Bessel function for complex variable

Kazuhito Mizuyama

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1 Bessel function

The Bessel function $Z_\nu(z)$ is the function wich satisfies

$$\left[\frac{\partial^2}{\partial z^2}+\frac{1}{z}\frac{\partial}{\partial z}+\left(1-\frac{\nu^2}{z^2}\right)\right]Z_\nu(z)=0. \tag{1}$$

Three kinds of the solutions of this equation are given by

$$\begin{aligned}
J_\nu(z) &\equiv \sqrt{\frac{2}{\pi z}}\left\{P_\nu(z)\cos\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]-Q_\nu(z)\sin\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]\right\} && (2)\\
N_\nu(z) &\equiv \sqrt{\frac{2}{\pi z}}\left\{P_\nu(z)\sin\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]+Q_\nu(z)\cos\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]\right\} && (3)\\
& \quad (-\pi<\arg z<\pi)\\
H_\nu^{(\pm)}(z) &\equiv J_\nu(z)\pm iN_\nu(z)\\
&= \sqrt{\frac{2}{\pi z}}e^{\pm i\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]}\left[P_\nu(z)\pm iQ_\nu(z)\right] \qquad \left\{\begin{array}{l}(-\pi<\arg z<2\pi)\\(-2\pi<\arg z<\pi)\end{array}\right. && (4)
\end{aligned}$$

where $P_\nu(z)$ and $Q_\nu(z)$ are polynomials given by

$$\begin{aligned}
P_\nu(z) &\equiv 1+\sum_{k=1}^{\infty}(-)^k C_{\nu,2k}(z) && (5)\\
Q_\nu(z) &\equiv \sum_{k=1}^{\infty}(-)^{k+1}C_{\nu,2k-1}(z) && (6)\\
C_{\nu,m}(z) &\equiv \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2m-1)^2)}{m!(8z)^m}, \qquad (m\geqq 1) && (7)
\end{aligned}$$

Note that J_ν, N_ν and $H_\nu^{(\pm)}$ are so-called Bessel, Neumann and Hankel functions as the solutions of Eq.(1).

2 Spherical Bessel function

By replacing ν by $n+\frac{1}{2}$ and defining $f_n(z)\equiv\sqrt{\frac{\pi}{2z}}Z_{n+\frac{1}{2}}(z)$, then Eq.(1) can be rewritten as

$$\left[\frac{\partial^2}{\partial z^2}+\frac{2}{z}\frac{\partial}{\partial z}+\left(1-\frac{n(n+1)}{z^2}\right)\right]f_n(z)=0. \tag{8}$$

The spherical Bessel $j_n(z)$, Neumann $n_n(z)$, and Hankel $h_n^{(\pm)}(z)$ functions are given by

$$j_n(z) \equiv \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z) = \frac{1}{z}\left\{P_{n+\frac{1}{2}}(z)\cos\left[z-(n+1)\frac{\pi}{2}\right] - Q_{n+\frac{1}{2}}(z)\sin\left[z-(n+1)\frac{\pi}{2}\right]\right\} \tag{9}$$

$$n_n(z) \equiv \sqrt{\frac{\pi}{2z}} N_{n+\frac{1}{2}}(z) = \frac{1}{z}\left\{P_{n+\frac{1}{2}}(z)\sin\left[z-(n+1)\frac{\pi}{2}\right] + Q_{n+\frac{1}{2}}(z)\cos\left[z-(n+1)\frac{\pi}{2}\right]\right\} \tag{10}$$

$$(-\pi < \arg z < \pi)$$

$$\begin{aligned} h_n^{(\pm)}(z) &\equiv \sqrt{\frac{\pi}{2z}} H^{(\pm)}_{n+\frac{1}{2}}(z) = j_n(z) \pm i n_n(z) \\ &= \frac{1}{z} e^{\pm i\left[z-(n+1)\frac{\pi}{2}\right]}\left[P_{n+\frac{1}{2}}(z) \pm i Q_{n+\frac{1}{2}}(z)\right] \qquad \begin{cases} (-\pi < \arg z < 2\pi) \\ (-2\pi < \arg z < \pi) \end{cases} \end{aligned} \tag{11}$$

where

$$P_{n+\frac{1}{2}}(z) \equiv 1 + \sum_{k=1}^{\left[\frac{n}{2}\right]} (-)^k C_{n+\frac{1}{2},2k}(z) \tag{12}$$

$$Q_{n+\frac{1}{2}}(z) \equiv \sum_{k=1}^{\left[\frac{n+1}{2}\right]} (-)^{k+1} C_{n+\frac{1}{2},2k-1}(z) \tag{13}$$

$$\begin{aligned} C_{n+\frac{1}{2},m}(z) &\equiv \frac{((2n+1)^2-1^2)((2n+1)^2-3^2)\cdots((2n+1)^2-(2m-1)^2)}{m!(8z)^m}, \\ &= \frac{(n+m)!}{m!(2z)^m(n-m)!}. \quad (1 \leqq m < n+1) \end{aligned} \tag{14}$$

Spherical bessel function

Therefore we, finally, obtain

$$h_n^{(\pm)}(z) = (\mp i)^{n+1} \frac{e^{\pm iz}}{z} \sum_{k=0}^{n} \left(\frac{\pm i}{2z}\right)^k \frac{(n+k)!}{k!(n-k)!} \tag{15}$$

Note that the symmetric properties of $h_n^{(\pm)}(z)$ can be obtained from Eq.(15) as

$$h_n^{(+)}(-z) = (-)^n(+i)^{n+1}\frac{e^{-iz}}{z}\sum_{k=0}^{n}\frac{(-)^k i^k (n+k)!}{k!(2z)^k(n-k)!} = (-)^n h_n^{(-)}(z) \tag{16}$$

$$h_n^{(+)*}(z^*) = (+i)^{n+1}\frac{e^{-iz}}{z}\sum_{k=0}^{n}\frac{(-i)^k(n+k)!}{k!(2z)^k(n-k)!} = h_n^{(-)}(z) \tag{17}$$

Also Eqs.(9) and (10) can be rewritten as

$$j_n(z) = \frac{1}{2}\left[h_n^{(+)}(z) + h_n^{(-)}(z)\right] \tag{18}$$

$$n_n(z) = \frac{1}{2i}\left[h_n^{(+)}(z) - h_n^{(-)}(z)\right] \tag{19}$$

Using Eq.(16), the symmetric properties of j_n and n_n can be obtained as

$$j_n(-z) = \frac{1}{2}\left[h_n^{(+)}(-z) + h_n^{(-)}(-z)\right] = (-)^n\frac{1}{2}\left[h_n^{(-)}(z) + h_n^{(+)}(z)\right] = (-)^n j_n(z) \tag{20}$$

$$n_n(-z) = \frac{1}{2i}\left[h_n^{(+)}(-z) - h_n^{(-)}(-z)\right] = (-)^n\frac{1}{2i}\left[h_n^{(-)}(z) - h_n^{(+)}(z)\right] = (-)^{n+1} n_n(z) \tag{21}$$

3 Asymtotic behaviour & Wronskian

From Eq.(15), we can get the asymtotic behabiour of the spherical Hankel function at the limit of $|z| \to \infty$ as

$$\lim_{|z|\to\infty} h_n^{(\pm)}(z) = \lim_{|z|\to\infty} (\mp i)^{n+1} \frac{e^{\pm iz}}{z} \sum_{k=0}^{n} \frac{(\pm i)^k (n+k)!}{k!(2z)^k (n-k)!} \sim (\mp i)^{n+1} \frac{e^{\pm iz}}{z} = (\mp i) \frac{e^{\pm i\left[z-\frac{n\pi}{2}\right]}}{z}. \quad (22)$$

Using Eq.(22), the asymtotic behaviour of j_n and n_n can be also obtained as

$$j_n(z) \sim \frac{\sin\left[z-\frac{n\pi}{2}\right]}{z}, \quad (23)$$

$$n_n(z) \sim -\frac{\cos\left[z-\frac{n\pi}{2}\right]}{z}. \quad (24)$$

Now let us introduce one of the very important quantity, the so-called "Wronskian".
The "Wronskian" is defined by using two kinds of the linearly independent functions as

$$W_n(f^{(1)}, f^{(2)}) \equiv z f_n^{(1)}(z) \frac{\partial}{\partial z} z f_n^{(2)}(z) - z f_n^{(2)}(z) \frac{\partial}{\partial z} z f_n^{(1)}(z) \quad (25)$$

Using the fact $z f_n(z)$ obey

$$\left[\frac{\partial^2}{\partial z^2} + \left(1 - \frac{n(n+1)}{z^2}\right)\right] z f_n(z) = 0, \quad (26)$$

the derivative of the Wronskian is given as

$$\begin{aligned}\frac{\partial W_n(f^{(1)}, f^{(2)})}{\partial z} &= z f_n^{(1)}(z) \frac{\partial^2}{\partial z^2} z f_n^{(2)}(z) - z f_n^{(2)}(z) \frac{\partial^2}{\partial z^2} z f_n^{(1)}(z) \\ &= -z f_n^{(1)}(z) \left[\left(1 - \frac{n(n+1)}{z^2}\right) z f_n^{(2)}(z)\right] \\ &\quad + z f_n^{(2)}(z) \left[\left(1 - \frac{n(n+1)}{z^2}\right) z f_n^{(1)}(z)\right] = 0. \quad (27)\end{aligned}$$

Then, we can find that the Wronskian is a constant for z. Therefore, the Wronskian can be calculated by using the asymtotic property of $z f_n(z)$ at the limit of $|z| \to \infty$ as

$$W_n(f^{(1)}, f^{(2)}) = \lim_{|z|\to\infty} \left[z f_n^{(1)}(z) \frac{\partial}{\partial z} z f_n^{(2)}(z) - z f_n^{(2)}(z) \frac{\partial}{\partial z} z f_n^{(1)}(z)\right] = const. \quad (28)$$

Wronskians for Bessel functions

Thus we can obtain the following results.

$$W_n(j, n) = \lim_{|z|\to\infty} \left[z j_n(z) \frac{\partial}{\partial z} z n_n(z) - z n_n(z) \frac{\partial}{\partial z} z j_n(z)\right] = 1, \quad (29)$$

$$W_n(j, h^{(\pm)}) = \lim_{|z|\to\infty} \left[z j_n(z) \frac{\partial}{\partial z} z h_n^{(\pm)}(z) - z h_n^{(\pm)}(z) \frac{\partial}{\partial z} z j_n(z)\right] = \pm i \quad (30)$$

$$W_n(n, h^{(\pm)}) = \lim_{|z|\to\infty} \left[z n_n(z) \frac{\partial}{\partial z} z h_n^{(\pm)}(z) - z h_n^{(\pm)}(z) \frac{\partial}{\partial z} z n_n(z)\right] = -1 \quad (31)$$

$$W_n(h^{(+)}, h^{(-)}) = \lim_{|z|\to\infty} \left[z h_n^{(+)}(z) \frac{\partial}{\partial z} z h_n^{(-)}(z) - z h_n^{(-)}(z) \frac{\partial}{\partial z} z h_n^{(+)}(z)\right] = -2i \quad (32)$$

4 Recurrence formula

Using Eq.(15), we can obtain

$$h_n^{(\pm)}(z) = (\mp i)^{n+1}\frac{e^{\pm iz}}{z}\sum_{k=0}^{n}\left(\frac{\pm i}{2z}\right)^k\frac{(n+k)!}{k!(n-k)!} \tag{33}$$

$$z^{n+1}h_n^{(\pm)}(z) = (\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n}\left(\frac{\pm i}{2}\right)^k z^{n-k}\frac{(n+k)!}{k!(n-k)!} \tag{34}$$

$$z^{-n}h_n^{(\pm)}(z) = (\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n}\left(\frac{\pm i}{2}\right)^k z^{-(n+k+1)}\frac{(n+k)!}{k!(n-k)!} \tag{35}$$

$$\begin{aligned}
\frac{\partial}{\partial z}\left[z^{n+1}h_n^{(\pm)}(z)\right] &= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n}\left(\frac{\pm i}{2}\right)^k z^{n-k}\frac{(n+k)!}{k!(n-k)!} \\
&\quad +(\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^k z^{n-k-1}\frac{(n-k)(n+k)!}{k!(n-k)!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^{k+1} z^{n-k-1}\frac{(n+k+1)!}{(k+1)!(n-k-1)!} + (\mp i)^n e^{\pm iz}z^n \\
&\quad +(\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^k z^{n-k-1}\frac{(n-k)(n+k)!}{k!(n-k)!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n-2}\left(\frac{\pm i}{2}\right)^{k+1} \frac{z^{n-k-1}}{(n-k-2)!}\frac{(n+k)!}{(k+1)!} + (\mp i)^n e^{\pm iz}z^n \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=1}^{n-1}\left(\frac{\pm i}{2}\right)^{k} \frac{z^{n-k}}{(n-k-1)!}\frac{(n+k-1)!}{k!} + (\mp i)^n e^{\pm iz}z^n \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^{k} \frac{z^{n-k}}{(n-k-1)!}\frac{(n+k-1)!}{k!} = z^{n+1}h_{n-1}^{(\pm)}(z)
\end{aligned} \tag{36}$$

$$\begin{aligned}
\frac{\partial}{\partial z}\left[z^{-n}h_n^{(\pm)}(z)\right] &= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n}\left(\frac{\pm i}{2}\right)^k z^{-(n+k+1)}\frac{(n+k)!}{k!(n-k)!} \\
&\quad -(\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n}\left(\frac{\pm i}{2}\right)^k z^{-(n+k+2)}\frac{(n+k+1)!}{k!(n-k)!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^{k+1} z^{-(n+k+2)}\frac{(n+k+1)!}{(k+1)!(n-k-1)!} + (\mp i)^n e^{\pm iz}z^{-(n+1)} \\
&\quad -(\mp i)^{n+1}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^k z^{-(n+k+2)}\frac{(n+k+1)!}{k!(n-k)!} - (\mp i)^{n+1}e^{\pm iz}\left(\frac{\pm i}{2}\right)^n z^{-(2n+2)}\frac{(2n+1)!}{n!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n-1}\left(\frac{\pm i}{2}\right)^{k+1} z^{-(n+k+2)}\frac{(n+k+2)!}{(k+1)!(n-k)!} \\
&\quad +(\mp i)^n e^{\pm iz}z^{-(n+1)} - (\mp i)^{n+1}e^{\pm iz}\left(\frac{\pm i}{2}\right)^n z^{-(2n+2)}\frac{(2n+1)!}{n!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=1}^{n}\left(\frac{\pm i}{2}\right)^{k} z^{-(n+k+1)}\frac{(n+k+1)!}{k!(n-k+1)!} \\
&\quad +(\mp i)^n e^{\pm iz}z^{-(n+1)} + (\mp i)^{n}e^{\pm iz}\left(\frac{\pm i}{2}\right)^{n+1} z^{-(2n+2)}\frac{(2n+2)!}{(n+1)!} \\
&= (\mp i)^{n}e^{\pm iz}\sum_{k=0}^{n+1}\left(\frac{\pm i}{2}\right)^{k} z^{-(n+k+1)}\frac{(n+k+1)!}{k!(n-k+1)!} = -z^{-n}h_{n+1}^{(\pm)}(z)
\end{aligned} \tag{37}$$

Recurrence formulas

We can derive the following recurrence formulas using Eqs.(36) and (37) as

$$\frac{2n+1}{z}h_n^{(\pm)}(z) = h_{n-1}^{(\pm)}(z) + h_{n+1}^{(\pm)}(z) \quad (38)$$

$$(2n+1)\frac{\partial}{\partial z}h_n^{(\pm)}(z) = nh_{n-1}^{(\pm)}(z) - (n+1)h_{n+1}^{(\pm)}(z) \quad (39)$$

5 Wave function of free particle

The Schrodinger equation for the free particle can be expressed as

$$-\frac{\hbar^2}{2m}\Delta\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r}) = E(k)\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r}), \quad (40)$$

$$-i\hbar\nabla\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r}) = \hbar\boldsymbol{k}\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r}), \quad (41)$$

where $\widehat{\boldsymbol{k}}$ is a unit vector which gives the direction of the momentum vector, *i.e.*, $\boldsymbol{k} = k\widehat{\boldsymbol{k}}$. Also $E(k) = \frac{\hbar^2 k^2}{2m}$ with complex k. $\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r})$ is known to be

$$\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{+i\boldsymbol{kr}} = \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{lm} i^l j_l(kr)Y_{lm}(\widehat{\boldsymbol{r}})Y_{lm}^*(\widehat{\boldsymbol{k}}). \quad (42)$$

The complex conjugate of this function is

$$\frac{1}{(2\pi)^{\frac{3}{2}}}e^{-i\boldsymbol{kr}} = \left\{\begin{array}{lcl} \chi^*(k^*,\widehat{\boldsymbol{k}};\boldsymbol{r}) & = & \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{lm}(-i)^l j_l^*(k^*r)Y_{lm}^*(\widehat{\boldsymbol{r}})Y_{lm}(\widehat{\boldsymbol{k}}) \\ \chi(-k,\widehat{\boldsymbol{k}};\boldsymbol{r}) & = & \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{lm} i^l j_l(-kr)Y_{lm}(\widehat{\boldsymbol{r}})Y_{lm}^*(\widehat{\boldsymbol{k}}) \\ \chi(k,-\widehat{\boldsymbol{k}};\boldsymbol{r}) & = & \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{lm} i^l j_l(-kr)Y_{lm}(\widehat{\boldsymbol{r}})Y_{lm}^*(-\widehat{\boldsymbol{k}}) \end{array}\right\}$$

(Using $j_l^*(k^*r) = j_l(kr)$, $j_l(-kr) = (-)^l j_l(kr)$ and $Y_{lm}(-\widehat{\boldsymbol{k}}) = (-)^l Y_{lm}(\widehat{\boldsymbol{k}})$)

$$= \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{lm}(-i)^l j_l(kr)Y_{lm}^*(\widehat{\boldsymbol{r}})Y_{lm}(\widehat{\boldsymbol{k}}) \quad (43)$$

(Note that $k^* = k$ if k is real. In this study, we consider the complex k in most of the cases.)

From the definition of the Delta function, we get

$$\delta(\boldsymbol{r}-\boldsymbol{r}') \equiv \frac{1}{(2\pi)^3}\int d\boldsymbol{k}e^{+i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} = \sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})\left[\frac{2}{\pi}\int_0^\infty dkk^2 j_l(kr)j_l(kr')\right]Y_{lm}^*(\widehat{\boldsymbol{r}}') \quad (44)$$

$$\delta(\boldsymbol{k}-\boldsymbol{k}') \equiv \frac{1}{(2\pi)^3}\int d\boldsymbol{r}e^{+i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}} = \sum_{lm} Y_{lm}(\widehat{\boldsymbol{k}})\left[\frac{2}{\pi}\int_0^\infty drr^2 j_l(kr)j_l(k'r)\right]Y_{lm}^*(\widehat{\boldsymbol{k}}'). \quad (45)$$

the orthogonal & completeness relation for the spherical Bessel function

Thus we obtain the orthogonal & completeness relation for the spherical Bessel function as

$$\delta(r-r') = \frac{2}{\pi}\int_0^\infty dk(kr)^2 j_l(kr)j_l(kr') \quad (46)$$

$$\delta(k-k') = \frac{2}{\pi}\int_0^\infty dr(kr)^2 j_l(kr)j_l(k'r) \quad (47)$$

the orthogonal and completeness relation for χ

In terms of the free particle wave function χ, the orthogonal and completeness relation is given by

$$\int d\boldsymbol{r}\chi^*(k^*,\widehat{\boldsymbol{k}};\boldsymbol{r})\chi(k',\widehat{\boldsymbol{k}}';\boldsymbol{r}) = \delta(\boldsymbol{k}-\boldsymbol{k}') \tag{48}$$

$$\int d\boldsymbol{k}\chi(k,\widehat{\boldsymbol{k}};\boldsymbol{r})\chi^*(k^*,\widehat{\boldsymbol{k}};\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{49}$$

6 Free particle Green's function

The free particle Green's function is defined by

$$\left[E(k)+\frac{\hbar^2}{2m}\Delta\right]G_F(\boldsymbol{r}\boldsymbol{r}';k)=\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{50}$$

It is very easy to prove $G_F(\boldsymbol{r}\boldsymbol{r}';k)$ can be represented as

$$G_F(\boldsymbol{r}\boldsymbol{r}';k) = \frac{2m}{\hbar^2}\int d\boldsymbol{q}\frac{\chi(q,\widehat{\boldsymbol{q}};\boldsymbol{r})\chi^*(q^*,\widehat{\boldsymbol{q}};\boldsymbol{r}')}{q^2-k^2}. \tag{51}$$

In the expression of the partial wave expansion, this can be rewritten as

$$\begin{aligned} G_F(\boldsymbol{r}\boldsymbol{r}';k) &= -\frac{2m}{\hbar^2}\frac{2}{\pi}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\int_0^\infty dq\frac{q^2j_l(qr)j_l(qr')}{q^2-k^2} \\ &= -\frac{2m}{\hbar^2}\frac{1}{2\pi}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\int_{-\infty}^\infty dq\left[\frac{1}{q-k}+\frac{1}{q+k}\right]qj_l(qr)j_l(qr') \\ &= -\frac{2m}{\hbar^2}\frac{1}{4\pi}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\int_{-\infty}^\infty dq\left[\frac{1}{q-k}+\frac{1}{q+k}\right]qj_l(qr_<)\left[h_l^{(+)}(qr_>)+h_l^{(-)}(qr_>)\right]. \end{aligned} \tag{52}$$

If we suppose $k=|k|e^{+i\theta}$ with $0<\theta<\pi$, then Eq.(52) can be calculated as

$$\begin{aligned} \text{Eq.(52)} &= -\frac{2m}{\hbar^2}\frac{1}{4\pi}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\oint_{C_+}dq\frac{1}{q-k}qj_l(qr_<)h_l^{(+)}(qr_>) \\ &\quad -\frac{2m}{\hbar^2}\frac{1}{4\pi}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\oint_{C_-}dq\frac{1}{q+k}qj_l(qr_<)h_l^{(-)}(qr_>) \\ &= -\frac{2m}{\hbar^2}\frac{i}{2}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')kj_l(kr_<)h_l^{(+)}(kr_>) \\ &\quad -\frac{2m}{\hbar^2}\frac{i}{2}\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')kj_l(-kr_<)h_l^{(-)}(-kr_>) \\ &= -\frac{2m}{\hbar^2}ik\sum_{lm}Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')j_l(kr_<)h_l^{(+)}(kr_>) = -\frac{2m}{\hbar^2}\frac{e^{+ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \end{aligned} \tag{53}$$

where $C_+(C_-)$ is the coutour integral path on the upper (lower) region of the complex momentum q plane (see Fig.1). We consider that $h^{(+)}(qr)(h^{(-)}(qr))$ is converged at the limit of $|q|\to\infty$ on the upper(lower) region of the complex q-plane.

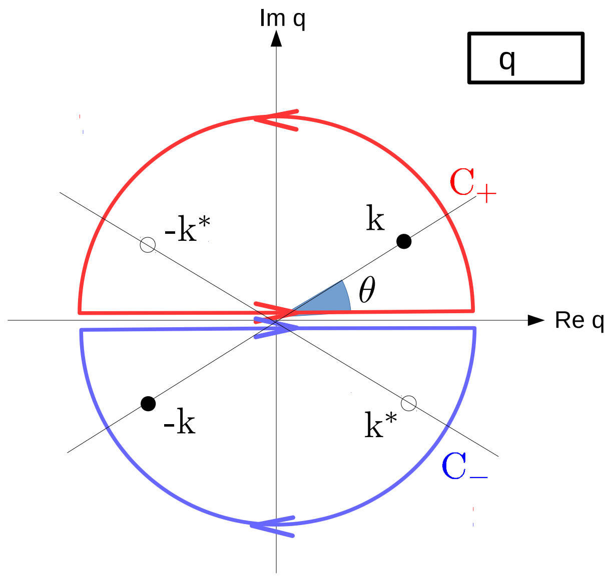

Figure 1: Contour path $C_\pm$ on the complex momentum q-plane.

Therefore

$$
\begin{aligned}
G_F(\boldsymbol{r}\boldsymbol{r}';k^*) &= -\frac{2m}{\hbar^2}\frac{1}{4\pi}\sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\oint_{C_+} dq\frac{1}{q+k^*}qj_l(qr_<)h_l^{(+)}(qr_>) \\
&\quad -\frac{2m}{\hbar^2}\frac{1}{4\pi}\sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')\oint_{C_-} dq\frac{1}{q-k^*}qj_l(qr_<)h_l^{(-)}(qr_>) \\
&= \frac{2m}{\hbar^2}\frac{i}{2}\sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')k^*j_l(-k^*r_<)h_l^{(+)}(-k^*r_>) \\
&\quad +\frac{2m}{\hbar^2}\frac{i}{2}\sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')k^*j_l(k^*r_<)h_l^{(-)}(k^*r_>) \\
&= \frac{2m}{\hbar^2}ik^*\sum_{lm} Y_{lm}(\widehat{\boldsymbol{r}})Y^*_{lm}(\widehat{\boldsymbol{r}}')j_l(k^*r_<)h_l^{(-)}(k^*r_>) = -\frac{2m}{\hbar^2}\frac{e^{-ik^*|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \qquad (54)
\end{aligned}
$$