Complex Scaling Method : Principles and Applications

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明 孝之

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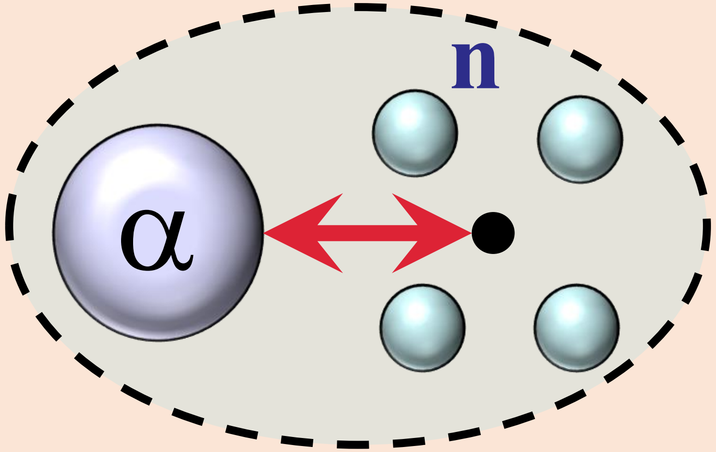
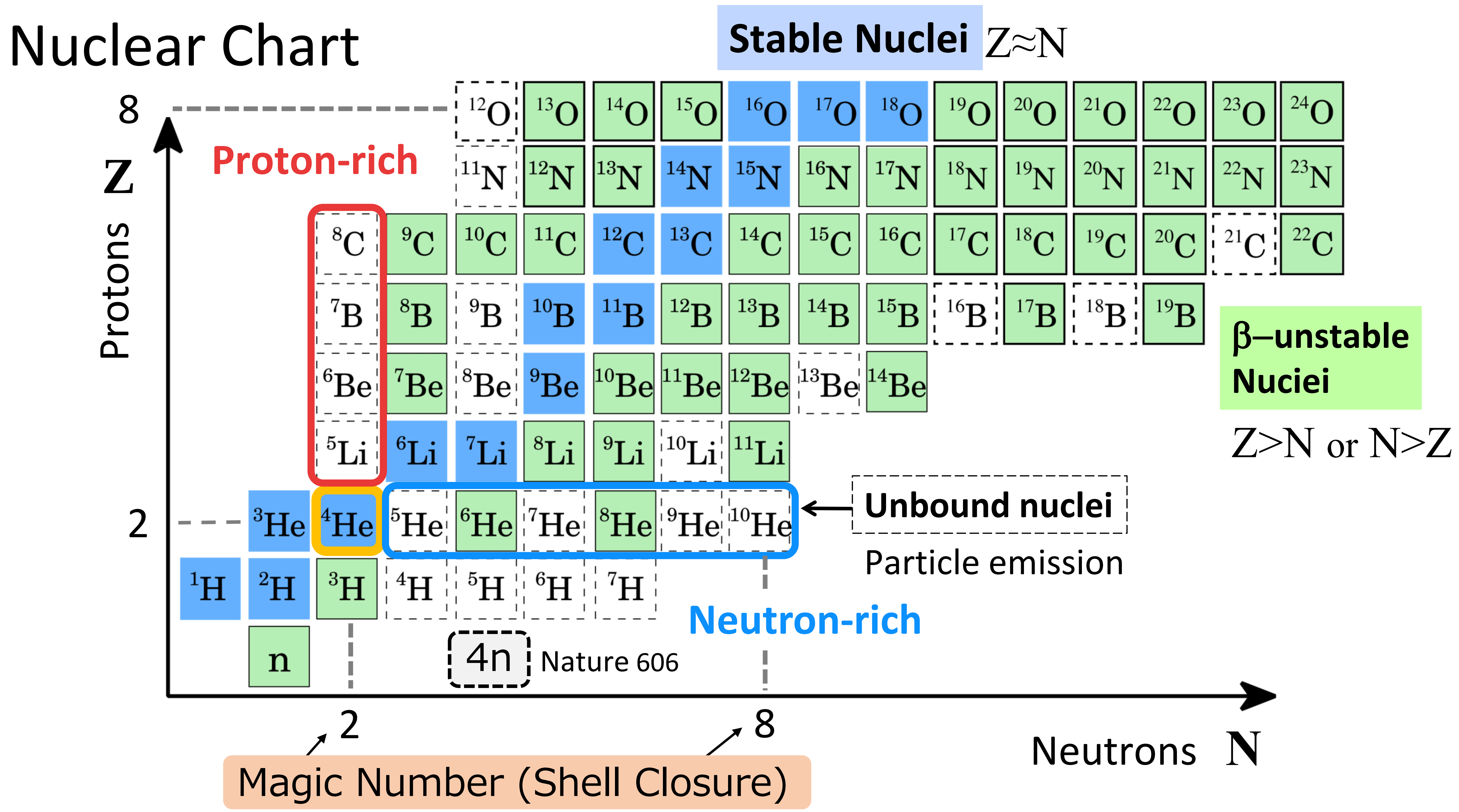
Outline

- Physics of light unstable nuclei
- Resonances and non-resonant continuum states
- Complex scaling Method (CSM)
  複素（座標）スケーリング法
- Nuclear model : Cluster model
- Basic properties of CSM
- Many-body resonances
- Level density, Green's function
- Strength functions, breakup reactions



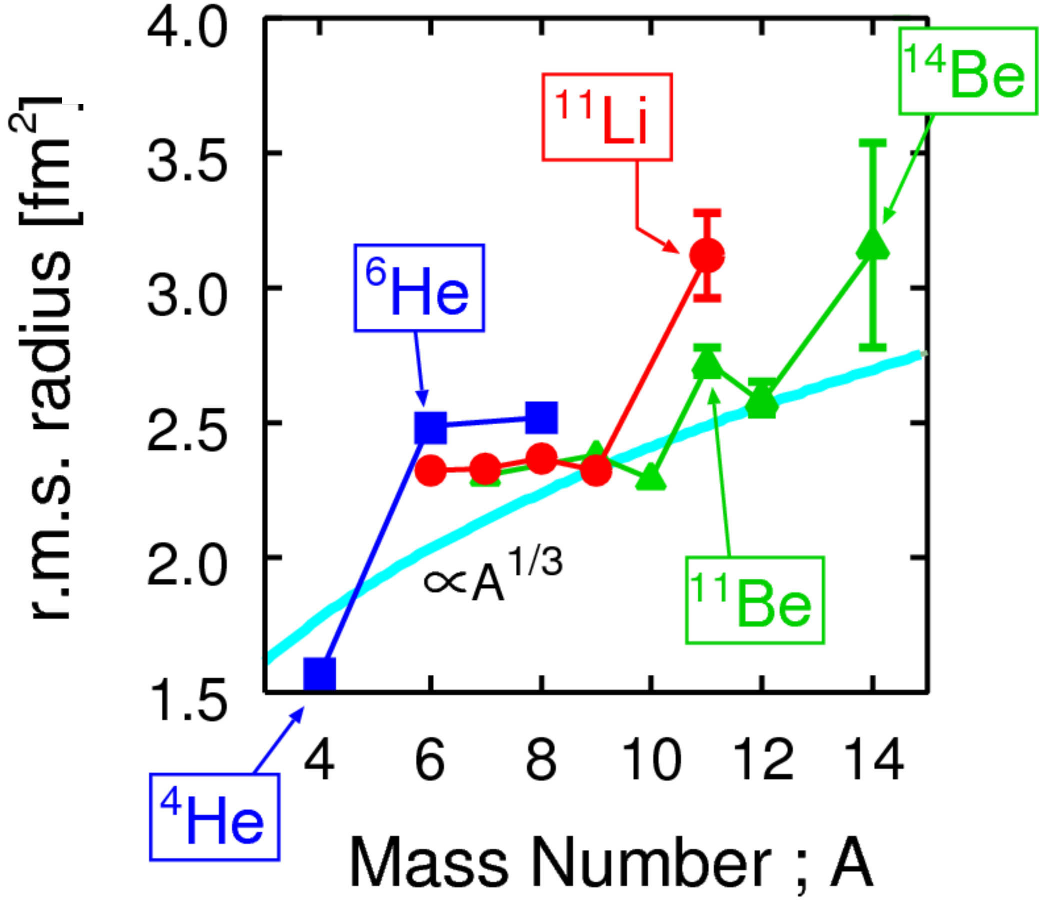
^{8}He

Introduction :
Physics of light unstable nuclei



Nuclear Chart
Stable Nuclei Z≈N
Proton-rich
Protons Z
β–unstable Nuciei
Z>N or N>Z
Unbound nuclei
Particle emission
Neutron-rich
4n Nature 606
Neutrons N
Magic Number (Shell Closure)

Nuclear Radius



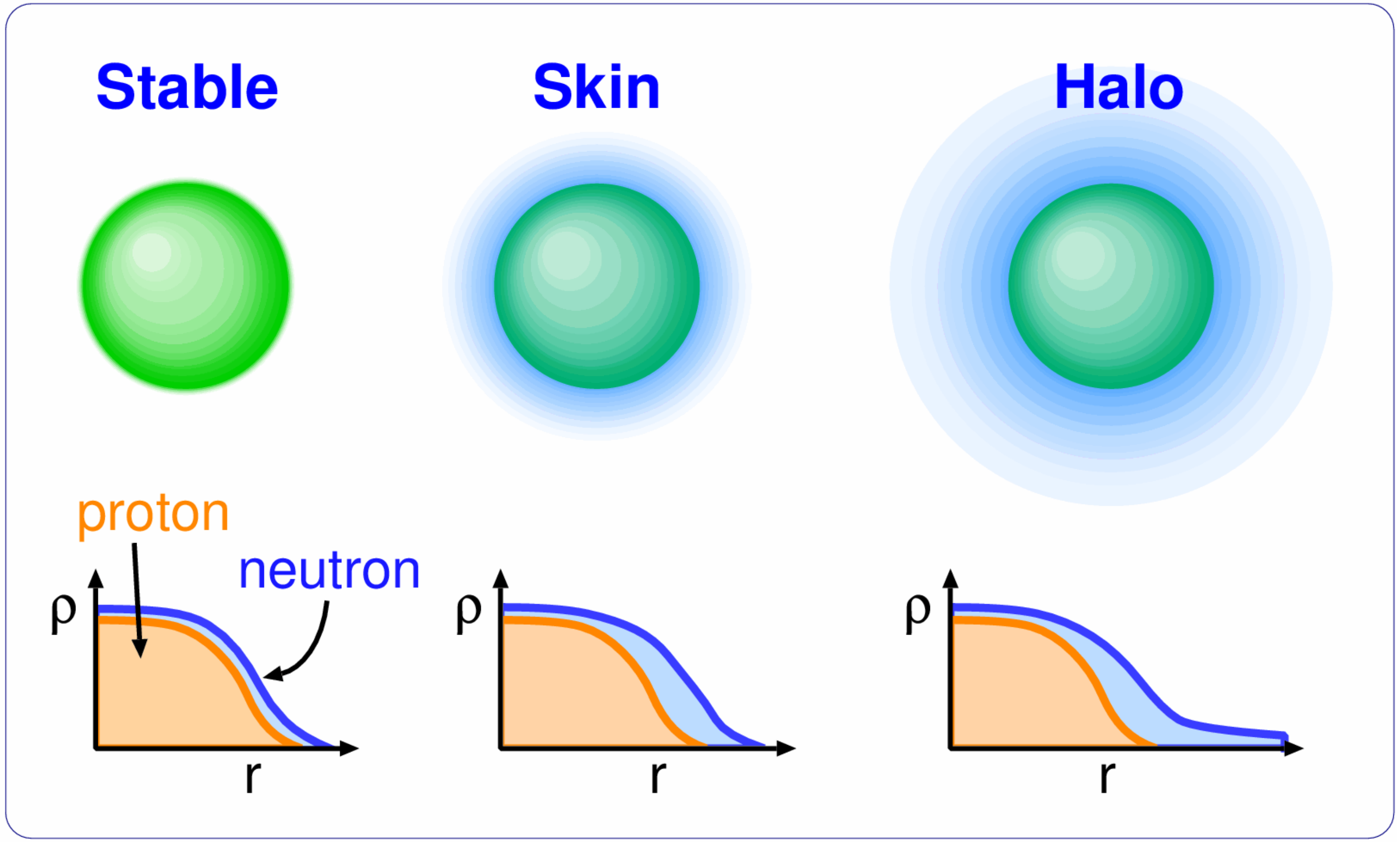
I. Tanihata et. al
PLB206(1988)592

Neutron(s) separation energy

	Z	N	S_n or S_{2n} (MeV)
^{6}He	2	4	0.98 $^{4}He+n+n$
^{11}Li	3	8	0.37 $^{9}Li+n+n$
^{11}Be	4	7	0.50 $^{10}Be+n$
^{14}Be	4	10	1.27 $^{12}Be+n+n$

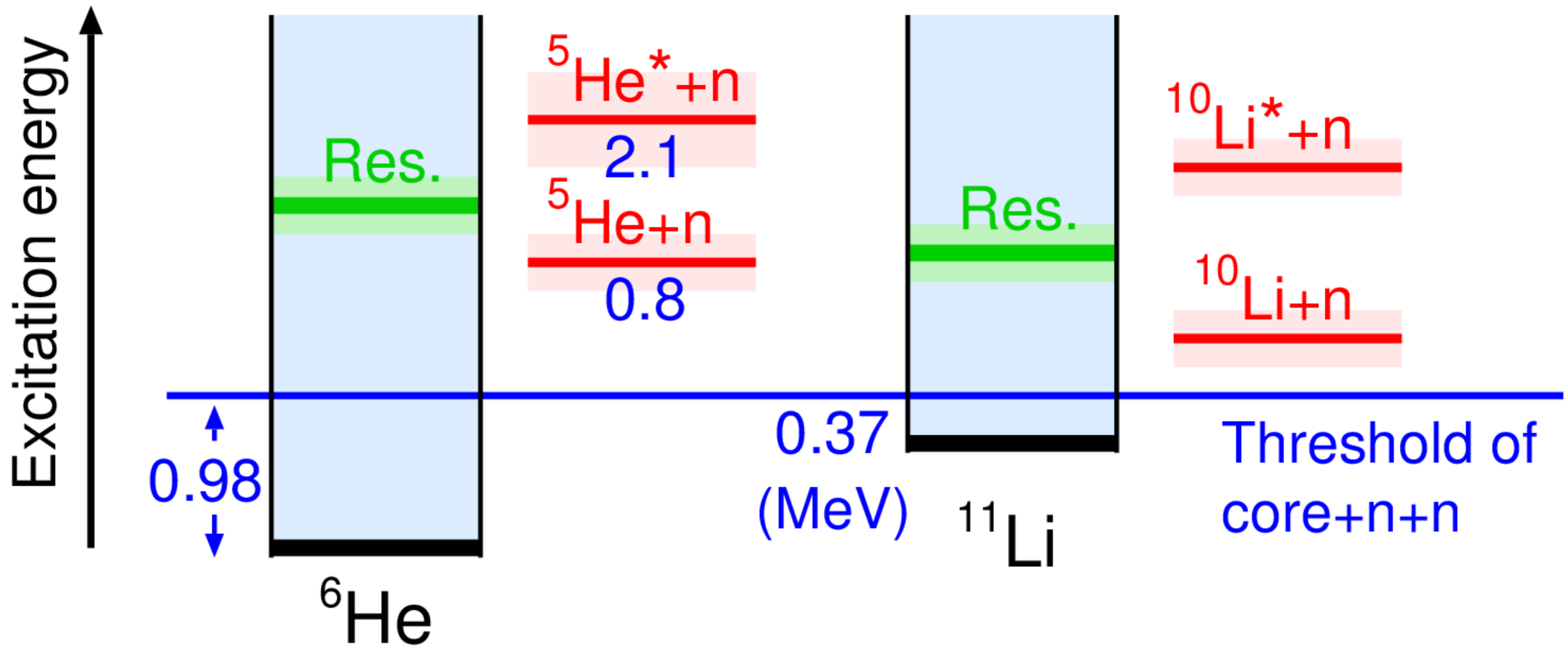
Stable nuclei -- 8 MeV/A

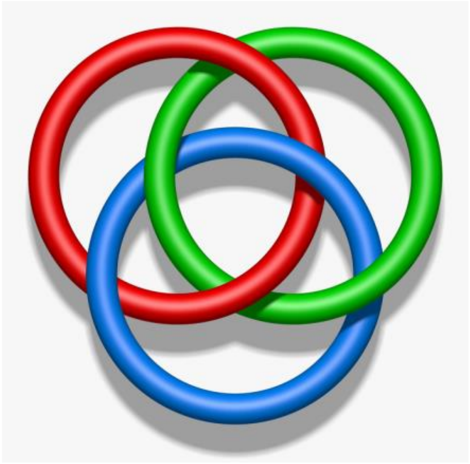
Neutron halo & skin structures



- Large matter radius deviated from empirical rule of $r_0 A^{1/3}$
- Small momentum distribution of halo/skin part (T. Kobayashi, NPA538(1992)343c)
- Small neutron(s) separation energy → (core + valence neutrons) picture

Borromean property of halo nuclei

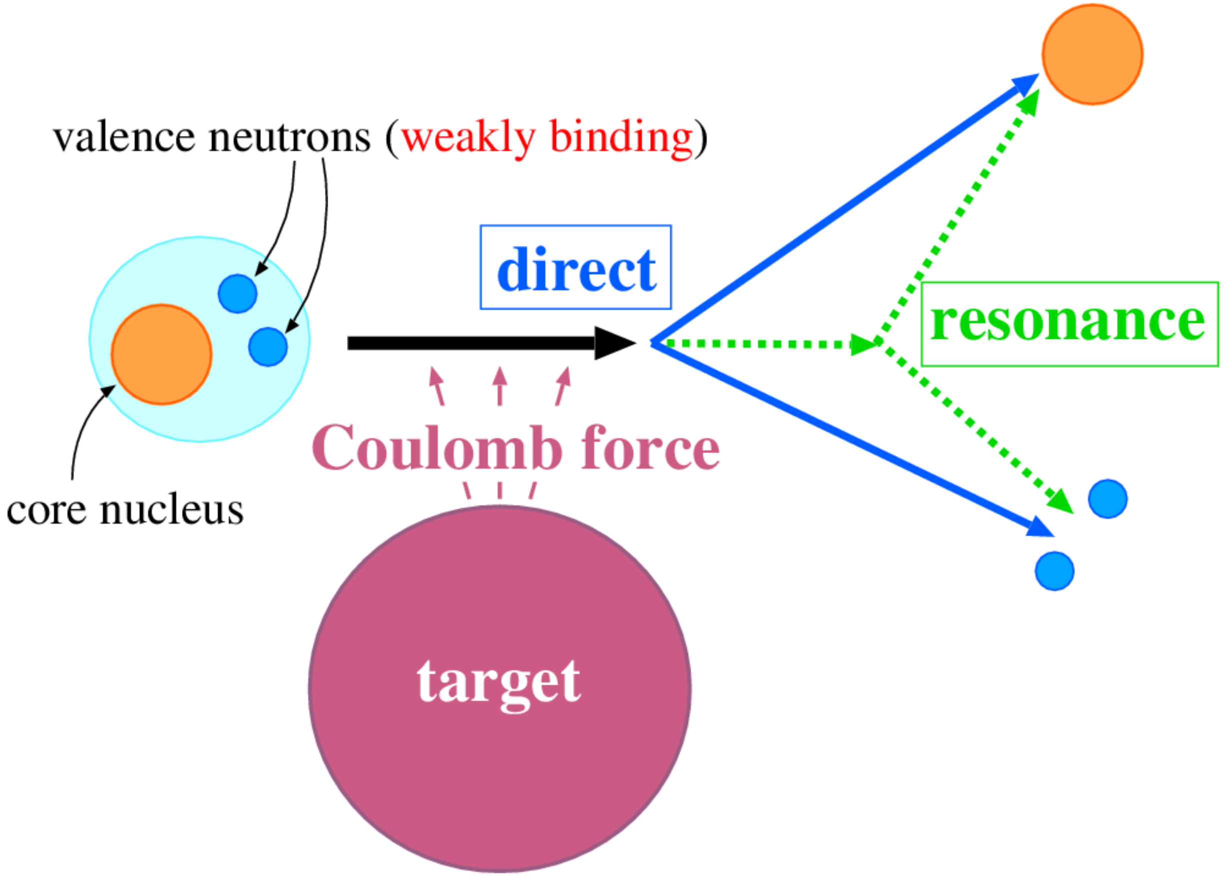


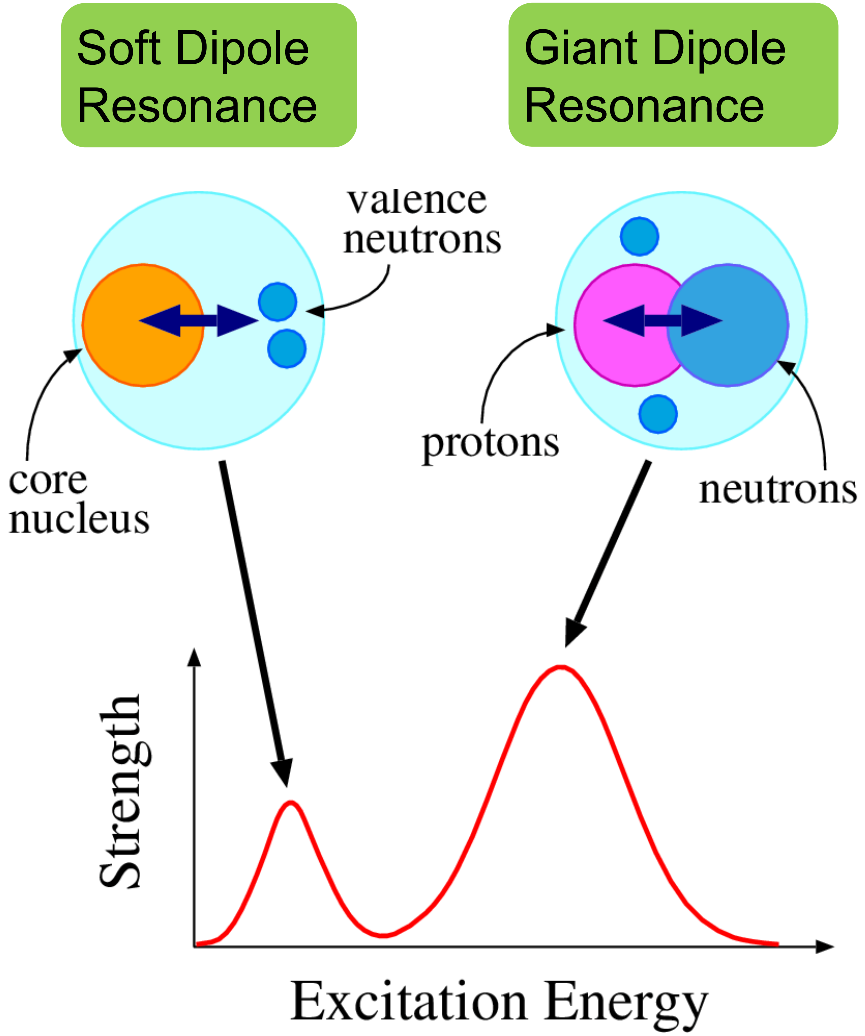


Borromean rings (Wiki)

- Binding mechanism is essential to understand the structures of halo nuclei.
- Theoretically, resonance description is inevitable for subsystem and the excitation of halo nuclei.

Coulomb breakup reactions & soft dipole resonance



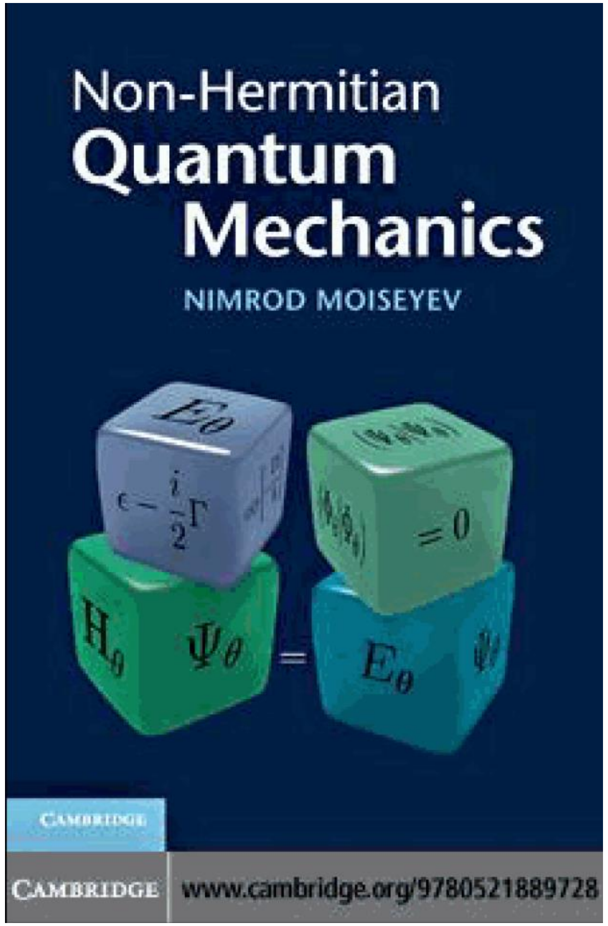


P.G. Hansen and B.Jonson, Europhys. Lett. 4(1987)409.
K. Ikeda, NPA538(1992)355c.
R. Kanungo, TM et al., PRL114 (2015) 192502

Complex Scaling Method (CSM)
to describe unbound states

Textbook for complex scaling

- Quantum theory of resonances: calculating energies, widths and cross-sections by complex scaling
  - **Prof. Nimrod Moiseyev (Israel)**
  - Physics Reports 302 (1998) 211-293
  - Application to atomic, molecular physics
- Non-Hermitian Quantum Mechanics
  - Prof. Nimrod Moiseyev
  - Cambridge University Press (2011)
  - 410 page



INVITED PAPERS

Progress of Theoretical Physics, Vol. 116, No. 1, July 2006

The Complex Scaling Method for Many-Body Resonances and Its Applications to Three-Body Resonances

Shigeyoshi Aoyama, Takayuki Myo, Kiyoshi Katō
and Kiyomi Ikeda

- Summary of recent studies of CSM to treat many-body resonances
- Applications to **three-body resonant states** in two-neutron halo nuclei and three-cluster systems.

Progress in Particle and Nuclear Physics 79 (2014) 1–56

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journal homepage: www.elsevier.com/locate/ppnp

Review

Recent development of complex scaling method for many-body resonances and continua in light nuclei

Takayuki Myo, Yuma Kikuchi, Hiroshi Masui, Kiyoshi Katō

- Investigate many-body resonant states in weakly bound nuclei very far from the stability lines.
- Using these complex eigenvalues and eigenstates in CSM, we construct the Green's functions to calculate strength functions and breakup cross sections.

PTEP

Progress of Theoretical and Experimental Physics, Special Sections

Complex scaling : physics of unbound light nuclei and perspective

Takayuki Myo and Kiyoshi Katō

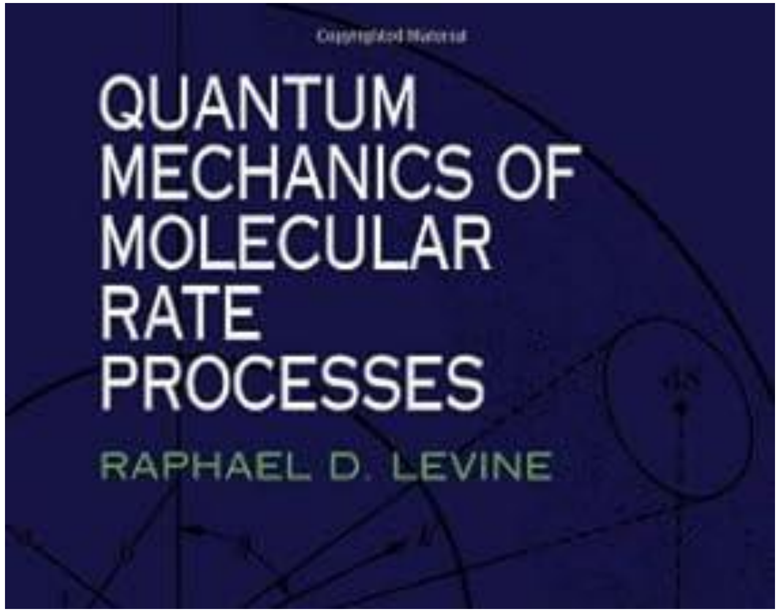
- Trends in non-Hermitian Quantum Mechanics
- Editor : Prof. Naomichi Hatano 羽田野直道

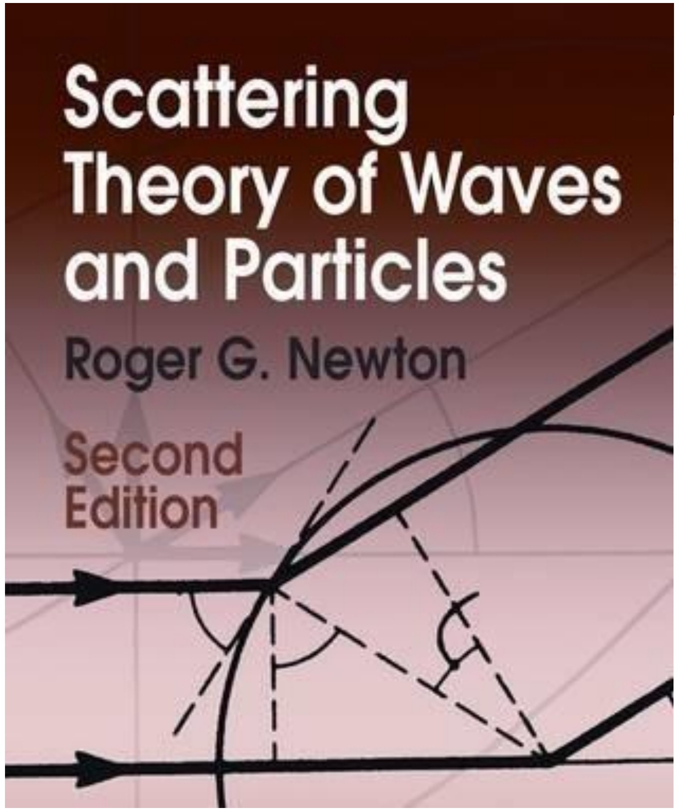
PTEP 2020 (2020) 12A101

- Basic application of CS to the unbound phenomena of light nuclei
- Resonant and non-resonant continuum states observed in unstable nuclei
- Continuum level density in the scattering problem using the Green's function

Textbook for scattering problem

- Scattering Theory of Waves and Particles,
  Roger G. Newton
- 大学演習　量子物理学,
  山内恭彦・武田　暁
- Quantum Mechanics of Molecular Rate Processes,
  Raphael D. Levine (1968)

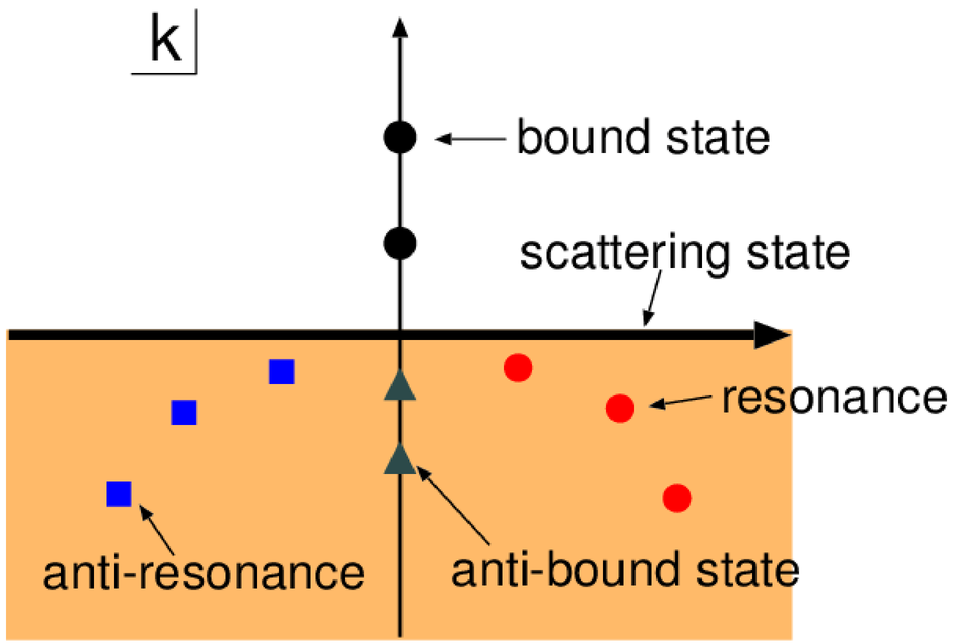




Boundary condition for resonances

- α decay : G. Gamow, Z. Phys. 51 (1928) 204.
  - Decaying state : Gamow state
  - Complex energy $E_R = E_r - \frac{i\Gamma}{2}$ $(k_R = \kappa - i\gamma)$
  - $\Psi(r,t) = \Psi(r) \cdot e^{-\frac{iE_R t}{\hbar}} = \Psi(r) \cdot e^{-\frac{iE_r}{\hbar}t} \cdot e^{-\frac{\Gamma}{2\hbar}t}$
  - $|\Psi(r,t)|^2 = |\Psi(r)|^2 \cdot e^{-\frac{\Gamma}{\hbar}t}$ $\quad \Gamma/\hbar$ Decay rate [1/s]
- Boundary condition of out-going waves
  - A.J.F. Siegert, Phys. Rev. 56 (1939) 750.
  - $\Psi(r) \to \frac{e^{ik_R r}}{r} = \frac{e^{i(\kappa - i\gamma)r}}{r} = \frac{e^{i\kappa r}}{r} \cdot e^{\gamma r} \to \infty \quad (r \to \infty)$ Divergent
  - Incoming wave : $\frac{e^{-ik_R r}}{r} = \frac{e^{-i(\kappa - i\gamma)r}}{r} = \frac{e^{-i\kappa r}}{r} \cdot e^{-\gamma r} \to 0$ Damping

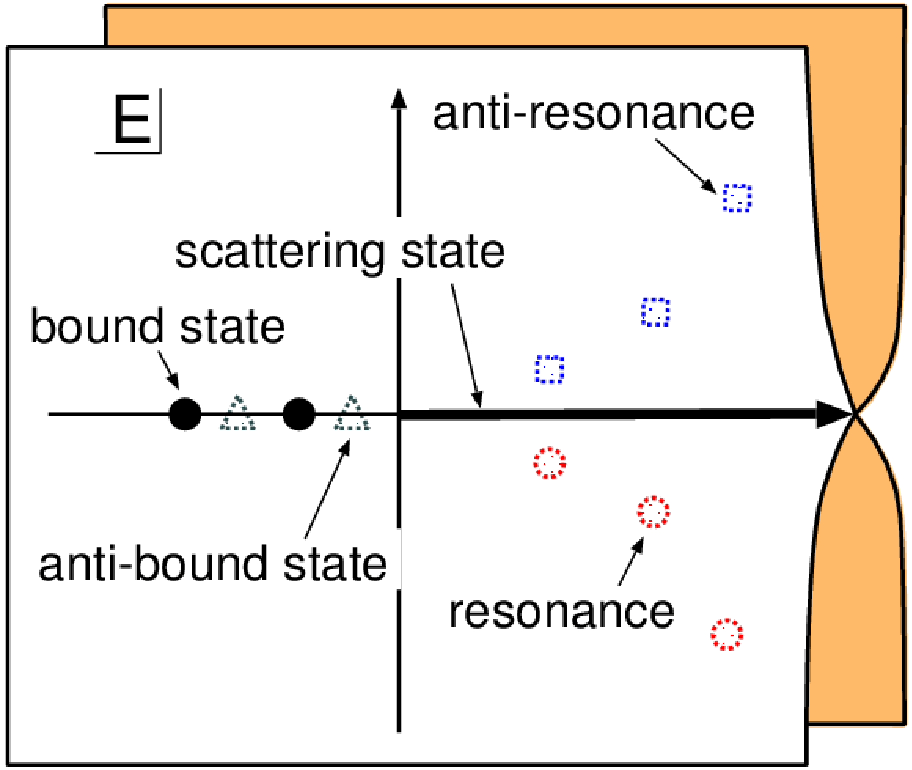
Discrete states in complex k & E planes



Complex momentum plane

$$k_{\mathrm{R}} = \kappa - i\gamma$$
$$k_{\mathrm{AR}} = -\kappa - i\gamma = -k_{\mathrm{R}}^{*} = \tilde{k}_{\mathrm{R}}$$

bi-orthogonal relation 双直交系



Complex energy plane

$$E_{\mathrm{R}} = E_r - i\,\Gamma/2$$ decay
$$E_{\mathrm{AR}} = E_r + i\,\Gamma/2 = E_{\mathrm{R}}^{*} = \tilde{E}_{\mathrm{R}}$$ grow

Hatano, Sasada, Nakamura, Petrosky, PTP119 (2008)187

Boundary condition for resonances

- Resonance as Decaying state

Quadrant-4 in *E* & *k*

  - $E_{\mathrm{R}} = E_r - \frac{i\Gamma}{2} \quad (k_{\mathrm{R}} = \kappa - i\gamma)$
  - $\Psi(r,t) = \Psi(r) \cdot e^{-\frac{iE_{\mathrm{R}}t}{\hbar}} = \Psi(r) \cdot e^{-\frac{iE_r t}{\hbar}} \cdot e^{-\frac{\Gamma t}{2\hbar}}$ **Decrease (decaying)**
  - $\Psi(r) \to \frac{e^{ik_{\mathrm{R}}r}}{r} = \frac{e^{i(\kappa - i\gamma)r}}{r} = \frac{e^{i\kappa r}}{r} \cdot e^{\gamma r} \to \infty \quad (r \to \infty)$ **Outgoing** ($\kappa > 0$)
  - Incoming wave : $\frac{e^{-ik_{\mathrm{R}}r}}{r} = \frac{e^{-i(\kappa - i\gamma)r}}{r} = \frac{e^{-i\kappa r}}{r} \cdot e^{-\gamma r} \to 0$

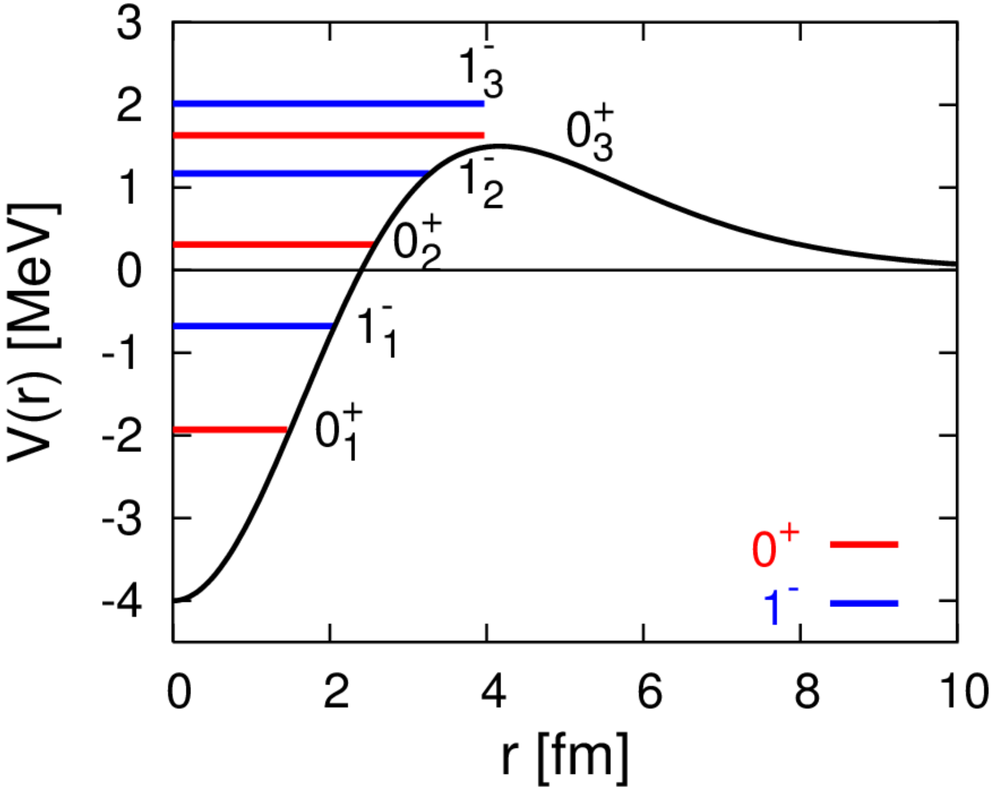
- Anti-Resonance as Capturing state

Quadrant-1 in *E*
Quadrant-3 in *k*

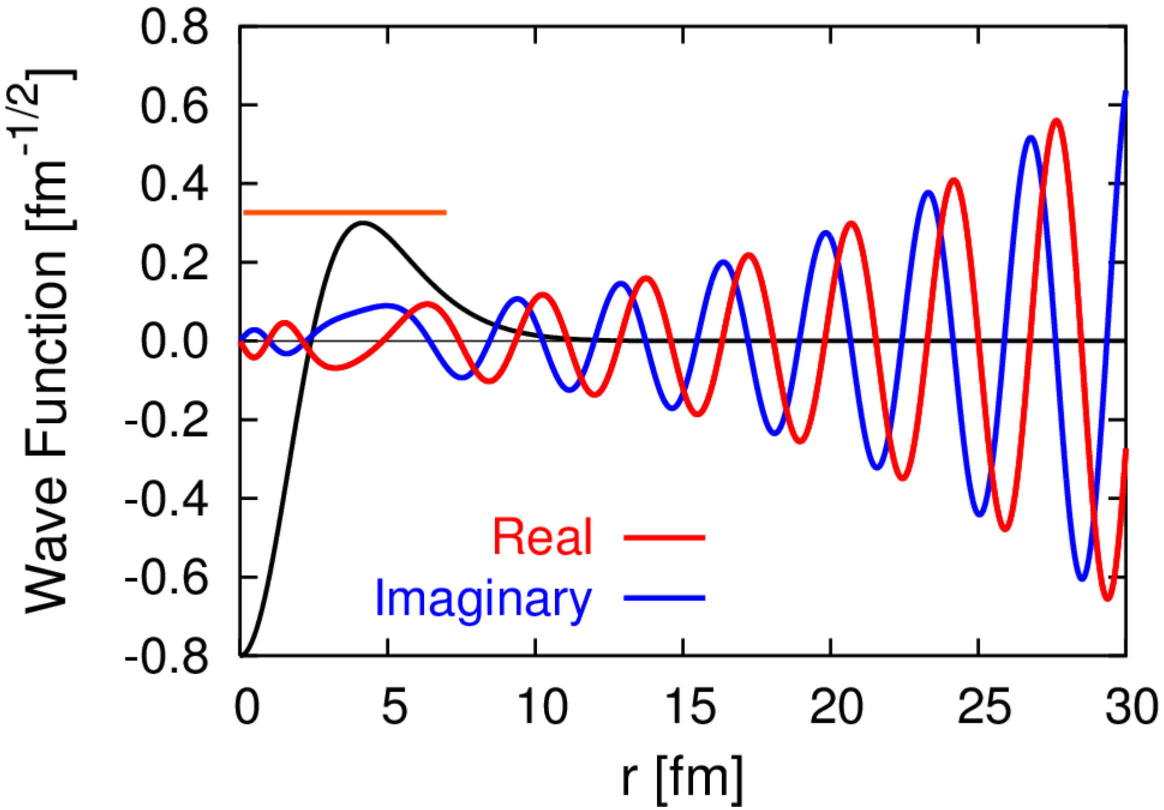
  - $E_{\mathrm{AR}} = E_r + \frac{i\Gamma}{2} \quad (k_{\mathrm{AR}} = -\kappa - i\gamma)$
  - $\Psi(r,t) = \Psi(r) \cdot e^{-\frac{iE_{\mathrm{AR}}t}{\hbar}} = \Psi(r) \cdot e^{-\frac{iE_r t}{\hbar}} \cdot e^{+\frac{\Gamma t}{2\hbar}}$ **Increase (growing)**
  - $\Psi(r) \to \frac{e^{ik_{\mathrm{AR}}r}}{r} = \frac{e^{i(-\kappa - i\gamma)r}}{r} = \frac{e^{-i\kappa r}}{r} \cdot e^{\gamma r} \to \infty \quad (r \to \infty)$ **Incoming** ($\kappa > 0$)
  - Incoming wave : $\frac{e^{-ik_{\mathrm{AR}}r}}{r} = \frac{e^{-i(-\kappa - i\gamma)r}}{r} = \frac{e^{i\kappa r}}{r} \cdot e^{-\gamma r} \to 0$ **Outgoing**

Schematic potential case

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V, \qquad \frac{\hbar^2}{\mu} = 1, \qquad V(r) = -8e^{-0.16r^2} + 4e^{-0.04r^2}$$



$$E(0_3^+) = 1.63 - i0.13 \text{ (MeV)}$$

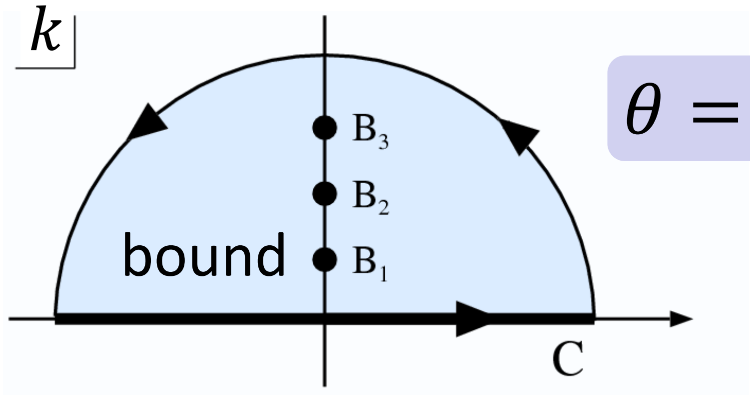


A. Csoto, B. Gyarmati, A. T. Kruppa, K. F. Pal, N. Moiseyev, Phys. Rev. A41 (1990) 3469

Complex Scaling

$$H_\theta \Phi_\theta = E_\theta \Phi_\theta$$

$$U(\theta): \boldsymbol{r} \to \boldsymbol{r}\exp(i\theta), \quad \boldsymbol{k} \to \boldsymbol{k}\exp(-i\theta), \quad H_\theta = U(\theta) H U^{-1}(\theta), \quad \theta \in \mathbb{R}$$

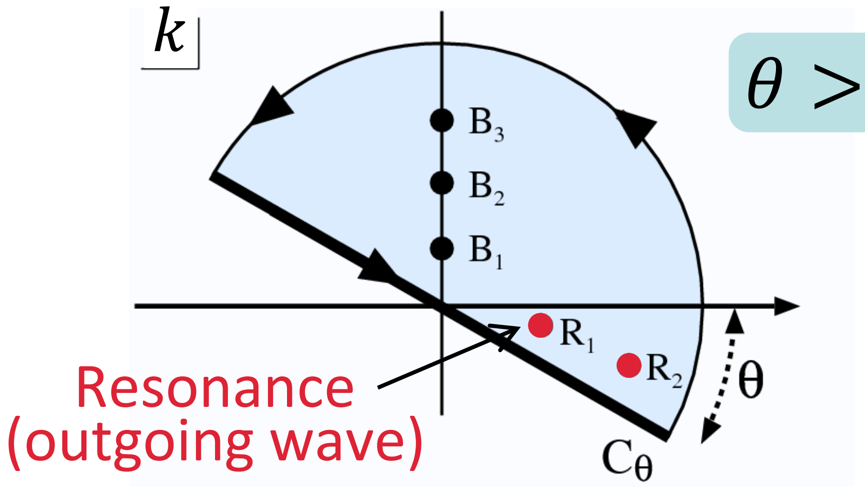


$\theta = 0$

Completeness relation

Textbook by R.G. Newton
J. Math. Phys. 1 (1960)319

$$1 = \sum_{\mathrm{B}} |\varphi_{\mathrm{B}}\rangle\langle\tilde{\varphi}_{\mathrm{B}}| + \int_C dk |\varphi_k\rangle\langle\tilde{\varphi}_k|$$



$\theta > 0$

Extended completeness relation

$$1 = \sum_{\mathrm{B}} |\varphi_{\mathrm{B}}\rangle\langle\tilde{\varphi}_{\mathrm{B}}| + \sum_{\mathrm{R}} |\varphi_{\mathrm{R}}\rangle\langle\tilde{\varphi}_{\mathrm{R}}| + \int_{C_\theta} dk_\theta |\varphi_{k_\theta}\rangle\langle\tilde{\varphi}_{k_\theta}|$$

Siegert state

$$E_{\mathrm{R}} = E_r - i\Gamma/2$$

Bi-orthogonal states 双直交系
T. Berggren, NPA109(1968)265.

J.Aguilar and J.M.Combes, Commun. Math. Phys.,22(1971)269.
E.Balslev and J.M.Combes, Commun. Math. Phys.,22(1971)280.

T. Myo, K. Katō, PTP98 (1997) 1275
B.G. Giraud, K. Katō, A. Ohnishi, J. Phys. A **37** (2004)11575

Schrödinger Eq. & wave func. with complex scaling

$$U(\theta)HU^{-1}(\theta) = H_\theta = T_\theta + V_\theta \qquad T_\theta = e^{-2i\theta}T, \qquad V_\theta = V(\boldsymbol{r}e^{i\theta})$$

Keep analyticity

$$H\Phi = E\Phi \rightarrow H_\theta\Phi_\theta = E_\theta\Phi_\theta \qquad \Phi_\theta(\boldsymbol{r}) = e^{\frac{3}{2}\cdot i\theta}\Phi(\boldsymbol{r}e^{i\theta})$$

$$\mathrm{d}\boldsymbol{r} \rightarrow \mathrm{d}\left(\boldsymbol{r}e^{i\theta}\right) = e^{3i\theta}\mathrm{d}\boldsymbol{r}$$

Jacobian

- Asymptotic condition ($\boldsymbol{r} \longrightarrow \infty$) in complex scaling
- Outgoing wave for decaying state : Φ_R

$$k_\mathrm{R} = \kappa - i\gamma = K_\mathrm{R}e^{-i\theta_\mathrm{R}}, \ \theta_\mathrm{R} > 0 \qquad E_\mathrm{R} = E_r - i\frac{\Gamma}{2}, \quad \Phi(t) = e^{-\frac{iE_\mathrm{R}t}{\hbar}} = e^{-\frac{iE_r t}{\hbar}} \cdot e^{-\frac{\Gamma t}{2\hbar}}$$

decay

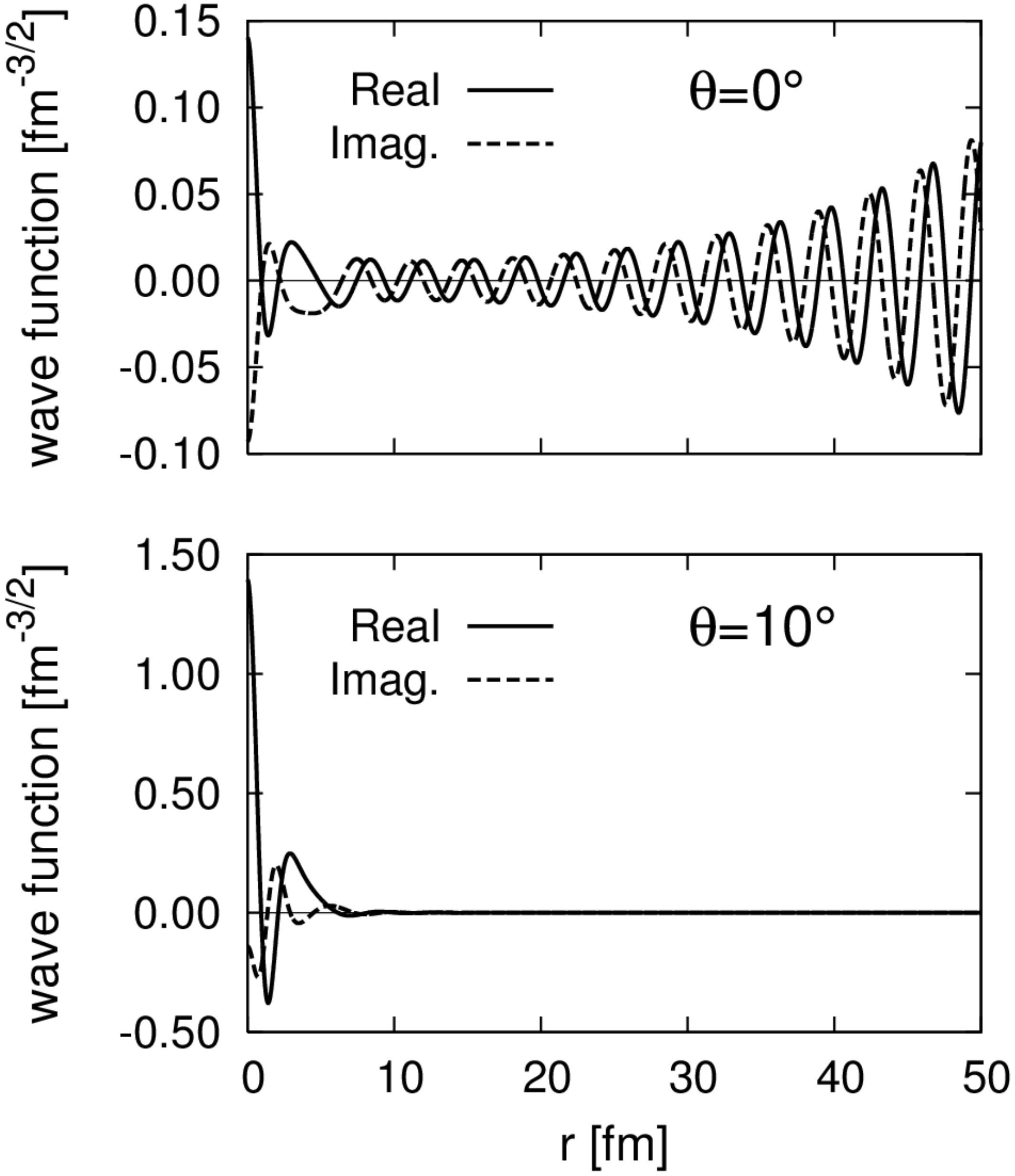
$$\Phi_\mathrm{R}(r) \sim \exp(ik_\mathrm{R}r) = \exp(iK_\mathrm{R}e^{-i\theta_\mathrm{R}}r)$$

$$U_\theta\Phi_\mathrm{R}(r) \sim \exp\left(iK_\mathrm{R}e^{-i\theta_\mathrm{R}}\cdot re^{i\theta}\right) = \exp(iK_\mathrm{R}re^{i(\theta-\theta_\mathrm{R})})$$

$$= \exp(iK_\mathrm{R}r\cos(\theta-\theta_\mathrm{R}))\cdot\exp(-K_\mathrm{R}r\sin(\theta-\theta_\mathrm{R}))$$

damping with $\theta > \theta_R$

Wave function with and w/o CSM



$E(0_3^+) = 1.63 - i0.13$ (MeV)

Divergent behavior

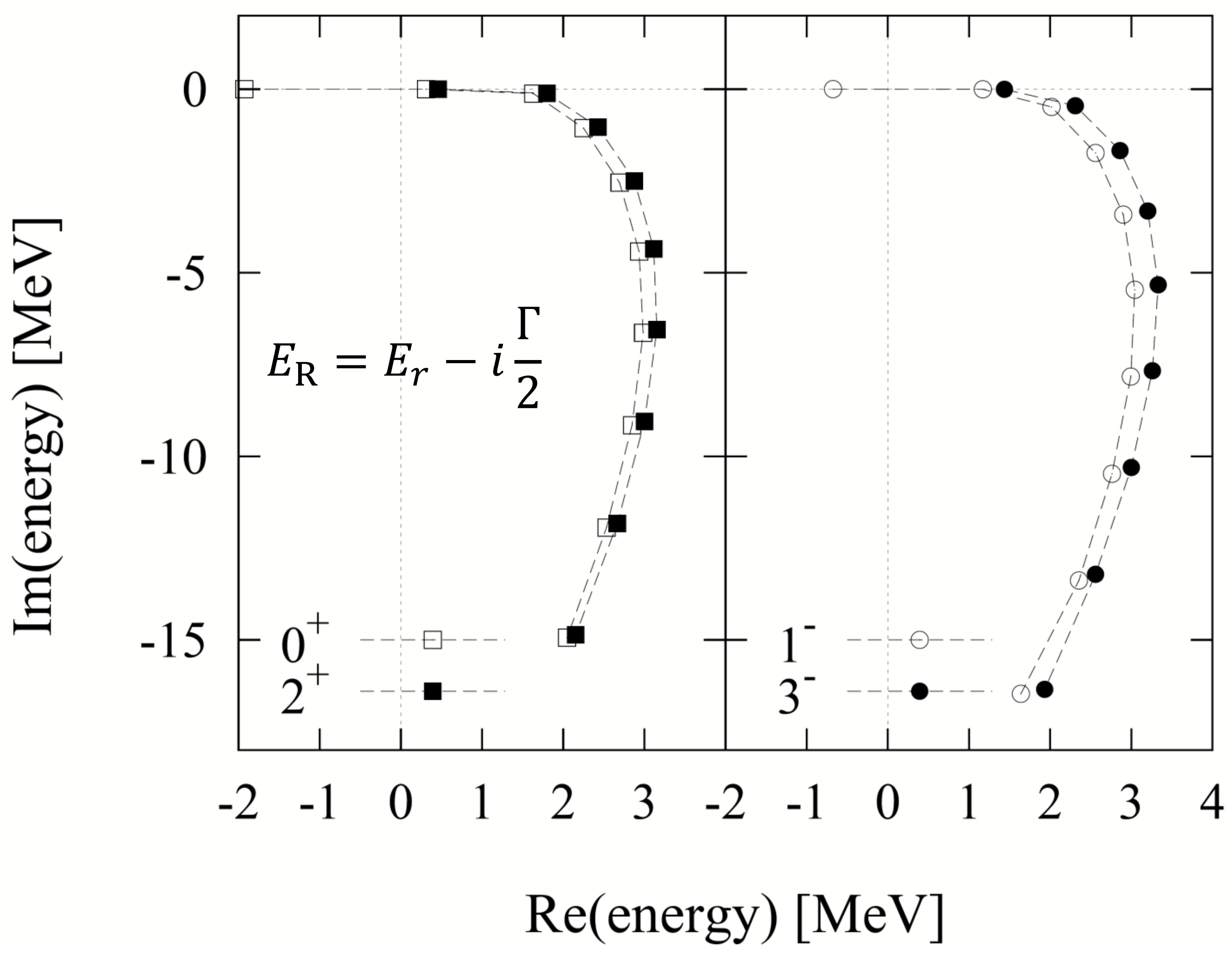
Damping behavior

日本物理学会誌61巻11号814 (2006)

解説「共鳴状態をめぐる理論と数値計算法の発展」

加藤幾芳、池田清美

Resonance poles with schematic potential



- Gaussian Potential : $V(r) = e^{-ar^2}$
- $V_\theta(r) = U_\theta V(r) U_\theta^{-1} = V(re^{i\theta})$
  $= \exp(-ar^2 e^{2i\theta})$
  $= \exp(-ar^2 \cos 2\theta - iar^2 \sin 2\theta)$
- Impose damping condition
- $\cos 2\theta > 0 \rightarrow 0 < 2\theta < \pi/2$
- Upper limit of θ
- $V(r) = \frac{V_0}{1+\exp\left(\frac{r-R}{a}\right)}$: Wood-Saxon pot.
  $\rightarrow \frac{re^{i\theta}-R}{a} = i(2n+1)\pi$: divergence

T. Myo, K. Kato, Prog. Theor. Phys. 98 (1998) 1275

Scattering problem in CSM

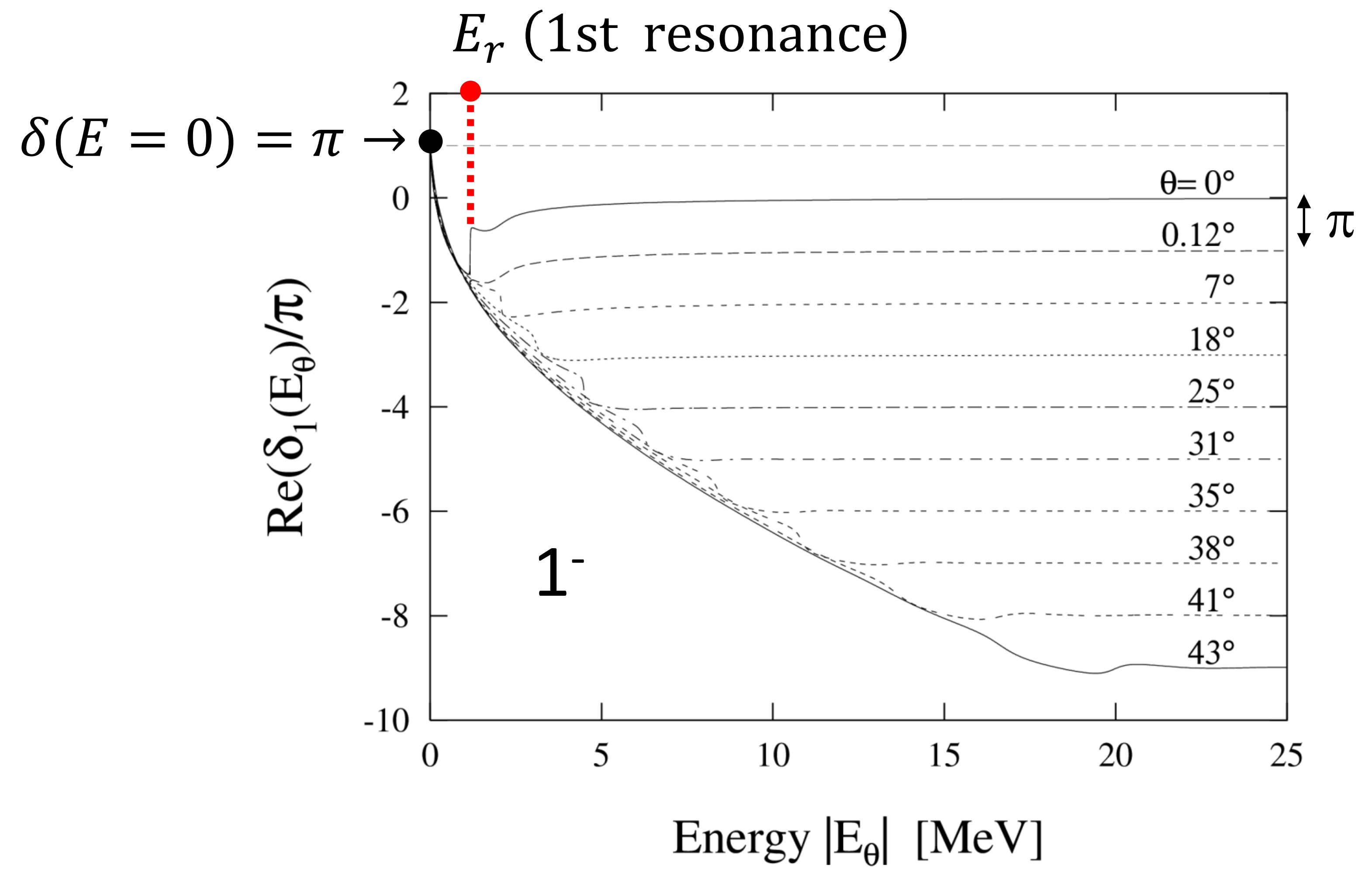
- Wave function : $\Phi_\ell(\boldsymbol{r}) = R_\ell\,(r)/r \cdot Y_{\ell m}(\hat{\boldsymbol{r}})$ (single channel)
- $\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2} + V(r)\right) R_\ell(r) = E R_\ell(r)$: radial part
- $R_\ell(r) \underset{r\to\infty}{\longrightarrow} u_\ell^{(-)}(kr) - S_\ell(k) u_\ell^{(+)}(kr)$, $E = \frac{\hbar^2 k^2}{2\mu}$, $S_\ell(k) = e^{2i\delta_\ell(k)}$
- $u_\ell^{(\pm)}(kr) = \rho(\,-n_\ell(kr) \pm i\, j_\ell(kr)\,)$: incoming/outgoing waves.

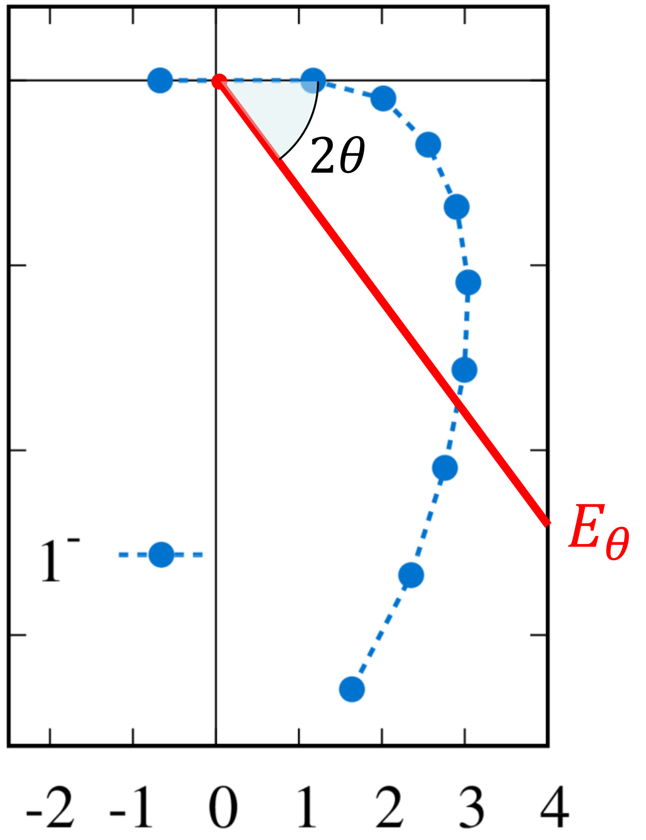
- In CSM, $r \to re^{i\theta}$
- $\left(-\frac{\hbar^2}{2\mu}\frac{1}{e^{2i\theta}}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2 e^{2i\theta}} + V(re^{i\theta})\right) R_\ell^\theta(r) = E_\theta R_\ell^\theta(r)$
- $R_\ell^\theta(r) \underset{r\to\infty}{\longrightarrow} u_\ell^{(-)}(kr) - S_\ell^\theta(k_\theta) u_\ell^{(+)}(kr)$

kr : invariant

- $E_\theta = \frac{\hbar^2 k_\theta^2}{2\mu}$, $k_\theta = ke^{-i\theta}$, $S_\ell^\theta(k_\theta) = e^{2i\delta_\ell^\theta(k_\theta)}$

Phase shift & Levinson Theorem





- $\delta(E=\infty)-\delta(E=0)=N_{\mathrm{B}}\,\pi,\quad N_{\mathrm{B}}$: # of bound states
- In CSM, $\delta(|E_\theta|=\infty)-\delta(|E_\theta|=0)=(N_{\mathrm{B}}+N_{\mathrm{R}}^{\theta})\cdot\pi$

M. Rittby, N. Elander and E. Branda, Phys. Rev. A**24**, 1636 (1981).

Linear basis expansion with complex scaling

- $\Phi(\boldsymbol{r}) = \sum_{n=1}^{N} C_n \, \phi_{n,\ell}(\boldsymbol{r}), \quad \phi_{n,\ell}(\boldsymbol{r}) = N_\ell(b_n)\, r^\ell e^{-(r/b_n)^2} Y_\ell(\hat{\boldsymbol{r}})$
- $H_\theta \Phi_\theta(\boldsymbol{r}) = E_\theta \Phi_\theta(\boldsymbol{r}), \quad \Phi_\theta(\boldsymbol{r}) = \sum_{n=1}^{N} C_n^\theta \, \phi_{n,\ell}(\boldsymbol{r})$
- $$\langle \tilde{\phi}_m | H_\theta | \phi_n \rangle = \int \tilde{\phi}_m^*(\boldsymbol{r}) H\left(\boldsymbol{r} e^{i\theta}\right) \phi_n(\boldsymbol{r})\, d\boldsymbol{r}$$
$$= \int \phi_m\left(\boldsymbol{r} e^{-i\theta}\right) H(\boldsymbol{r}) \phi_n\left(\boldsymbol{r} e^{-i\theta}\right) d\left(\boldsymbol{r} e^{-i\theta}\right)$$
$$= \langle \phi_{m,-\theta}^* | H | \phi_{n,-\theta} \rangle = H_{m,n}^\theta = H_{n,m}^\theta$$
- Norm $N_{m,n} = \langle \tilde{\phi}_m | \phi_n \rangle$, Energy $E_\theta = \dfrac{\langle \tilde{\Phi}_\theta | H_\theta | \Phi_\theta \rangle}{\langle \tilde{\Phi}_\theta | \Phi_\theta \rangle}$
- Bi-variational principle $\delta E_\theta = 0$ (stationary condition with respect to θ)
- Generalized eigenvalue problem $\sum_n^N \left(H_{m,n}^\theta - E_\theta N_{m,n}\right) C_n^\theta = 0$

Gaussian

$b_1, b_2, \cdots, b_N$ [fm]

geometric progression

$b_n = b_0 \gamma^{n-1}$

$\tilde{\phi}_m^*(r) = \phi_m(r)$

radial only

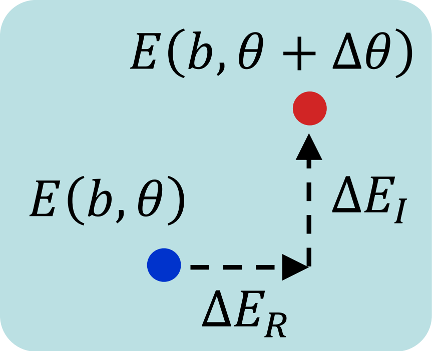
Applicable to Many-body system

Cauchy–Riemann equations in CSM

$$\begin{cases} x = b\cos\theta \\ y = b\sin\theta \end{cases}$$

- $E = E(b,\theta) = E\left(be^{i\theta}\right) = E_R\left(be^{i\theta}\right) + iE_I\left(be^{i\theta}\right)$
- $b\dfrac{\partial E_R}{\partial b} = \dfrac{\partial E_I}{\partial \theta},\quad b\dfrac{\partial E_I}{\partial b} = -\dfrac{\partial E_R}{\partial \theta},\quad \left(\dfrac{\partial E_R}{\partial x} = \dfrac{\partial E_I}{\partial y},\ \dfrac{\partial E_I}{\partial x} = -\dfrac{\partial E_R}{\partial y}\right)$
- $\dfrac{\partial E}{\partial \theta} = \dfrac{\partial E_R}{\partial \theta} + i\dfrac{\partial E_I}{\partial \theta} = -b\dfrac{\partial E_I}{\partial b} + ib\dfrac{\partial E_R}{\partial b} = ib\left(\dfrac{\partial E_R}{\partial b} + i\dfrac{\partial E_I}{\partial b}\right) = ib\dfrac{\partial E}{\partial b}$
- Search for stationary point for both variables
- $\dfrac{\partial E}{\partial \theta} = 0 \Leftrightarrow \dfrac{\partial E}{\partial b} = 0$
- "θ–trajectory" ⊥ "b–trajectory" (Orthogonality)
  - $\dfrac{\partial E}{\partial b} \equiv E_b,\ \dfrac{\partial E}{\partial \theta} \equiv E_\theta,\quad \dfrac{E_{I,b}}{E_{R,b}} \times \dfrac{E_{I,\theta}}{E_{R,\theta}} = \dfrac{E_{I,b}}{E_{R,b}} \times \dfrac{b\cdot E_{R,b}}{(-b\cdot E_{I,b})} = -1$

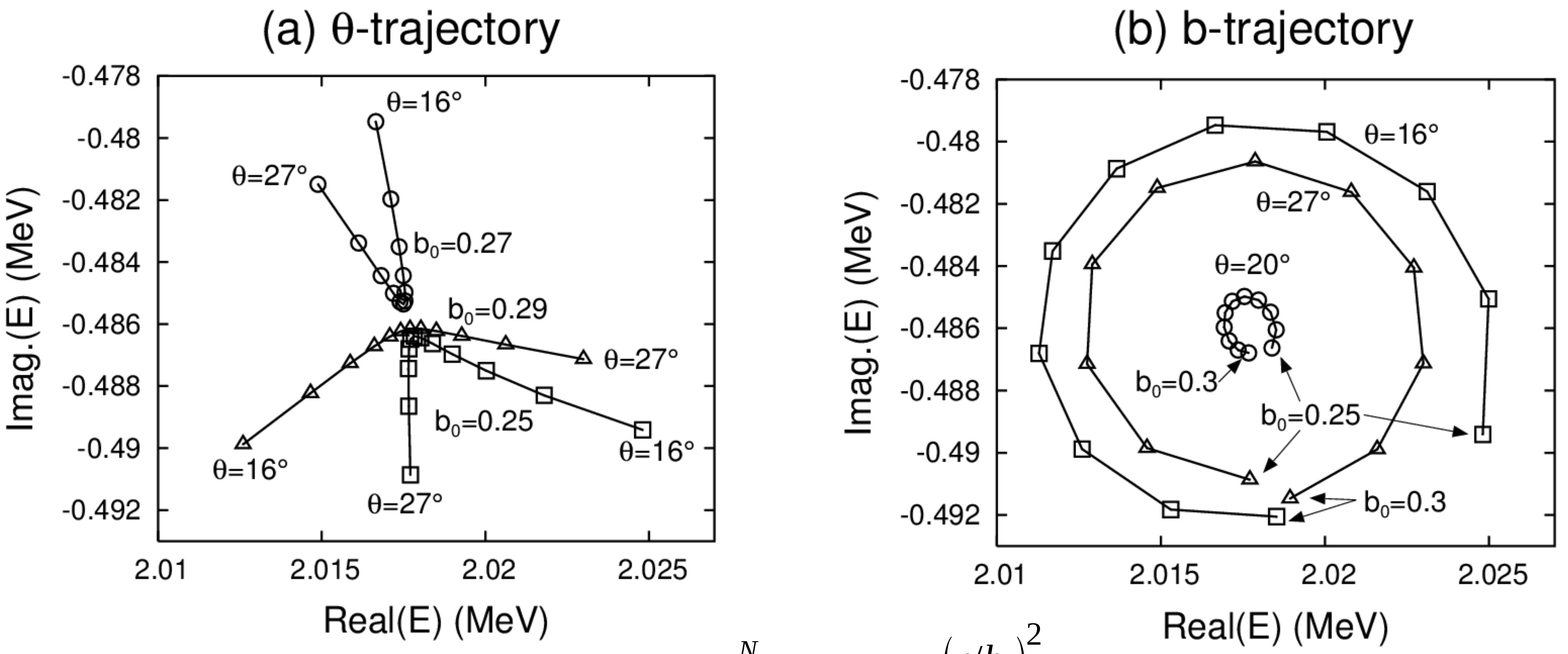
bの傾き　θの傾き　直交



b- and ϑ-trajectory in CSM

- Solutions depends on the basis parameter and θ because of the approximation of the wave functions.
- Search for the stationary point for complex energy $E(b,\theta)$.

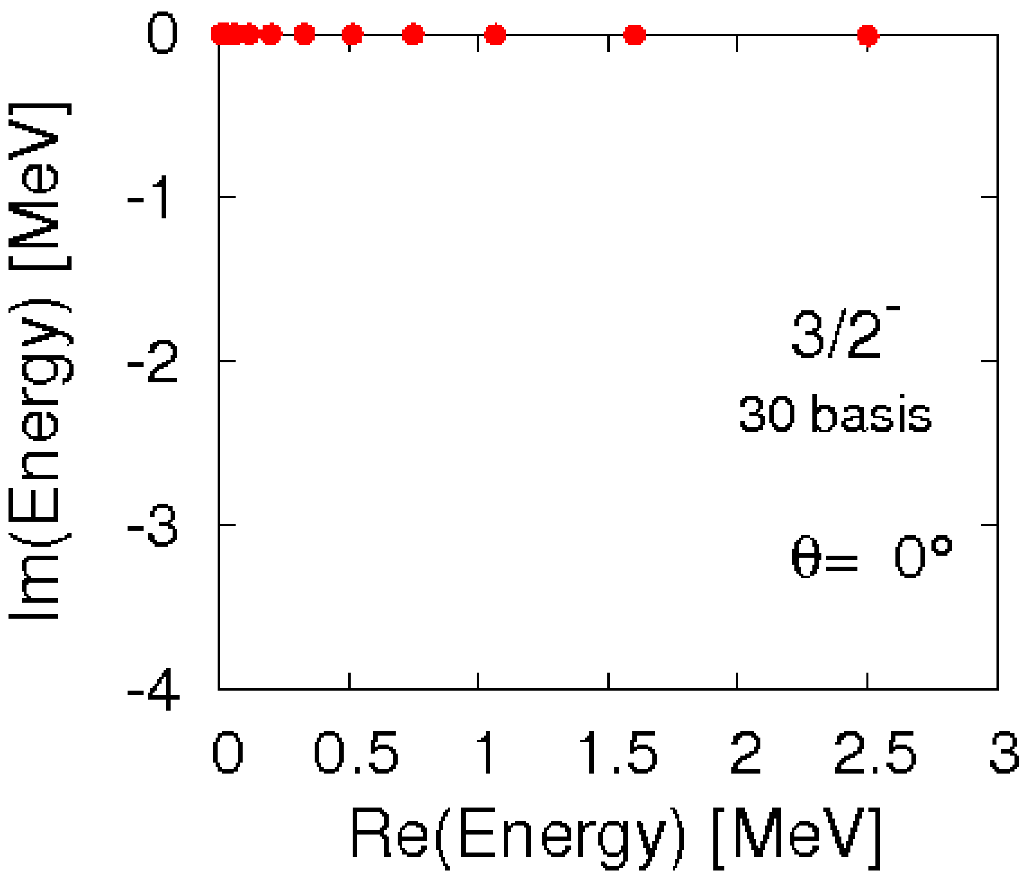
$$\left|\frac{\partial E}{\partial \theta}\right|_{\theta_{\rm opt}} = 0$$

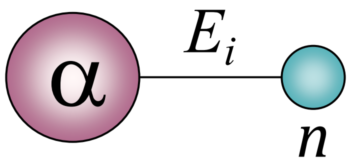


$$\phi_\ell(\mathbf{r}) = \sum_n^N C_n \cdot r^\ell e^{-(r/b_n)^2} Y_\ell(\hat{\mathbf{r}})$$

CS-Review : PTP116 (2006) 1

^{5}He : α+n system using discretized states



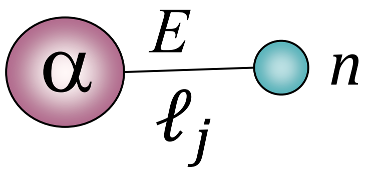


energy eigenvalues

30 Gaussian basis functions

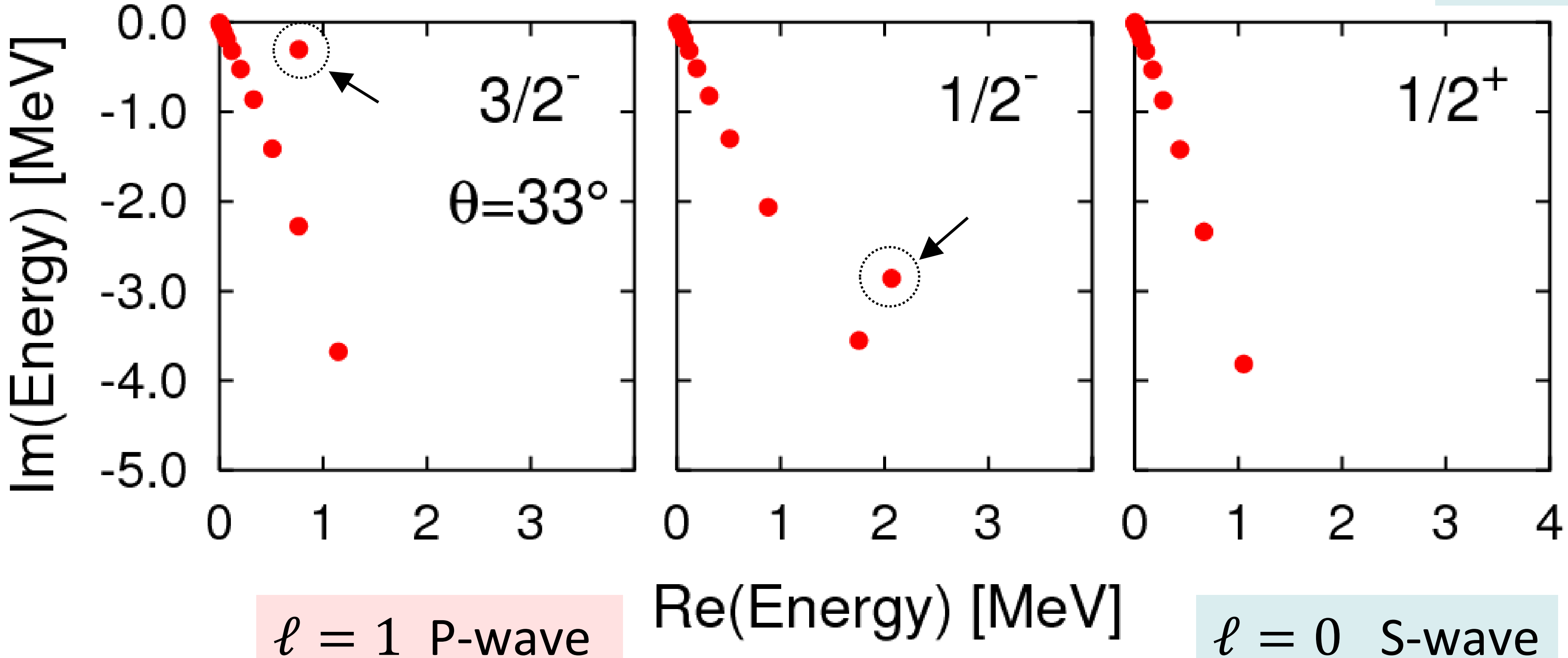
^{5}He : α+n system with discretized continuum states

Energy eigenvalues



$j = \ell \pm \frac{1}{2}$

No resonance



$\ell = 1$ P-wave

$\ell = 0$ S-wave

Matrix elements of resonances

Convergence factor method

To avoid the singularity of Φ_R at $r = \infty$

$$\langle O \rangle = \langle \widetilde{\Phi}_R | O | \Phi_R \rangle = \lim_{\alpha \to 0} \int d\boldsymbol{r}\, \widetilde{\Phi}_R^*\, O\, \Phi_R \cdot e^{-\alpha r^2}$$

Ya.B. Zel'dovich, Sov. Phys. JETP 12 (1961) 542.
N. Hokkyo, Prog. Theor. Phys. 33(1965)1116.

$\boldsymbol{r} \to \boldsymbol{r}e^{i\theta}$ Integration along $re^{i\theta}$

damping condition at $r \to \infty$

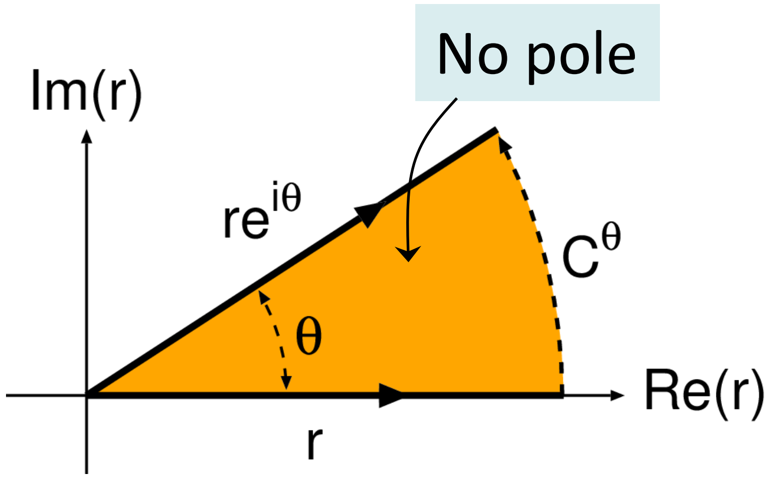
$$\langle O \rangle = \lim_{\alpha \to 0} \int_{C^\theta} d(\boldsymbol{r}e^{i\theta})\, \widetilde{\Phi}_{\mathrm{R}}^*(\boldsymbol{r}e^{i\theta}) O(\boldsymbol{r}e^{i\theta}) \Phi_{\mathrm{R}}(\boldsymbol{r}e^{i\theta}) \cdot e^{-\alpha r^2 e^{2i\theta}}$$

$$= \int_{C^\theta} d(\boldsymbol{r}e^{i\theta})\, \widetilde{\Phi}_{\mathrm{R}}^*(\boldsymbol{r}e^{i\theta}) O(\boldsymbol{r}e^{i\theta}) \Phi_{\mathrm{R}}(\boldsymbol{r}e^{i\theta})$$

$$= \langle U_\theta \widetilde{\Phi}_{\mathrm{R}} | U_\theta O U_\theta^{-1} | U_\theta \Phi_{\mathrm{R}} \rangle$$

$$= \langle \widetilde{\Phi}_{\mathrm{R}}^\theta | O_\theta | \Phi_{\mathrm{R}}^\theta \rangle$$

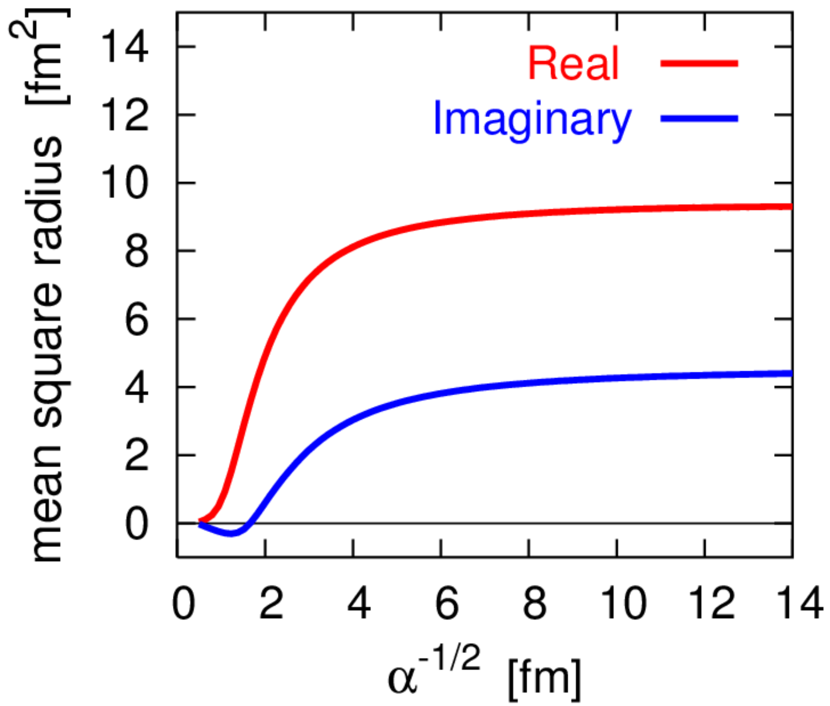
θ-independent



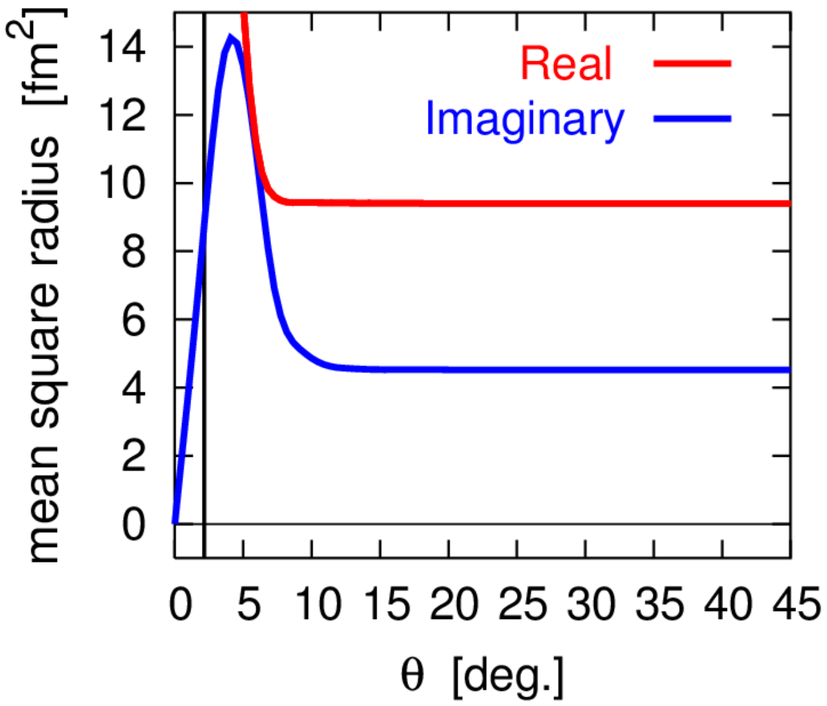
B. Gyarmati and T. Vertse,
Nucl. Phys. A160, 523 (1971).

Matrix elements of resonances

Squared radius with schematic potential



$E(0_3^+) = 1.63 - i0.13$ (MeV)



Convergence factor method ⟷ agree ⟷ Complex scaling method

$$\langle O \rangle = \lim_{\alpha \to 0} \int d\vec{r}\, \widetilde{\Phi}_R^* O \Phi_R \cdot e^{-\alpha r^2}$$

To avoid the singularity of Φ_R at $r = \infty$

Ya.B. Zel'dovich, Sov. Phys. JETP 12 (1961) 542.
N. Hokkyo, Prog. Theor. Phys. 33(1965)1116.
M. Homma, T. Myo, K. Katō, Prog. Theor. Phys. 97 (1997) 561

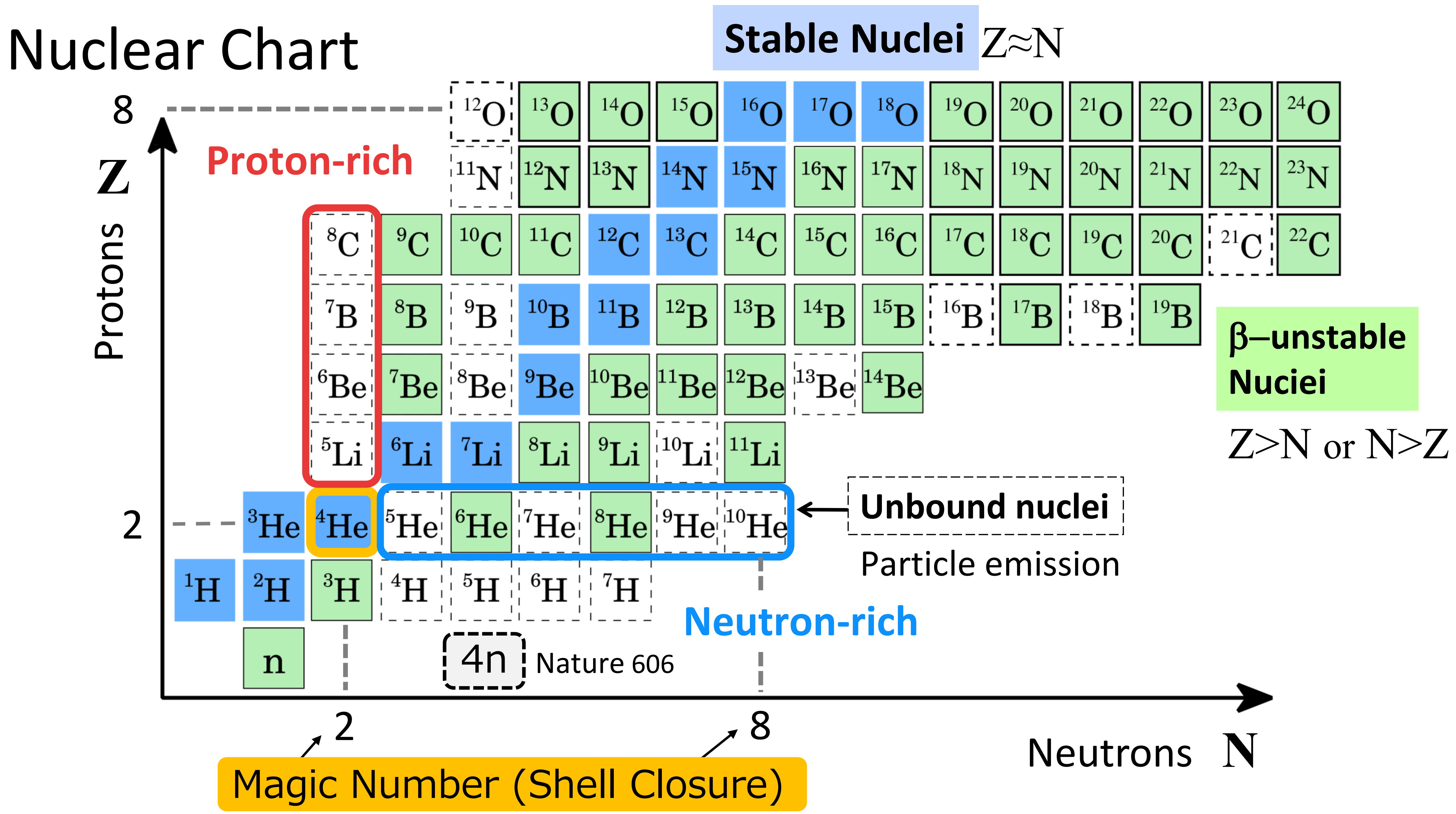
C-product for notation

- $\left\langle\Phi_{\mathrm{AR}}\middle|\hat{O}\middle|\Phi_{\mathrm{R}}\right\rangle = \left\langle\tilde{\Phi}_{\mathrm{R}}\middle|\hat{O}\middle|\Phi_{\mathrm{R}}\right\rangle = \left(\Phi_{\mathrm{R}}\middle|\hat{O}\middle|\Phi_{\mathrm{R}}\right) = \{\text{radial}\} \times \{\text{angular}\}$
- $\{\text{radial}\} \quad = \int_0^\infty \phi_{\mathrm{R}}(r)\hat{O}(r)\phi_{\mathrm{R}}(r)\, r^2 dr$: No-complex conjugate
- $\{\text{angular}\} = \int \phi_{\mathrm{R}}^*(\hat{\boldsymbol{r}})\hat{O}(\hat{\boldsymbol{r}})\phi_{\mathrm{R}}(\hat{\boldsymbol{r}})\, d\hat{\boldsymbol{r}}$: Ordinary definition
- $\left\langle\Phi_{\mathrm{AR}}^{-\theta}\middle|\hat{O}^{\theta}\middle|\Phi_{\mathrm{R}}^{\theta}\right\rangle = \left\langle\tilde{\Phi}_{\mathrm{R}}^{\theta}\middle|\hat{O}^{\theta}\middle|\Phi_{\mathrm{R}}^{\theta}\right\rangle = \left(\Phi_{\mathrm{R}}^{\theta}\middle|\hat{O}^{\theta}\middle|\Phi_{\mathrm{R}}^{\theta}\right)$

Nimrod Moiseyev, Physics Reports 302 (1998) 211-293

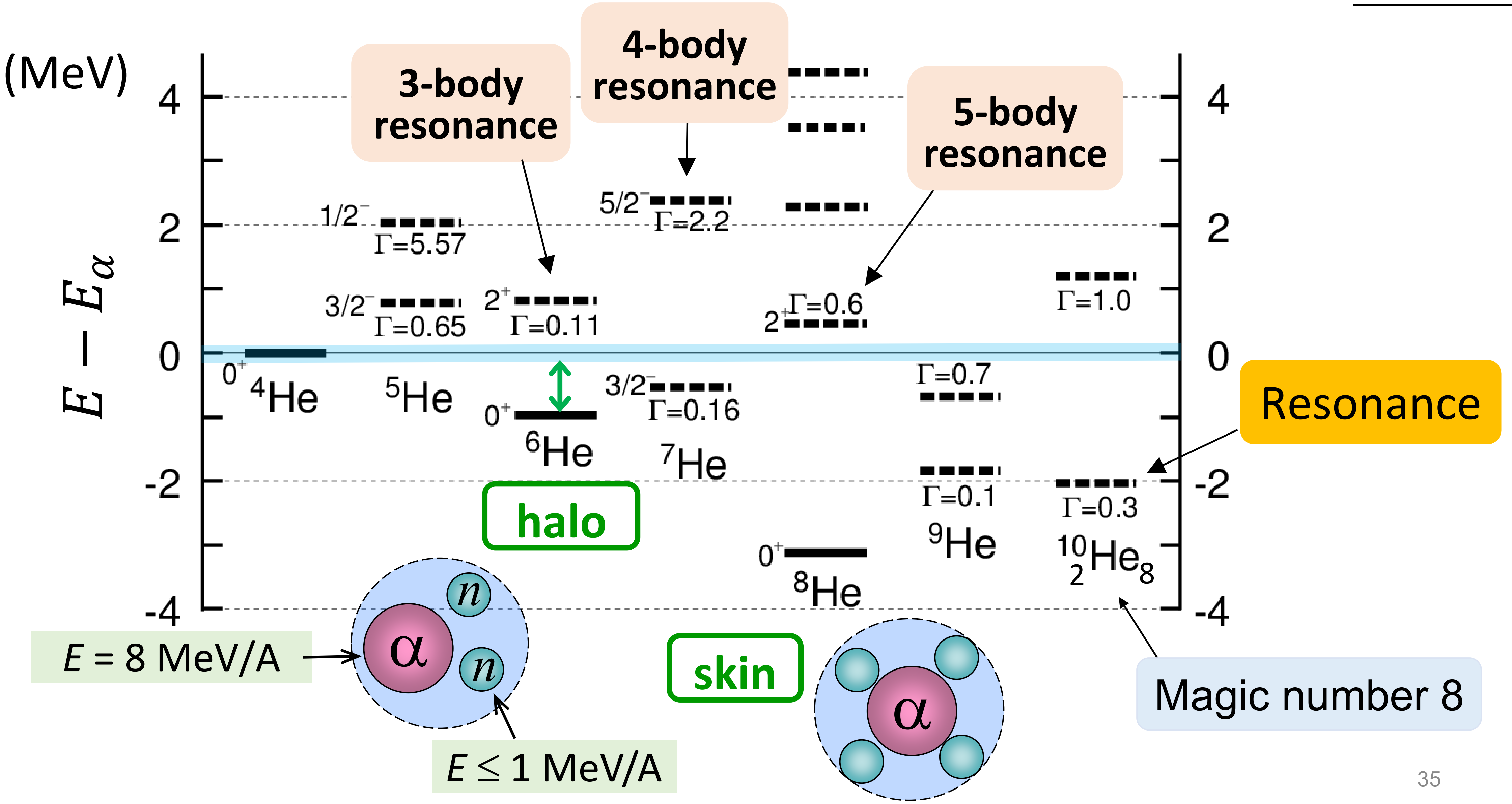
1. Resonance spectroscopy with CSM
2. Scattering states with CSM
   - Complex-scaled Green's function

Nuclear Chart

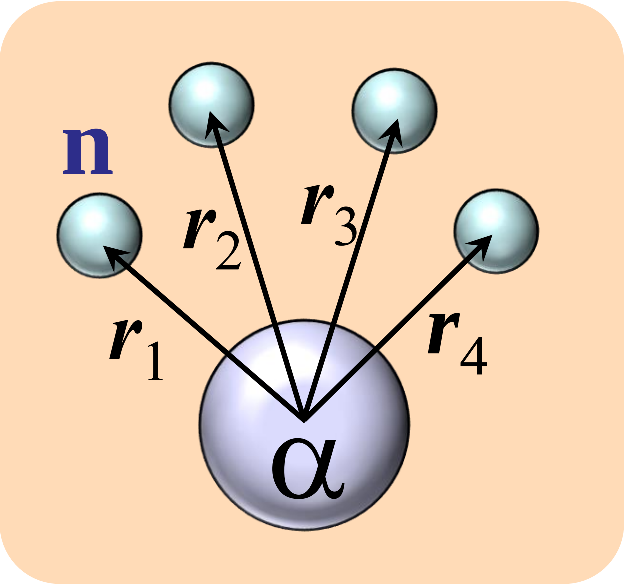


Neutron-rich He isotopes , Experiments

TUNL Nuclear Data Evaluation



Nuclear Model



- Stable core nucleus + valence nucleons
  - ^{8}He = α+n+n+n+n   (5-body problem)
  - core nucleus is changeable such as ^{16}O
- Method
  - Particle coordinates : core nucleus is a center
    - Single-particle picture with configuration mixing (s-wave, p-wave, ….)
  - Relative motion : few-body approach with Gaussian basis expansion
    - Unified description of strong/weak bindings of valence particles
  - Complex Scaling for resonances (Siegert condition) in many-body systems
    - $\boldsymbol{r}_i \to \boldsymbol{r}_i e^{i\theta}$, $\boldsymbol{k}_i \to \boldsymbol{k}_i e^{-i\theta}$ with common scaling angle θ

Hamiltonian & Wave function for N-rich systems

$$H_A = H_{\text{core}} + H_{\text{val.}} + H_{\text{core-val.}}$$

valence neutron number : $N_V = 1, 2, \cdots$

$$\Phi_A = \mathcal{A}\{\Psi_{\text{core}} \cdot \sum_n^N C_n \Psi_{\text{val},n}\}$$

n : configuration

$$\Psi_{\text{val},n} = \mathcal{A}\{\phi_{n1}(\boldsymbol{r}_1)\phi_{n2}(\boldsymbol{r}_2)\phi_{n3}(\boldsymbol{r}_3)\cdots\}$$

single nucleon state, $s_{1/2}, p_{1/2}, p_{3/2}, \cdots$

$$\phi_{n\ell j}(\boldsymbol{r}) = r^\ell e^{-a_n r^2}\left[Y_\ell(\hat{\boldsymbol{r}}), \chi^\sigma_{1/2}\right]_j$$

Range parameter $a_1, a_2, \cdots, a_N$

$$H_A\Phi_A = E\Phi_A \qquad H_{\text{core}}\Psi_{\text{core}} = E_{\text{core}}\Psi_{\text{core}}$$

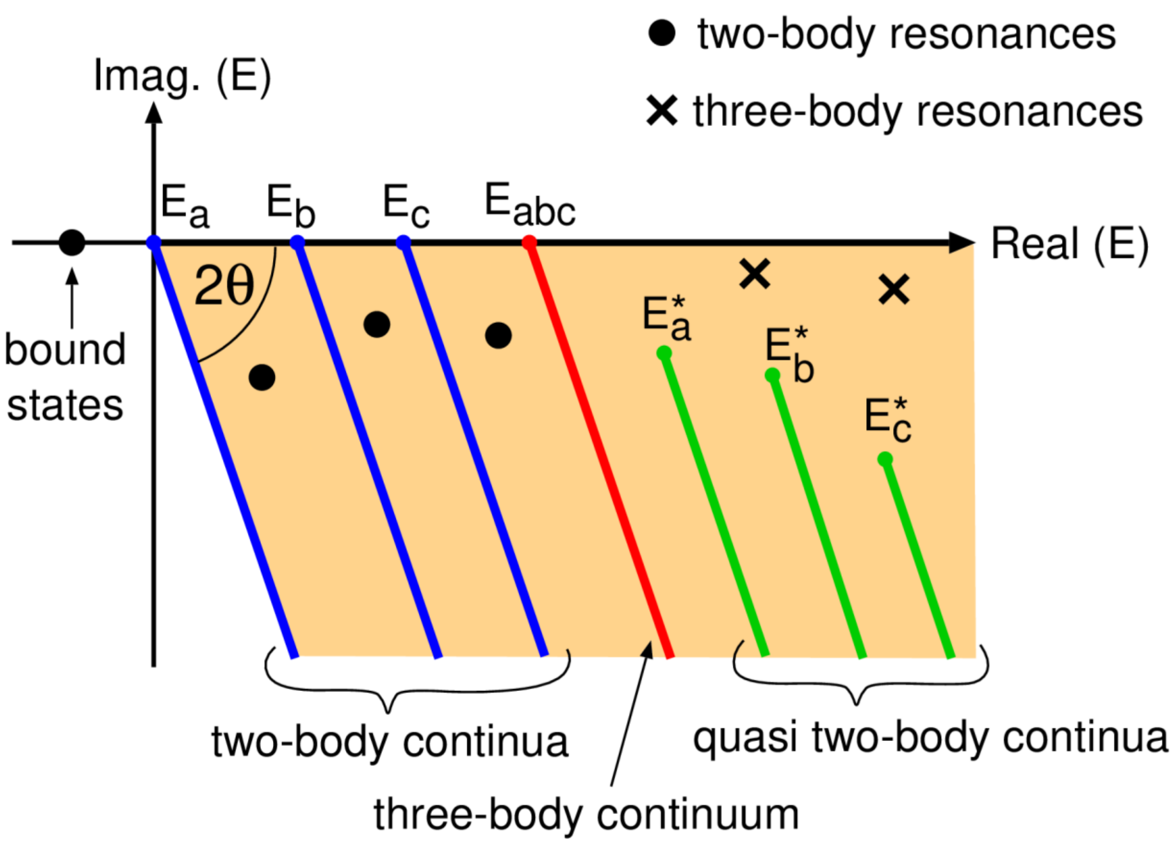
$\delta E = 0$: Solve eigenvalue problem

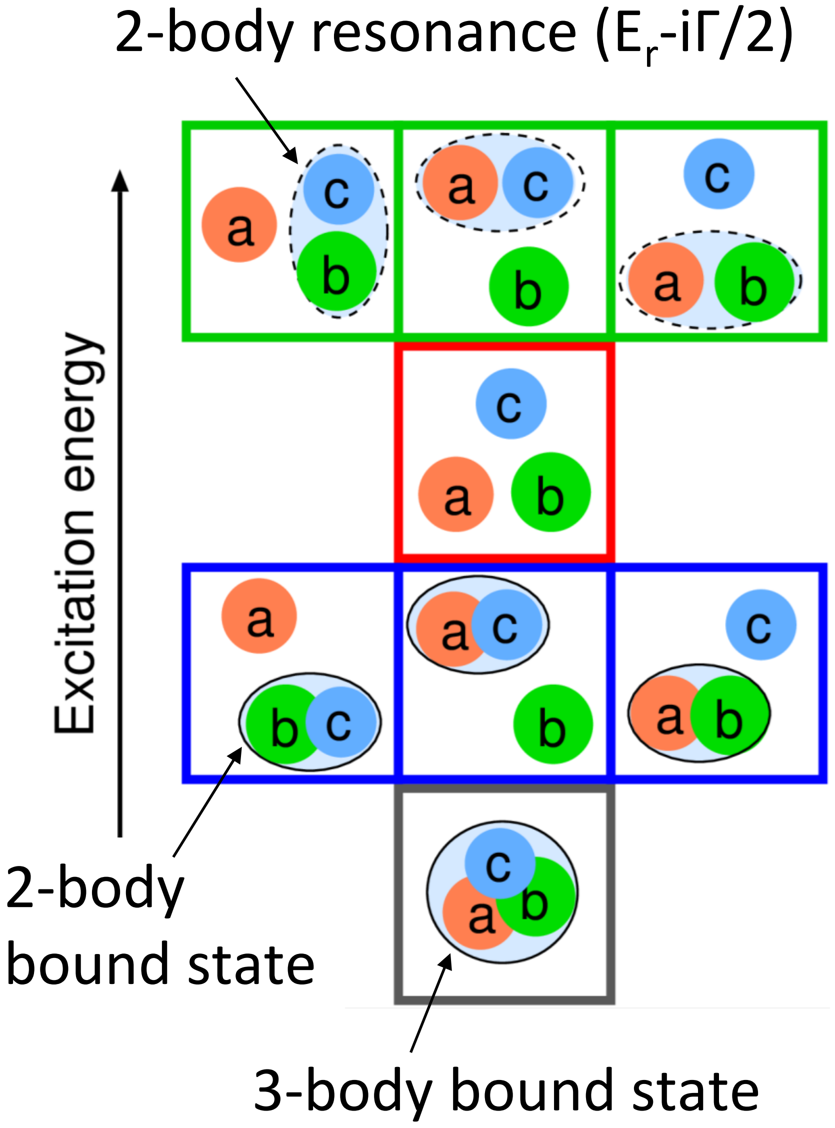
$$\sum_n^N \left\langle \Psi_{\text{val},n'} \middle| \sum_i^{N_V+\text{core}} t_i - T_G + \sum_i^{N_V} V_i^{\text{core}-N} + \sum_i^{N_V} V_{ij}^{NN} - (E - E_{\text{core}}) \middle| \Psi_{\text{val},n} \right\rangle C_n = 0$$

Orthogonality Condition Model

Y. Suzuki, K. Ikeda, PRC38(1988)410
H. Masui, K. Kato, K. Ikeda, PRC73(2006)034318

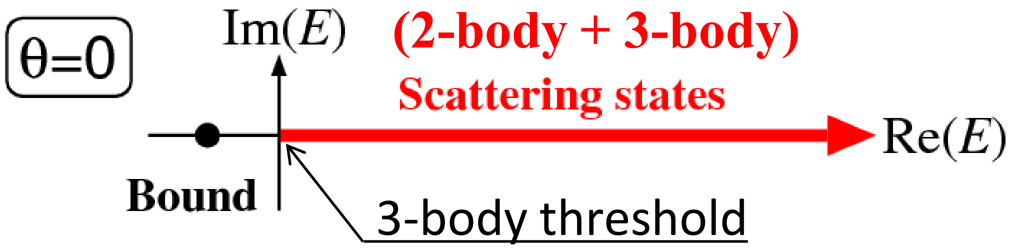
Complex Scaling for 3-body case





Complex Scaling for 3-body Borromean case

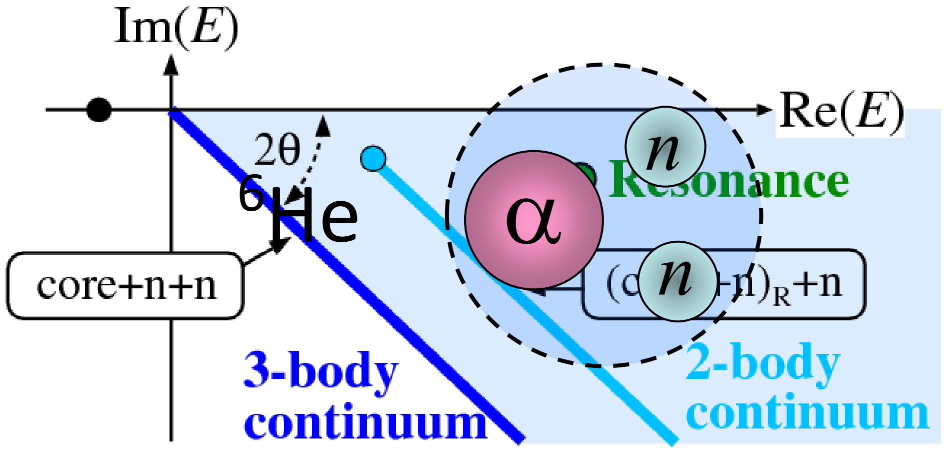
$$U(\theta)\ :\ \mathbf{r}\to\mathbf{r}\cdot\exp(i\theta)\ ,\quad \mathbf{k}\to\mathbf{k}\cdot\exp(-i\theta)\ ,\quad \theta\in \mathrm{R}$$



Completeness relation

$$1=\sum_B |\varphi_B\rangle\langle\tilde{\varphi}_B| + \int_C dE\,|\varphi_E\rangle\langle\tilde{\varphi}_E|$$

T. Berggren, NPA109('68)265.



$$1=\sum_B \ldots$$

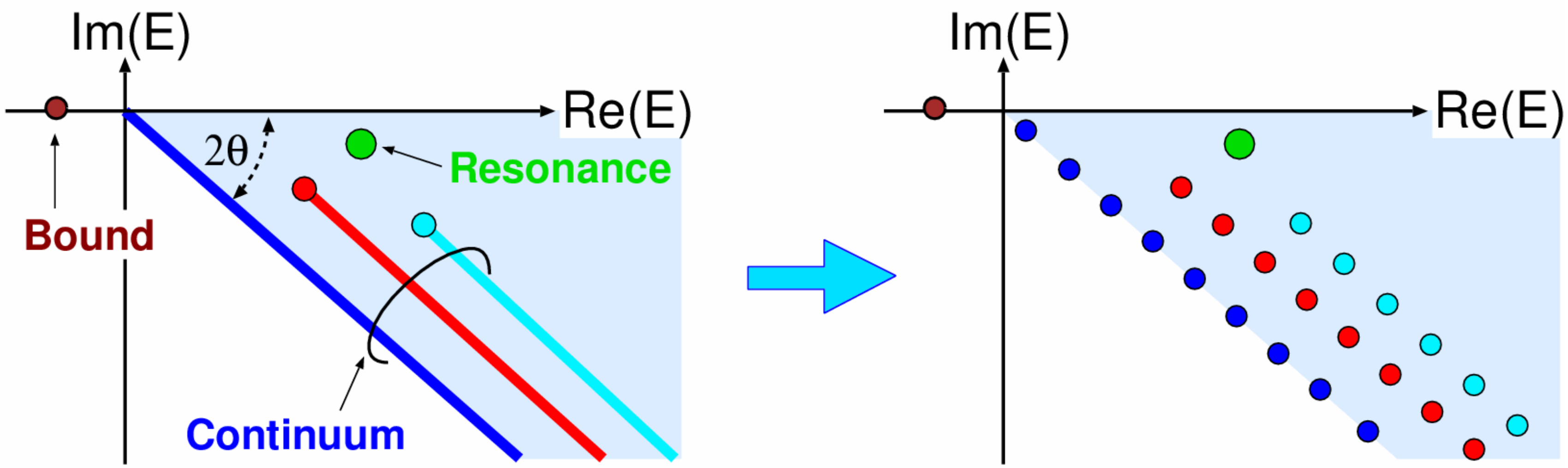
$$+\ \ldots\ |\varphi_{E_\theta}\rangle\langle\tilde{\varphi}_{E_\theta}|$$

Borromean rings

Halo nuclei : "core+n+n" with Borromean condition

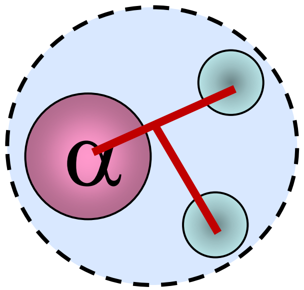
$^6He = {}^4He + n + n$, $^{11}Li = {}^9Li + n + n$, $^{14}Be = {}^{12}Be + n + n$, …

Treatments of unbound states in CSM



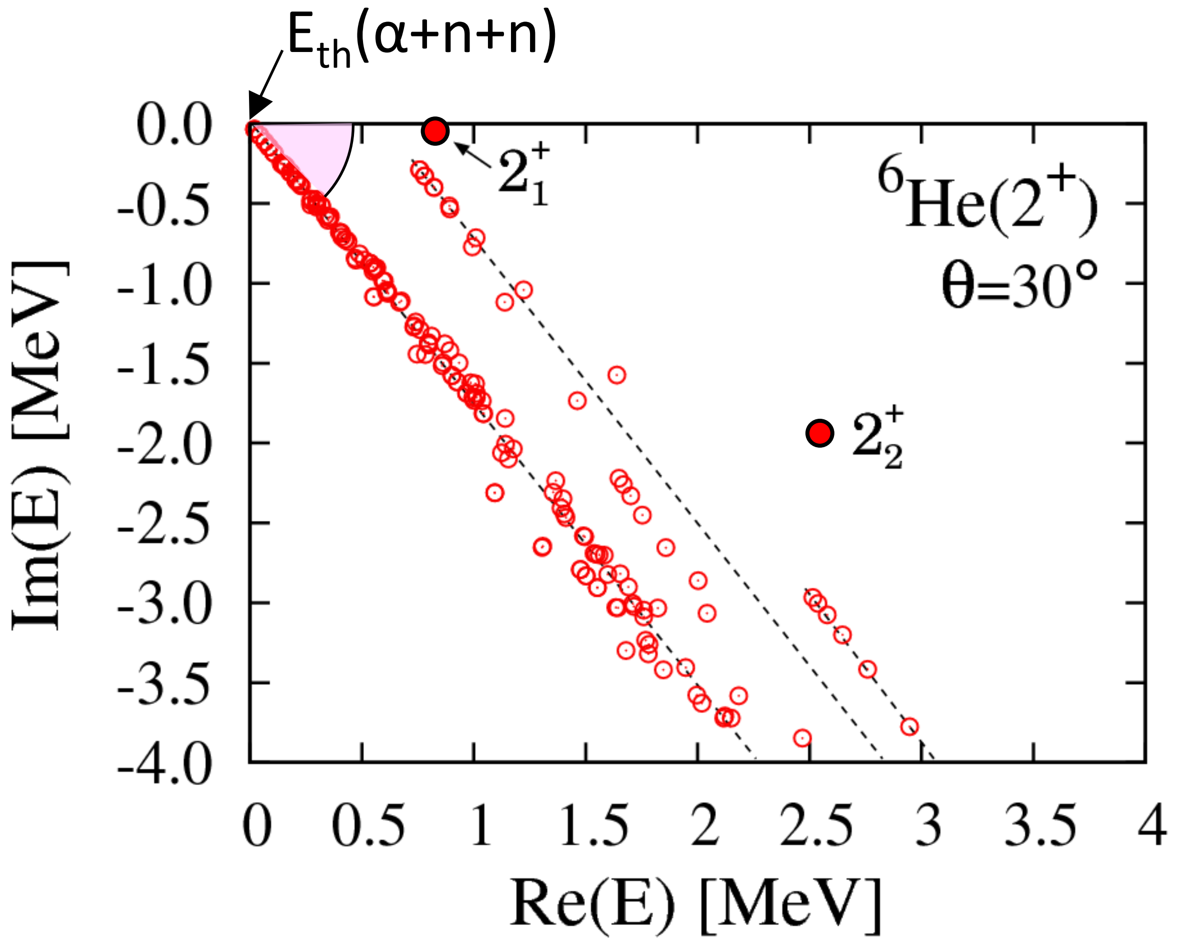
- Each relative motion is solved
  with Gaussian expansion

$$\phi_\ell(\mathbf{r}) = \sum_n^N C_n \cdot r^\ell e^{-(r/b_n)^2} Y_\ell(\hat{\mathbf{r}})$$

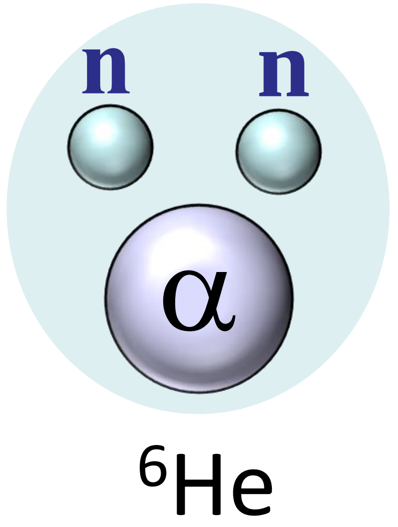


- Exact asymptotic condition for resonances (2-body & 3-body).
- 2- & 3-body continuum states are discretized.

Energy spectrum of ^{6}He with α+n+n model



^{6}He$^{(*)}$
^{5}He+n
^{4}He+n+n



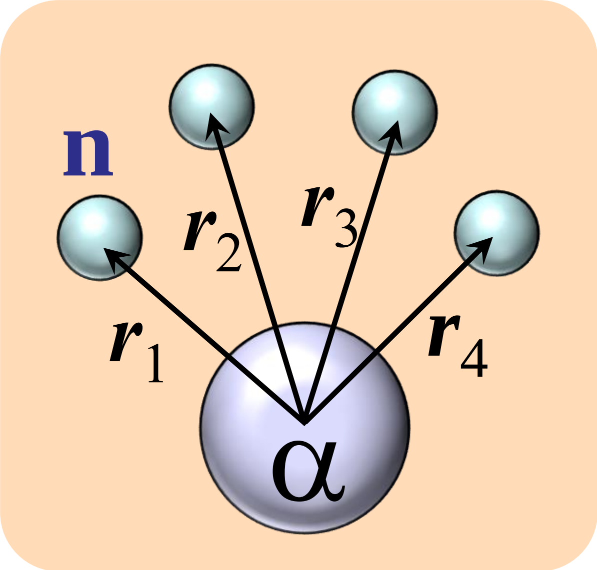
Continuum states are discretized using **Gaussian basis functions**

$$\phi_\ell(\mathbf{r}) = \sum_n C_n \cdot r^\ell e^{-(r/b_n)^2} Y_\ell(\hat{\mathbf{r}})$$

A. Csoto, PRC49 ('94) 3035,
S. Aoyama et al. PTP94('95)343, T. Myo et al. PRC63('01)054313

Hamiltonian for He isotopes

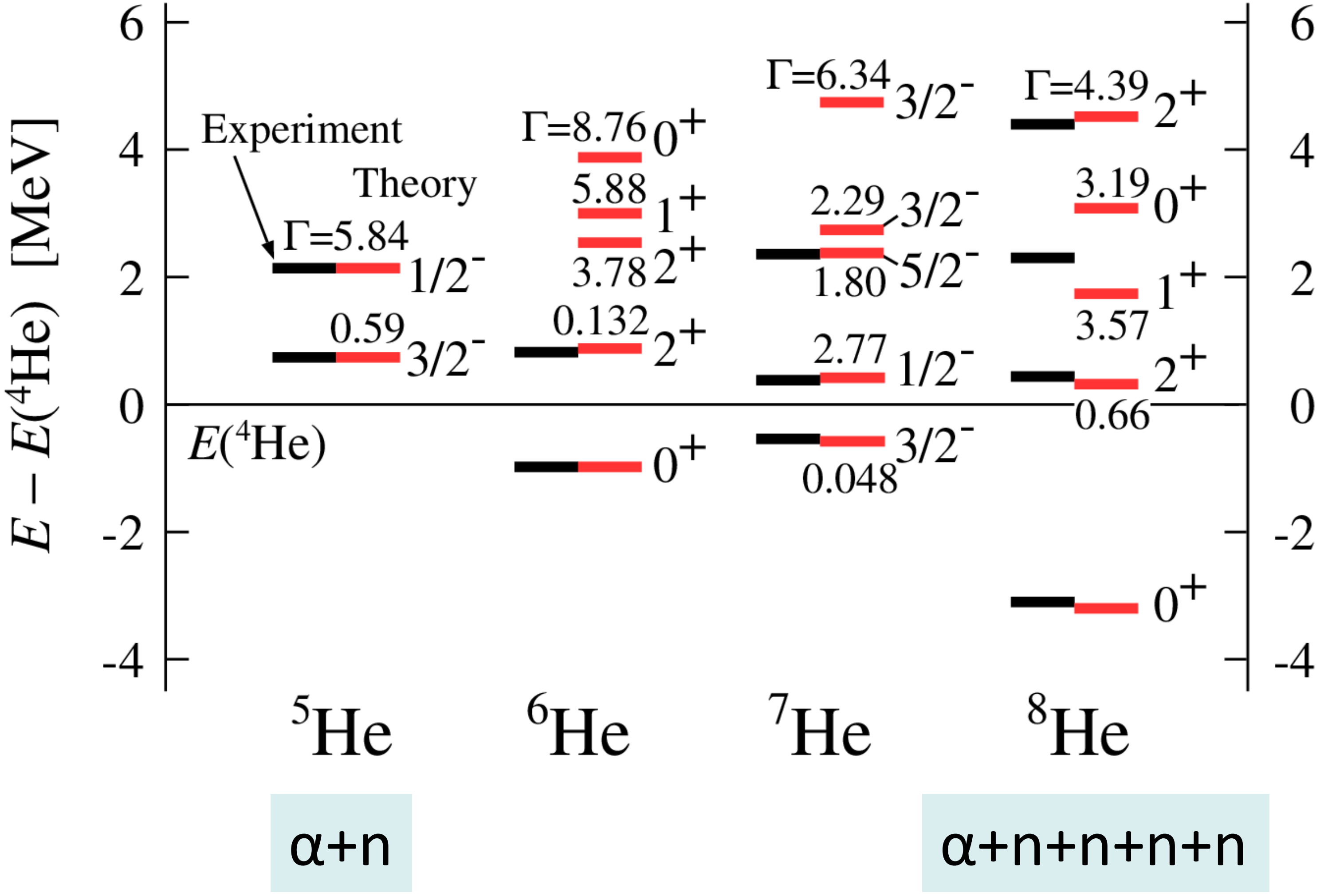
- $V_{\alpha n}$: microscopic KKNN potential
  - s,p,d,f-waves of α-n scattering phase shifts
- V_{nn} : Minnesota central potential
  + Coulomb for *p*-rich nuclei



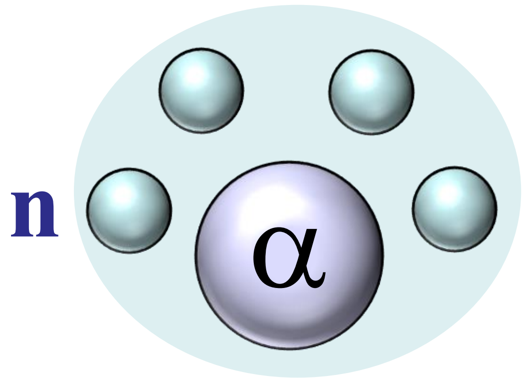
Fit energy of the ground state of ^{6}He

A. Csoto, PRC48(1993)165.
K. Arai, Y. Suzuki and R.G. Lovas, PRC59(1999)1432.
TM, S. Aoyama, K. Kato, K. Ikeda, PRC63(2001)054313.
TM et al. PTP113(2005)763.

He isotopes : Experiment & Theory



α+n+n+n+n

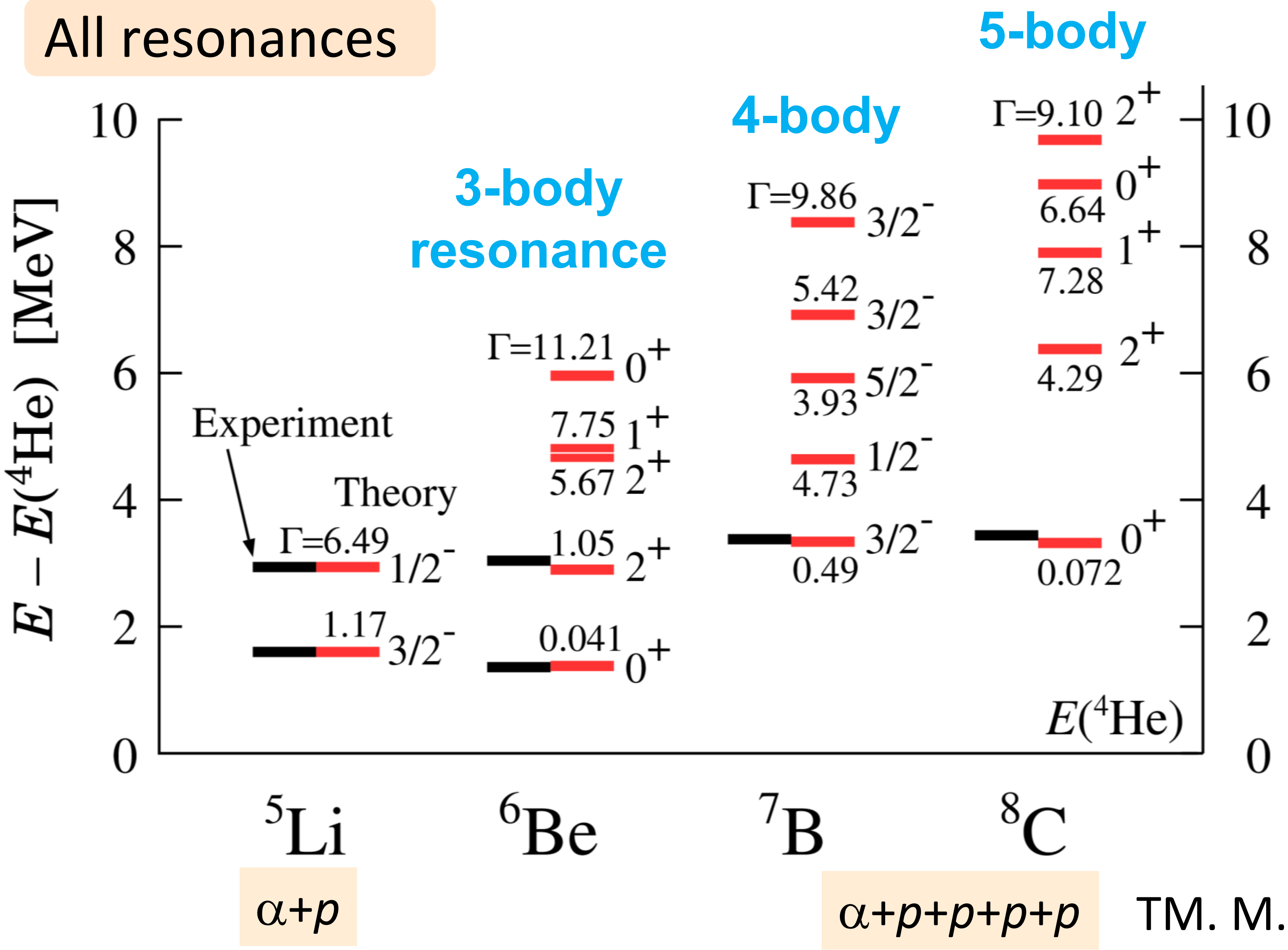


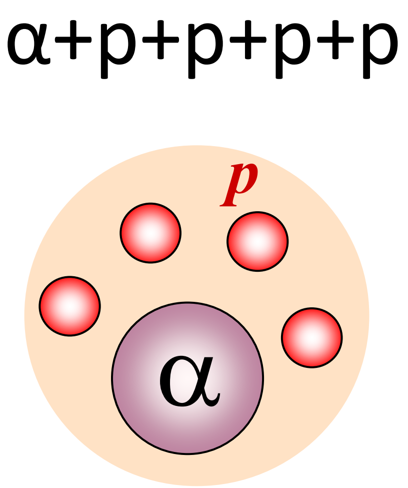
- Solve the motion of valence neutron precisely
- Reproduction & Prediction

TM. M. Odsuren, K. Katō, PRC104 (2021)

Proton-rich : α+p+p+p+p

All resonances

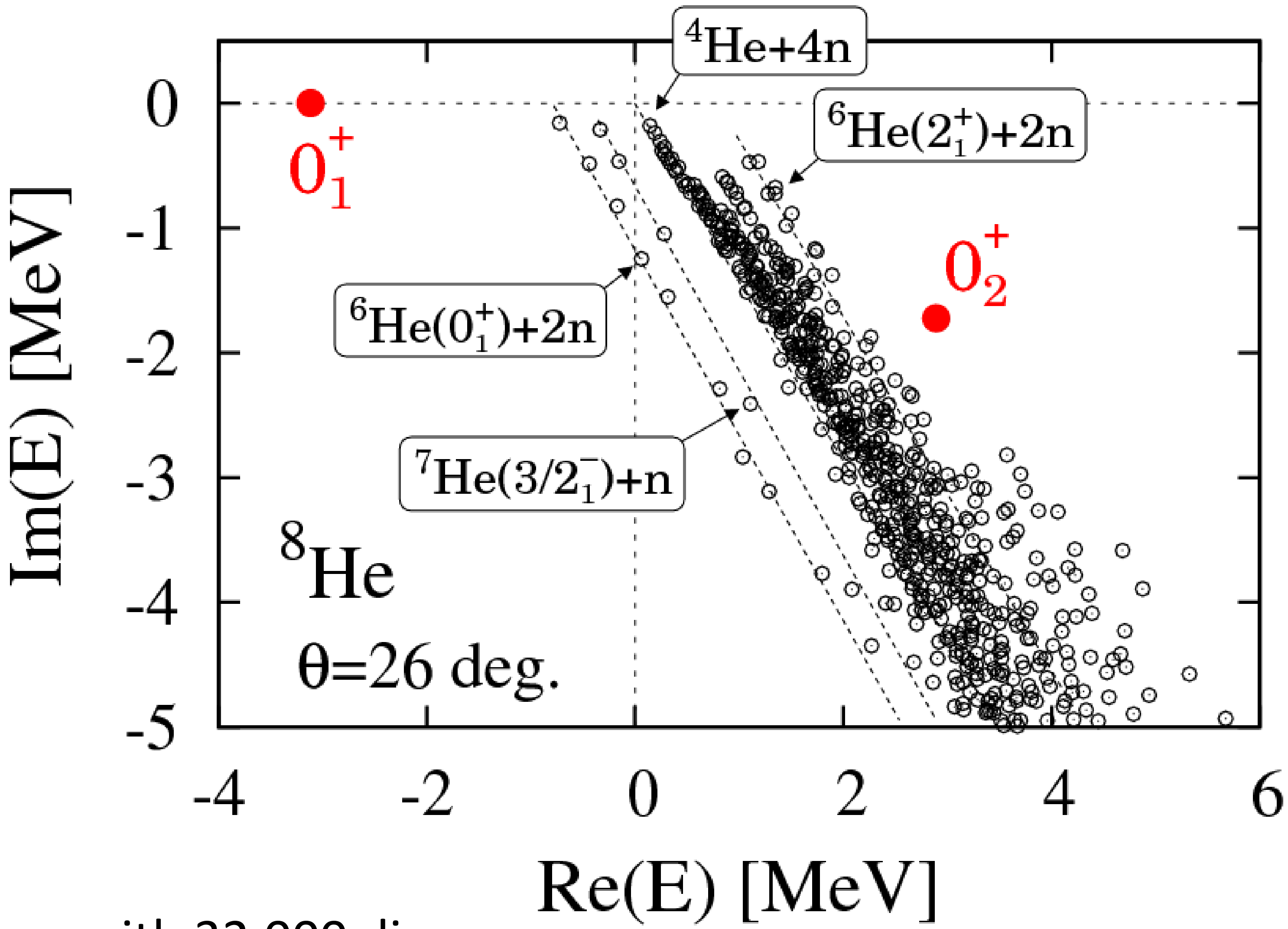




- Coulombic 5-body problem
- Isospin symmetry breaking between n-rich & p-rich

R. J. Charity *et al.*, PRC84 (2011)
TM. M. Odsuren, K. Katō, PRC104 (2021)

Energy of ^{8}He with 5-body complex scaling



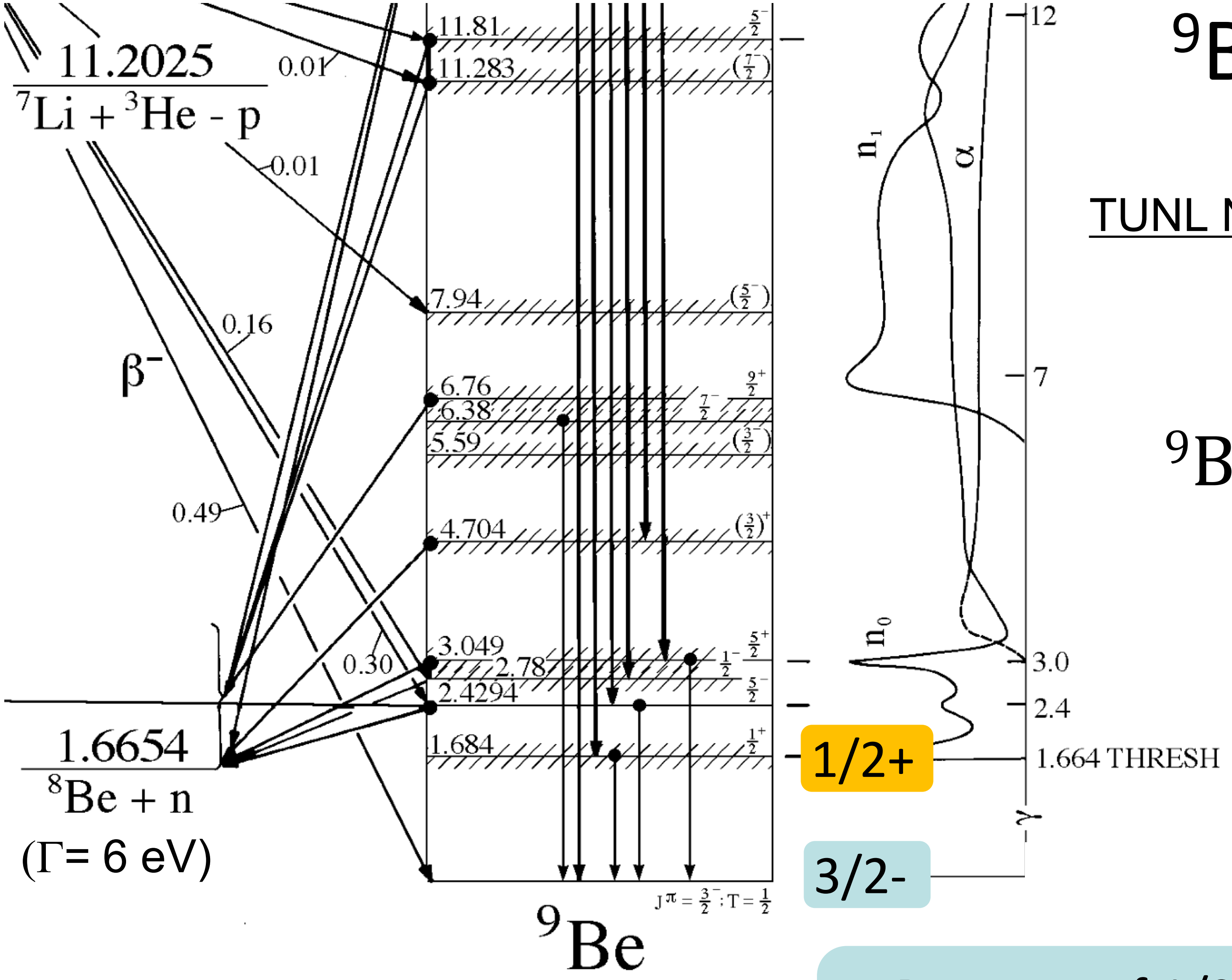
5-body resonance

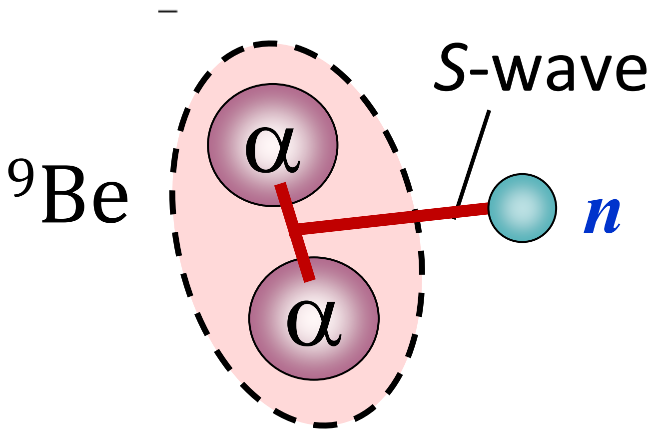
Eigenvalue problem with 32,000 dim.

Full diagonalization of complex matrix @ SX8R of NEC (Osaka Univ.)

^{9}Be photodisintegration

TUNL Nuclear Data evaluation



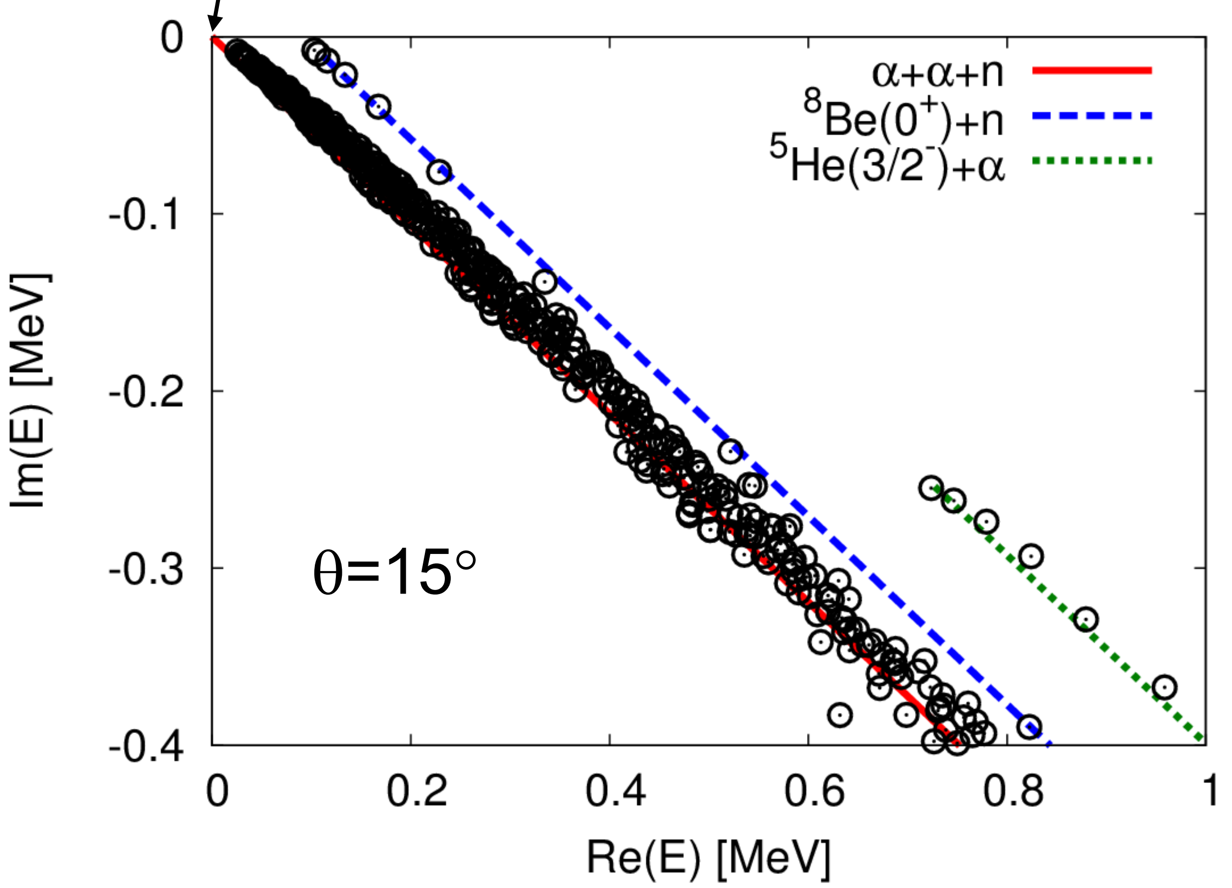


1.5736 (MeV)
^{4}He + ^{4}He + n

- Property of $1/2^+$ S-state - resonance ? / virtual state ?
- Photodisintegration with α+α+n 3-body approach

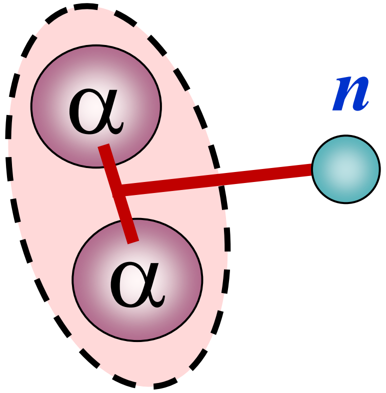
^{9}Be property of $1/2^{+}$ state

α+α+n threshold energy



α+α+n 3-body model (OCM+CSM)

No 3-body resonance



3-body potential
(1 range Gaussian with hyper-radius)

$$H(\,^{9}\mathrm{Be}) = T_R + T_r + V_{\alpha_1 n} + V_{\alpha_2 n} + V_{\alpha\alpha} + V_{\mathrm{PF}} + V_3$$

Odsuren, Kikuchi, Myo, Aikawa, Katō, Physical Review C (2015) 014322

Backup

Matrix Elements

- Radial component : $\phi_{\mathrm{AR}}(r) = \tilde{\phi}_{\mathrm{R}}(r) = \phi_{\mathrm{R}}^*(r), \quad \phi_{\mathrm{AR}}^{-\theta}(r) = \{\phi_{\mathrm{R}}^{\theta}(r)\}^*$
- Angular component : $\tilde{\phi}_{\mathrm{AR}}(\hat{\boldsymbol{r}}) = \phi_{\mathrm{R}}(\hat{\boldsymbol{r}})$
- $\langle \Phi_{\mathrm{AR}} | \hat{O} | \Phi_{\mathrm{R}} \rangle = \int \Phi_{\mathrm{AR}}^*(\boldsymbol{r}) \hat{O}(\boldsymbol{r}) \Phi_{\mathrm{R}}(\boldsymbol{r})\, d\boldsymbol{r}$

$$= \int_0^\infty \phi_{\mathrm{AR}}^*(r) \hat{O}(r) \phi_{\mathrm{R}}(r)\, r^2 dr \times \int \phi_{\mathrm{AR}}^*(\hat{\boldsymbol{r}}) \hat{O}(\hat{\boldsymbol{r}}) \phi_{\mathrm{R}}(\hat{\boldsymbol{r}})\, d\hat{\boldsymbol{r}}$$

$$= \int_0^\infty \phi_{\mathrm{R}}(r) \hat{O}(r) \phi_{\mathrm{R}}(r)\, r^2 dr \times \int \phi_{\mathrm{R}}^*(\hat{\boldsymbol{r}}) \hat{O}(\hat{\boldsymbol{r}}) \phi_{\mathrm{R}}(\hat{\boldsymbol{r}})\, d\hat{\boldsymbol{r}}$$

radial

angular

- $\langle \Phi_{\mathrm{AR}} | \hat{O} | \Phi_{\mathrm{R}} \rangle = \int [\Phi_{\mathrm{AR}}^*(\boldsymbol{r})]^{\theta} \hat{O}^{\theta}(\boldsymbol{r}) \Phi_{\mathrm{R}}^{\theta}(\boldsymbol{r})\, d\boldsymbol{r}$

$$= \int \left[\Phi_{\mathrm{AR}}^{-\theta}(\boldsymbol{r})\right]^* \hat{O}^{\theta}(\boldsymbol{r}) \Phi_{\mathrm{R}}^{\theta}(\boldsymbol{r})\, d\boldsymbol{r} = \langle \Phi_{\mathrm{AR}}^{-\theta} | \hat{O}^{\theta} | \Phi_{\mathrm{R}}^{\theta} \rangle$$

$\theta > 0$

Normalization of resonance w.f.

- Bi-orthogonal states by T. Berggren, NPA109(1968)265.
- Pole : $\tilde{k} = -k^*$, $\tilde{\Psi}^*(\boldsymbol{r}) = \Psi(\boldsymbol{r})$ from *S*-matrix property
- ket-state : $\Psi_R = \sum_{n=1}^{N} C_n \phi_n$, n : channel $(1, \ldots, N)$
- bra-state : $\tilde{\Psi}_R = \sum_{n=1}^{N} \tilde{C}_n \tilde{\phi}_n$ (bi-orthogonal state)
- Norm : $\langle \tilde{\Psi}_R | \Psi_R \rangle = \sum_m^N \sum_n^N \tilde{C}_m^* C_n \langle \tilde{\phi}_m | \phi_n \rangle$

$$= \sum_n^N C_n^2 \qquad \langle \tilde{\phi}_m | \phi_n \rangle = \delta_{m,n} \qquad \tilde{C}_m^* = C_m$$

$$= C_1^2 + C_2^2 + \cdots + C_N^2$$

$$= 1$$

complex

- $\mathbf{Re}(\sum_n^N C_n^2) = 1$, $\mathbf{Im}(\sum_n^N C_n^2) = 0$.

Hermite case

$1 = \sum_n^N |C_n|^2$