol{r}}) j_{l-1}(kr)\right] \tag{82}$$

Where the recursion formula of the spherical Bessel function is given by

$$\left(\frac{\partial}{\partial r} - \frac{l}{r}\right) j_l(kr) = -k j_{l+1}(kr) \tag{83}$$

$$\left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right) j_l(kr) = k j_{l-1}(kr) \tag{84}$$

By applying the following formula to Eq.(82)

$$\sum_m Y^*_{lm}(\widehat{\boldsymbol{k}}) \boldsymbol{Y}_{ll+1m}(\widehat{\boldsymbol{r}}) = \sum_{m,n,q} Y^*_{lm}(\widehat{\boldsymbol{k}}) \left[(-)^{-l-m}\sqrt{2l+1} \begin{pmatrix} l+1 & 1 & l \\ n & q & -m \end{pmatrix} \boldsymbol{e}_q Y_{l+1,n}(\widehat{\boldsymbol{r}})\right]$$

$$= \sum_n (-)^{n+1}\sqrt{\frac{2l+1}{2l+3}} \left[\sum_{m,q} (-)^{-l+1+n}\sqrt{2l+3} \begin{pmatrix} l & 1 & l+1 \\ -m & q & n \end{pmatrix} \boldsymbol{e}_q Y_{l,-m}(\widehat{\boldsymbol{k}})\right] Y_{l+1,n}(\widehat{\boldsymbol{r}})$$

$$= -\sum_n \sqrt{\frac{2l+1}{2l+3}} \boldsymbol{Y}_{l+1,l,n}(\widehat{\boldsymbol{k}}) Y^*_{l+1,n}(\widehat{\boldsymbol{r}}) \tag{85}$$

$$\sum_m Y^*_{lm}(\widehat{\boldsymbol{k}})\boldsymbol{Y}_{ll-1m}(\widehat{\boldsymbol{r}}) = \sum_{m,n,q} Y^*_{lm}(\widehat{\boldsymbol{k}})\left[(-)^{-l-m}\sqrt{2l+1}\begin{pmatrix} l-1 & 1 & l \\ n & q & -m\end{pmatrix}\boldsymbol{e}_q Y_{l-1,n}(\widehat{\boldsymbol{r}})\right]$$

$$= \sum_n (-)^{n+1}\sqrt{\frac{2l+1}{2l-1}}\left[\sum_{m,q}(-)^{-l+1+n}\sqrt{2l-1}\begin{pmatrix} l & 1 & l-1 \\ -m & q & n\end{pmatrix}\boldsymbol{e}_q Y_{l,-m}(\widehat{\boldsymbol{k}})\right] Y_{l-1,n}(\widehat{\boldsymbol{r}})$$

$$= -\sum_n \sqrt{\frac{2l+1}{2l-1}}\boldsymbol{Y}_{l-1,l,n}(\widehat{\boldsymbol{k}})Y^*_{l-1,n}(\widehat{\boldsymbol{r}}) \tag{86}$$

we get

$$\nabla e^{+i\boldsymbol{kr}} = -4\pi k\sum_{lm} i^l\left[\sqrt{\frac{l+1}{2l+3}}\boldsymbol{Y}_{l+1,l,m}(\widehat{\boldsymbol{k}})Y^*_{l+1,m}(\widehat{\boldsymbol{r}})j_{l+1}(kr) + \sqrt{\frac{l}{2l-1}}\boldsymbol{Y}_{l-1,l,m}(\widehat{\boldsymbol{k}})Y^*_{l-1,m}(\widehat{\boldsymbol{r}})j_{l-1}(kr)\right]$$

($l \to l-1$ and $l \to l+1$ for the 1st and 2nd terms, respectively.)

$$= 4\pi k\sum_{lm} i^{l+1}\left[\sqrt{\frac{l}{2l+1}}\boldsymbol{Y}_{l,l-1,m}(\widehat{\boldsymbol{k}}) - \sqrt{\frac{l+1}{2l+1}}\boldsymbol{Y}_{l,l+1,m}(\widehat{\boldsymbol{k}})\right] Y^*_{l,m}(\widehat{\boldsymbol{r}})j_l(kr) \tag{87}$$

Using the formula,

$$\widehat{\boldsymbol{k}}Y_{lm}(\widehat{\boldsymbol{k}}) = \sqrt{\frac{l}{2l+1}}\boldsymbol{Y}_{l,l-1,m}(\widehat{\boldsymbol{k}}) - \sqrt{\frac{l+1}{2l+1}}\boldsymbol{Y}_{l,l+1,m}(\widehat{\boldsymbol{k}}) \tag{88}$$

finally we can obtain

$$\nabla e^{+i\boldsymbol{kr}} = ik\widehat{\boldsymbol{k}}4\pi\sum_{lm} i^l Y_{lm}(\widehat{\boldsymbol{k}})Y^*_{l,m}(\widehat{\boldsymbol{r}})j_l(kr) = i\boldsymbol{k}e^{+i\boldsymbol{kr}} \tag{89}$$

If we choose $\widehat{\boldsymbol{k}} = \boldsymbol{e}_z$, *i.e.* $\theta_k = \varphi_k = 0$, then we have

$$Y_{lm}(\widehat{\boldsymbol{k}}) = Y_{lm}(\boldsymbol{e}_z) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m0}. \tag{90}$$

With this condition, Eqs.(85) and (86) become

$$\sum_m Y^*_{lm}(\widehat{\boldsymbol{k}})\boldsymbol{Y}_{ll+1m}(\widehat{\boldsymbol{r}}) = \sum_{q=0,\pm1}\sqrt{\frac{2l+1}{4\pi}}\left[(-)^{-l}\sqrt{2l+1}\begin{pmatrix} l+1 & 1 & l \\ -q & q & 0\end{pmatrix}\boldsymbol{e}_q Y_{l+1,-q}(\widehat{\boldsymbol{r}})\right]$$

$$= -\sqrt{\frac{2l+1}{4\pi(2l+3)}}\left[\sqrt{l+1}\boldsymbol{e}_0 Y_{l+1,0}(\widehat{\boldsymbol{r}}) + \sqrt{\frac{l+2}{2}}\left(\boldsymbol{e}^*_{+1}Y_{l+1,1}(\widehat{\boldsymbol{r}}) + \boldsymbol{e}_{+1}Y^*_{l+1,1}(\widehat{\boldsymbol{r}})\right)\right] \tag{91}$$

$$\sum_m Y^*_{lm}(\widehat{\boldsymbol{k}})\boldsymbol{Y}_{ll-1m}(\widehat{\boldsymbol{r}}) = \sum_{q=0,\pm1}\sqrt{\frac{2l+1}{4\pi}}\left[(-)^{-l}\sqrt{2l+1}\begin{pmatrix} l-1 & 1 & l \\ -q & q & 0\end{pmatrix}\boldsymbol{e}_q Y_{l-1,-q}(\widehat{\boldsymbol{r}})\right]$$

$$= \sqrt{\frac{2l+1}{4\pi(2l-1)}}\left[\sqrt{l}\boldsymbol{e}_0 Y_{l-1,0}(\widehat{\boldsymbol{r}}) - \sqrt{\frac{l-1}{2}}\left(\boldsymbol{e}_{+1}Y^*_{l-1,1}(\widehat{\boldsymbol{r}}) + \boldsymbol{e}^*_{+1}Y_{l-1,1}(\widehat{\boldsymbol{r}})\right)\right] \tag{92}$$

Therefore we can obtain

$$e^{+i\boldsymbol{kr}} = 4\pi\sum_l i^l\sqrt{\frac{2l+1}{4\pi}}Y_{l0}(\widehat{\boldsymbol{r}})j_l(kr) \tag{93}$$

$$\nabla e^{+i\boldsymbol{kr}} = 4\pi k\sum_l i^l\left[-\sqrt{\frac{l+1}{4\pi(2l+3)}}\left\{\sqrt{l+1}\boldsymbol{e}_0 Y_{l+1,0}(\widehat{\boldsymbol{r}}) + \sqrt{\frac{l+2}{2}}\left(\boldsymbol{e}^*_{+1}Y_{l+1,1}(\widehat{\boldsymbol{r}}) + \boldsymbol{e}_{+1}Y^*_{l+1,1}(\widehat{\boldsymbol{r}})\right)\right\}j_{l+1}(kr)\right.$$

$$\left. + \sqrt{\frac{l}{4\pi(2l-1)}}\left\{\sqrt{l}\boldsymbol{e}_0 Y_{l-1,0}(\widehat{\boldsymbol{r}}) - \sqrt{\frac{l-1}{2}}\left(\boldsymbol{e}_{+1}Y^*_{l-1,1}(\widehat{\boldsymbol{r}}) + \boldsymbol{e}^*_{+1}Y_{l-1,1}(\widehat{\boldsymbol{r}})\right)\right\}j_{l-1}(kr)\right] \tag{94}$$

$$= 4\pi k \sum_l \frac{i^{l+1}}{\sqrt{4\pi(2l+1)}} \left[\left\{ l\boldsymbol{e}_0 Y_{l,0}(\widehat{\boldsymbol{r}}) + \sqrt{\frac{l(l+1)}{2}} \left(\boldsymbol{e}_{+1}^* Y_{l,1}(\widehat{\boldsymbol{r}}) + \boldsymbol{e}_{+1} Y_{l,1}^*(\widehat{\boldsymbol{r}}) \right) \right\} \right.$$
$$\left. + \left\{ (l+1)\boldsymbol{e}_0 Y_{l,0}(\widehat{\boldsymbol{r}}) - \sqrt{\frac{l(l+1)}{2}} \left(\boldsymbol{e}_{+1} Y_{l,1}^*(\widehat{\boldsymbol{r}}) + \boldsymbol{e}_{+1}^* Y_{l,1}(\widehat{\boldsymbol{r}}) \right) \right\} \right] j_l(kr) \quad (95)$$

$$= ik\boldsymbol{e}_0 \left[4\pi \sum_l i^l \sqrt{\frac{2l+1}{4\pi}} Y_{l,0}(\widehat{\boldsymbol{r}}) j_l(kr) \right] = i\boldsymbol{k} e^{+i\boldsymbol{k}\boldsymbol{r}} \quad (96)$$

Operator		Density
Y_{LM}	$\to$	ρ
$Y_{LM}\nabla_\mu$	$\to$	$\boldsymbol{j}$
$Y_{LM}\sigma_\mu$	$\to$	$\boldsymbol{s}$
$Y_{LM}\sigma_\mu\nabla_\nu$	$\to$	$J_{\mu\nu}$

TABLE I:

IV. REDUCED MATRIX ELEMENT(WIGNER-ECKART'S THEOREM)

The definition of the reduced matrix element is given by

$$\langle j'm'|T_{LM}|jm\rangle = (-)^{j'-m'}\begin{pmatrix} j' & L & j \\ -m' & M & m \end{pmatrix}\langle j'||T_L||j\rangle \tag{97}$$

where T_{LM} is the so-called "*polarization operator*, and this relation between the matrix element and the reduced one is called as "*Wick's theorem*". There are some correspondences between the linear response quantities of the various density(density matrices) and the (polarization) operators. We show the correspondences on the Table.I.

A. Reduced matrix element of the Spin-independent operator

$$\begin{aligned}
\langle l'j'm'|Y_{LM}|ljm\rangle &= \sum_\sigma \int d\hat{\boldsymbol{r}}\mathcal{Y}^*_{l'j'm'}(\hat{\boldsymbol{r}}\sigma)Y_{LM}(\hat{\boldsymbol{r}})\mathcal{Y}_{ljm}(\hat{\boldsymbol{r}}\sigma) \\
&= \sum_{m'_l,m_l,m_s}\langle l'm'_l:\frac{1}{2}m_s|j'm'\rangle\langle lm_l:\frac{1}{2}m_s|jm\rangle\langle l'm'_l|Y_{LM}|lm_l\rangle \\
&= \sum_{m'_l,m_l,m_s}(-)^{l+l'+m-m'+l'-m'_l}\begin{pmatrix} l' & \frac{1}{2} & j' \\ m'_l & m_s & -m' \end{pmatrix}\begin{pmatrix} l & \frac{1}{2} & j \\ m_l & m_s & -m \end{pmatrix} \\
&\qquad \times \begin{pmatrix} l' & L & l \\ -m'_l & M & m_l \end{pmatrix}\langle l'||Y_L||l\rangle \\
&= (-)^{j'-m'}\begin{pmatrix} j' & L & j \\ -m' & M & m \end{pmatrix}\langle l'j'||Y_L||lj\rangle
\end{aligned} \tag{98}$$

where

$$\langle l'||Y_L||l\rangle = (-)^{l'}\frac{\widehat{L}\widehat{l'}\widehat{l}}{\sqrt{4\pi}}\begin{pmatrix} L & l' & l \\ 0 & 0 & 0 \end{pmatrix} \tag{99}$$

$$\langle l'j'||Y_L||lj\rangle = \left[(-)^{j+l'+L+1/2}\widehat{j'}\widehat{j}\begin{Bmatrix} j & j' & L \\ l' & l & \frac{1}{2} \end{Bmatrix}\langle l'||Y_L||l\rangle\right] \tag{100}$$

Using Eq.(99), Eq.(100) and Eq.(10), one can get

$$\langle l'j'||Y_L||lj\rangle = (-)^{j'+L-1/2}\frac{\widehat{j'}\widehat{j}}{\sqrt{4\pi}}\left\{\frac{1+(-)^{L+l+l'}}{2}\right\}\langle j1/2;j'-1/2|L0\rangle \tag{101}$$

1. Example of the application: $\langle l'j'||\boldsymbol{Y}_{L\lambda}\cdot\nabla||lj\rangle$

Using Eq.(80) and Eq.(73), one can get

$$\begin{aligned}
\langle l'm_l'|\boldsymbol{Y}_{L\lambda M}\cdot\overrightarrow{\nabla}|lm_l\rangle &= \int d\hat{\boldsymbol{r}} Y^*_{l'm_l'}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{L\lambda M}\cdot\overrightarrow{\nabla}Y_{lm_l}(\hat{\boldsymbol{r}}) && (102)\\
&= \sum_{l''=l\pm1}\frac{(-)^{\frac{1}{2}(l''-l+1)}}{\widehat{l}}\sqrt{\frac{l''+l+1}{2}}\int d\hat{\boldsymbol{r}} Y^*_{l'm_l'}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}_{ll''m_l}(\hat{\boldsymbol{r}})\\
&\quad\times\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-l'')}\left(\frac{3l+1-l''}{2}\right)\frac{1}{r}\right] && (103)\\
&= (-)^{l'-m_l'}\begin{pmatrix} l' & L & l\\ -m_l' & M & m_l\end{pmatrix}\\
&\quad\times\sum_{l''=l\pm1}(-)^{L+\lambda+l'+\frac{1}{2}(l''-l+1)}\widehat{L}\widehat{\lambda}\widehat{l''}\widehat{l'}\sqrt{\frac{l''+l+1}{8\pi}}\begin{pmatrix}\lambda & l'' & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} L & l & l'\\ l'' & \lambda & 1\end{Bmatrix}\\
&\quad\times\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-l'')}\left(\frac{3l+1-l''}{2}\right)\frac{1}{r}\right] && (104)\\
&= (-)^{l'-m_l'}\begin{pmatrix} l' & L & l\\ -m_l' & M & m_l\end{pmatrix}\langle l'||\boldsymbol{Y}_{L\lambda}\cdot\overrightarrow{\nabla}||l\rangle && (105)
\end{aligned}$$

where

$$\begin{aligned}
\langle l'||\boldsymbol{Y}_{L\lambda}\cdot\overrightarrow{\nabla}||l\rangle &= \sum_{l''=l\pm1}(-)^{L+\lambda+l'+\frac{1}{2}(l''-l+1)}\widehat{L}\widehat{\lambda}\widehat{l''}\widehat{l'}\sqrt{\frac{l''+l+1}{8\pi}}\begin{pmatrix}\lambda & l'' & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} L & l & l'\\ l'' & \lambda & 1\end{Bmatrix}\\
&\quad\times\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-l'')}\left(\frac{3l+1-l''}{2}\right)\frac{1}{r}\right] && (106)\\
&= (-)^{\lambda+l'+1}\frac{\widehat{L}\widehat{\lambda}\widehat{l}\widehat{l'}}{\sqrt{4\pi}}\left[\begin{pmatrix}\lambda & 1 & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & L\\ 0 & 0 & 0\end{pmatrix}\frac{\overrightarrow{\partial}}{\partial r}\right.\\
&\quad\left.+\frac{1+(-)^{\lambda+l+l'+1}}{2}\sqrt{2l(l+1)}\begin{pmatrix}\lambda & 1 & L\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix} l' & l & L\\ 0 & 1 & -1\end{pmatrix}\frac{1}{r}\right] && (107)\\
&= \langle l'||Y_L||l\rangle(-)^L\widehat{\lambda}\begin{pmatrix}\lambda & 1 & L\\ 0 & 0 & 0\end{pmatrix}\\
&\quad\times\left[\frac{\overrightarrow{\partial}}{\partial r}+\left\{\frac{\lambda(\lambda+1)-2-L(L+1)}{4L(L+1)}\right\}\frac{l'(l'+1)-l(l+1)-L(L+1)}{r}\right] && (108)
\end{aligned}$$

$$\begin{aligned}
\langle l'm_l'|\overleftarrow{\nabla}\cdot\boldsymbol{Y}_{L\lambda M}|lm_l\rangle &= \int d\hat{\boldsymbol{r}} Y^*_{l'm_l'}(\hat{\boldsymbol{r}})\overleftarrow{\nabla}\cdot\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}})Y_{lm_l}(\hat{\boldsymbol{r}}) && (109)\\
&= \sum_{l''=l'\pm1}\left[\frac{\overleftarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l'+1-l'')}\left(\frac{3l'+1-l''}{2}\right)\frac{1}{r}\right]\frac{(-)^{\frac{1}{2}(l''-l'+1)}}{\widehat{l'}}\sqrt{\frac{l''+l'+1}{2}}\\
&\quad\times\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'l''m_l'}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}})Y_{lm_l}(\hat{\boldsymbol{r}}) && (110)\\
&= \sum_{l''=l'\pm1}\left[\frac{\overleftarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l'+1-l'')}\left(\frac{3l'+1-l''}{2}\right)\frac{1}{r}\right]\frac{(-)^{\frac{1}{2}(l''-l'+1)}}{\widehat{l'}}\sqrt{\frac{l''+l'+1}{2}}\\
&\quad\times(-)^{l'+l''+1+m_l'+m_l}\int d\hat{\boldsymbol{r}} Y^*_{l-m_l}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{L\lambda M}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}_{l'l''-m_l'}(\hat{\boldsymbol{r}}) && (111)
\end{aligned}$$

$$
\begin{aligned}
&= (-)^{l'-m_l'} \begin{pmatrix} l' & L & l \\ -m_l' & M & m_l \end{pmatrix} \\
&\quad \times \sum_{l''=l'\pm 1} \left[\frac{\overleftarrow{\partial}}{\partial r} - (-)^{\frac{1}{2}(l'+1-l'')} \left(\frac{3l'+1-l''}{2} \right) \frac{1}{r} \right] \sqrt{\frac{l''+l'+1}{2}} \\
&\qquad \times (-)^{\lambda+l+l'+l''+1+\frac{1}{2}(l''-l'+1)} \frac{\widehat{L}\widehat{\lambda}\widehat{l''}\widehat{l}}{\sqrt{4\pi}} \begin{pmatrix} \lambda & l'' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & l' & l \\ l'' & \lambda & 1 \end{Bmatrix} \qquad (112) \\
&= (-)^{l'-m_l'} \begin{pmatrix} l' & L & l \\ -m_l' & M & m_l \end{pmatrix} \langle l' || \overleftarrow{\nabla} \cdot \boldsymbol{Y}_{L\lambda} || l \rangle \qquad (113)
\end{aligned}
$$

where

$$
\begin{aligned}
\langle l' || \overleftarrow{\nabla} \cdot \boldsymbol{Y}_{L\lambda} || l \rangle &= \sum_{l''=l'\pm 1} \left[\frac{\overleftarrow{\partial}}{\partial r} - (-)^{\frac{1}{2}(l'+1-l'')} \left(\frac{3l'+1-l''}{2} \right) \frac{1}{r} \right] \sqrt{\frac{l''+l'+1}{2}} \\
&\qquad \times (-)^{\lambda+l+l'+l''+1+\frac{1}{2}(l''-l'+1)} \frac{\widehat{L}\widehat{\lambda}\widehat{l''}\widehat{l}}{\sqrt{4\pi}} \begin{pmatrix} \lambda & l'' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & l' & l \\ l'' & \lambda & 1 \end{Bmatrix} \qquad (114) \\
&= (-)^{L+\lambda+l+1} \frac{\widehat{L}\widehat{\lambda}\widehat{l}\widehat{l'}}{\sqrt{4\pi}} \left[\begin{pmatrix} \lambda & 1 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & L \\ 0 & 0 & 0 \end{pmatrix} \frac{\overleftarrow{\partial}}{\partial r} \right. \\
&\quad \left. + \frac{1+(-)^{\lambda+l+l'+1}}{2} \sqrt{2l'(l'+1)} \begin{pmatrix} \lambda & 1 & L \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} l' & l & L \\ -1 & 0 & 1 \end{pmatrix} \frac{1}{r} \right] \qquad (115) \\
&= \langle l' || Y_L || l \rangle (-)^L \widehat{\lambda} \begin{pmatrix} \lambda & 1 & L \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times \left[\frac{\overleftarrow{\partial}}{\partial r} + \left\{ \frac{\lambda(\lambda+1) - 2 - L(L+1)}{4L(L+1)} \right\} \frac{l(l+1) - l'(l'+1) - L(L+1)}{r} \right] \qquad (116)
\end{aligned}
$$

2.

$$
\begin{aligned}
&\langle l'j'm' | \sigma_\mu Y_{LM} | ljm \rangle \\
&= \sum_{JM_J} (-)^{j'-m'} \begin{pmatrix} j' & J & j \\ -m' & M_J & m \end{pmatrix} \widehat{J}\widehat{j}\widehat{j'} \sqrt{6} \langle l' || Y_L || l \rangle \begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ L & l & l' \\ J & j & j' \end{Bmatrix} \widehat{J} (-)^{-L+1-M_J} \begin{pmatrix} L & 1 & J \\ M & \mu & -M_J \end{pmatrix} \qquad (117)
\end{aligned}
$$

On the other hand,

$$
\sigma_\mu Y_{LM} = \sum_{JM_J} \widehat{J} (-)^{-L+1-M_J} \begin{pmatrix} L & 1 & J \\ M & \mu & -M_J \end{pmatrix} \boldsymbol{Y}_{JLM_J} \cdot \sigma \qquad (118)
$$

Therefore we can obtain

$$\langle l'j'm'|\boldsymbol{Y}_{JLM_J}\cdot\sigma|ljm\rangle \;=\; (-)^{j'-m'}\begin{pmatrix} j' & J & j \\ -m' & M_J & m \end{pmatrix}\langle l'j'||\boldsymbol{Y}_{JL}\cdot\sigma||lj\rangle \tag{119}$$

where

$$\langle l'j'||\boldsymbol{Y}_{JL}\cdot\sigma||lj\rangle = \widehat{J}\widehat{j}\widehat{j'}\sqrt{6}\begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ L & l & l' \\ J & j & j' \end{Bmatrix}\langle l'||Y_L||l\rangle \tag{120}$$

and also

$$\begin{aligned}&\langle l'j'm'|\sigma_\mu Y_{LM}|ljm\rangle \\ &= \sum_{JM_J}(-)^{j'-m'}\begin{pmatrix} j' & J & j \\ -m' & M_J & m \end{pmatrix}\widehat{J}(-)^{-L+1-M_J}\begin{pmatrix} L & 1 & J \\ M & \mu & -M_J \end{pmatrix}\langle l'j'||\boldsymbol{Y}_{JL}\cdot\sigma||lj\rangle\end{aligned} \tag{121}$$

B.

$$\langle l'm_l'|\overleftarrow{\nabla}\cdot Y_{LM}\overrightarrow{\nabla}|lm_l\rangle = \int d\hat{\boldsymbol{r}}Y^*_{l'm_l'}(\hat{\boldsymbol{r}})\overleftarrow{\nabla}\cdot Y_{LM}(\hat{\boldsymbol{r}})\overrightarrow{\nabla}Y_{lm_l}(\hat{\boldsymbol{r}}) \tag{122}$$

$$\begin{aligned}&= \sum_{\eta'=l'\pm1}\sum_{\eta=l\pm1}\frac{(-)^{\frac{1}{2}(\eta'+\eta-l'-l+2)}}{2\widehat{l'}\widehat{l}}\sqrt{(\eta'+l'+1)(\eta+l+1)}\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot Y_{LM}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{l\eta m_l}(\hat{\boldsymbol{r}}) \\ &\times\left[\frac{\overleftarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l'+1-\eta')}\left(\frac{3l'+1-\eta'}{2}\right)\frac{1}{r}\right]\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-\eta)}\left(\frac{3l+1-\eta}{2}\right)\frac{1}{r}\right]\end{aligned} \tag{123}$$

$$\begin{aligned}&\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot Y_{LM}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{l\eta m_l}(\hat{\boldsymbol{r}}) \\ &= \sum_{IJM_J}(-)^{J+I}\frac{\widehat{l}\widehat{L}\widehat{\eta}\widehat{I}\widehat{J}}{\sqrt{4\pi}}\begin{pmatrix} \eta & L & I \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} 1 & I & J \\ L & l & \eta \end{Bmatrix}\begin{pmatrix} L & J & l \\ M & M_J & m_l \end{pmatrix}\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}^*_{JIM_J}(\hat{\boldsymbol{r}}) \\ &= (-)^{-m_l'}\frac{\widehat{l}\widehat{L}\widehat{\eta}\widehat{\eta'}\widehat{l'}}{\sqrt{4\pi}}\begin{pmatrix} \eta & L & \eta' \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} 1 & \eta' & l' \\ L & l & \eta \end{Bmatrix}\begin{pmatrix} L & l' & l \\ M & -m_l' & m_l \end{pmatrix}\end{aligned} \tag{124}$$

where

$$\begin{aligned}&Y_{LM}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{l\eta m_l}(\hat{\boldsymbol{r}}) \\ &= \sum_{IJM_J}(-)^{J+I}\frac{\widehat{l}\widehat{L}\widehat{\eta}\widehat{I}\widehat{J}}{\sqrt{4\pi}}\begin{pmatrix} \eta & L & I \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} 1 & I & J \\ L & l & \eta \end{Bmatrix}\begin{pmatrix} L & J & l \\ M & M_J & m_l \end{pmatrix}\boldsymbol{Y}^*_{JIM_J}(\hat{\boldsymbol{r}})\end{aligned} \tag{125}$$

and

$$\begin{aligned}&\int d\hat{\boldsymbol{r}}\boldsymbol{Y}_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}_{JIM_J}(\hat{\boldsymbol{r}}) \\ &= \sum_{J'M_J'}(-)^{l'+\eta}\frac{\widehat{l'}\widehat{J}\widehat{\eta'}\widehat{I}\widehat{J'}}{\sqrt{4\pi}}\begin{pmatrix} \eta' & I & J' \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l' & J & J' \\ I & \eta' & 1 \end{Bmatrix}\begin{pmatrix} l' & J & J' \\ m_l' & M_J & M_J' \end{pmatrix}\int d\hat{\boldsymbol{r}}Y^*_{J'M_J'}(\hat{\boldsymbol{r}}) \\ &= (-)^{l'+\eta'}\widehat{l'}\widehat{J}\widehat{\eta'}\widehat{I}\begin{pmatrix} \eta & I & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l' & J & 0 \\ I & \eta' & 1 \end{Bmatrix}\begin{pmatrix} l' & J & 0 \\ m_l' & M_J & 0 \end{pmatrix} \\ &= (-)^{-m_l'}\widehat{l'}\widehat{\eta'}\begin{Bmatrix} l' & l' & 0 \\ \eta' & \eta' & 1 \end{Bmatrix}\delta_{l'J}\delta_{m_l',-M_J}\delta_{\eta'I} = (-)^{l'+\eta'-m_l'}\delta_{l'J}\delta_{\eta'I}\delta_{m_l',-M_J}\end{aligned} \tag{126}$$

1. In case of L=0.

$$\langle l'm_l'|\overleftarrow{\nabla}\cdot Y_{00}\overrightarrow{\nabla}|lm_l\rangle = \int d\hat{\boldsymbol{r}} Y^*_{l'm_l'}(\hat{\boldsymbol{r}})\overleftarrow{\nabla}\cdot Y_{00}(\hat{\boldsymbol{r}})\overrightarrow{\nabla} Y_{lm_l}(\hat{\boldsymbol{r}}) \tag{127}$$

$$= \sum_{\eta'=l'\pm1}\sum_{\eta=l\pm1}\frac{(-)^{\frac{1}{2}(\eta'+\eta-l'-l+2)}}{2\widehat{l'}\widehat{l}}\sqrt{(\eta'+l'+1)(\eta+l+1)}\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot Y_{00}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{l\eta m_l}(\hat{\boldsymbol{r}})$$

$$\times\left[\frac{\overleftarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l'+1-\eta')}\left(\frac{3l'+1-\eta'}{2}\right)\frac{1}{r}\right]\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-\eta)}\left(\frac{3l+1-\eta}{2}\right)\frac{1}{r}\right] \tag{128}$$

$$= \sum_{\eta=l\pm1}\frac{(-)^{\eta-l}}{\widehat{l}^2\sqrt{16\pi}}(\eta+l+1)$$

$$\times\left[\frac{\overleftarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-\eta)}\left(\frac{3l+1-\eta}{2}\right)\frac{1}{r}\right]\left[\frac{\overrightarrow{\partial}}{\partial r}-(-)^{\frac{1}{2}(l+1-\eta)}\left(\frac{3l+1-\eta}{2}\right)\frac{1}{r}\right] \tag{129}$$

$$\int d\hat{\boldsymbol{r}}\boldsymbol{Y}^*_{l'\eta'm_l'}(\hat{\boldsymbol{r}})\cdot Y_{00}(\hat{\boldsymbol{r}})\boldsymbol{Y}_{l\eta m_l}(\hat{\boldsymbol{r}}) = (-)^{-m_l'}\frac{\widehat{l}\widehat{\eta}\widehat{\eta'}\widehat{l'}}{\sqrt{4\pi}}\begin{pmatrix}\eta & 0 & \eta'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}1 & \eta' & l'\\ 0 & l & \eta\end{Bmatrix}\begin{pmatrix}0 & l' & l\\ 0 & -m_l' & m_l\end{pmatrix} \tag{130}$$

$$= (-)\frac{1}{\sqrt{4\pi}}\delta_{\eta\eta'}\delta_{ll'}\delta_{m_l m_l'} \tag{131}$$

V. SPECIAL REDUCTION FORMULAS FOR 6J, 9J SYMBOL

There is a relation between 6j and 9j symbol,

$$\sum_X(-)^{2X}\widehat{X}^2\begin{Bmatrix}a & b & X\\ c & d & p\end{Bmatrix}\begin{Bmatrix}c & d & X\\ e & f & q\end{Bmatrix}\begin{Bmatrix}e & f & X\\ b & a & r\end{Bmatrix} = \begin{Bmatrix}a & f & r\\ d & q & e\\ p & c & b\end{Bmatrix} \tag{132}$$

One can obtain a formula as following,

$$\sum_r\widehat{r}^2\begin{Bmatrix}e & a & r\\ b & f & X\end{Bmatrix}\begin{Bmatrix}a & f & r\\ d & q & e\\ p & c & b\end{Bmatrix} = \sum_X(-)^{2X}\widehat{X}^2\begin{Bmatrix}a & b & X\\ c & d & p\end{Bmatrix}\begin{Bmatrix}c & d & X\\ e & f & q\end{Bmatrix}\sum_r\widehat{r}^2\begin{Bmatrix}e & a & r\\ b & f & X\end{Bmatrix}\begin{Bmatrix}e & a & r\\ b & f & X\end{Bmatrix} \tag{133}$$

$$= \sum_X(-)^{2X}\begin{Bmatrix}a & b & X\\ c & d & p\end{Bmatrix}\begin{Bmatrix}c & d & X\\ e & f & q\end{Bmatrix}\delta(e,f,X)\delta(a,b,X) \tag{134}$$

1. Special case

$$\sum_{j_f j_i}\widehat{j}_f\widehat{j}_i\langle l_f j_f||Y_L||l_i j_i\rangle\begin{Bmatrix}l_f & 1/2 & j_f\\ l_i & 1/2 & j_i\\ \lambda & S & L\end{Bmatrix} = \langle l_f||Y_L||l_i\rangle\sum_{j_f j_i}(-)^{j_i+l_f+L+1/2}\widehat{j}_f^2\widehat{j}_i^2\begin{Bmatrix}j_i & j_f & L\\ l_f & l_i & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}l_f & 1/2 & j_f\\ l_i & 1/2 & j_i\\ \lambda & S & L\end{Bmatrix} \tag{135}$$

$$= \langle l_f||Y_L||l_i\rangle\sum_{j_i}(-)^{j_i+l_f+L+1/2}\widehat{j}_i^2\sum_{j_f}\widehat{j}_f^2\begin{Bmatrix}\frac{1}{2} & l_f & j_f\\ L & j_i & l_i\end{Bmatrix}\begin{Bmatrix}l_f & 1/2 & j_f\\ l_i & 1/2 & j_i\\ \lambda & S & L\end{Bmatrix} \tag{136}$$