NUCLEAR STRUCTURE ASPECT OF DOUBLE BETA DECAY

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The nuclear physics in general and the nuclear structure calculations in particular have played a major role in the understanding of weak and strong interactions and hence, in the development of QFD, QCD and string theories.

Presently, the nuclear structure physics is going to play an important role in extracting the mass of neutrinos from the observed nuclear double beta decay data.

So, our aim is to review the nuclear structure aspect of $\beta\beta$ decay.

The present talk is organized as follows.

1. Introduction
2. Nuclear double $\beta\beta$ decay
3. Nuclear models
4. Nuclear structure aspect of $\beta\beta$ decay
5. Conclusions

1. Introduction

Wolfgang Pauli, postulated the existence of the electron neutrino in 1930 and twenty-five years after in 1955, Cowan and Reines experimentally detected it at Savannah River reactor site.

However the properties of the neutrino are not completely known till today.

The mass and nature of the neutrino- Dirac or Majorana character- play a unique role in modern gauge field theories namely GUTs, SUSYs, SUGRAs etc.

Further, the neutrino is a candidate for many astrophysical and geophysical issues.

Experimental search for the neutrino mass
(1) Terrestrial

- accelerator: π, τ- decay and neutrino oscillations
- non-accelerator: tritium decay, neutrino oscillations (at reactors), neutrino decay (at reactors) and the $\beta\beta$ decay

(2) Extra-terrestrial

- solar neutrinos
- atmospheric neutrinos
- neutrinos from supernovae explosions
- neutrino cosmic background radiation (to be observed)

Evidence for neutrino mass

(a) Cosmological constraints: $m_i$ 25 to 100 eV

(b) SN1987a Constraints: $m_{\tau} < 10$ to 30 eV

(c) Neutrino Oscillation

- atmospheric: $\Delta m^2_{\text{atm}} \sim 2.5 \times 10^{-3}$ eV$^2$
  (Kamiokande, IMB and Super-Kamiokande)

- solar: $\Delta m^2_{\text{sun}} \sim 6.6 \times 10^{-5}$ eV$^2$
  (Chlorine at the Homestake mine, Kamiokande at Kamioka in Japan, Gallex at Gran Sasso in Italy, SAGE at Baksan in Russia and Super-Kamiokande)

- LSND: $\Delta m^2_{\text{LSND}} \geq 0.1$ eV$^2$

- The results from CHOOZ and Palo Verde, K2K, SNO and Kam-LAND are consistent with the above observations.

(d) Tritium beta decay: $m_e < 2.3$ eV (MAINZ)

- However, actual neutrino mass can not be extracted from these data.
On the other hand, the study of tritium single beta decay and $\beta \beta$ decay together can provide sharpest limits on the mass and the nature of the electron neutrino.

Specifically, to distinguish the Dirac or Majorana nature of neutrino, the study of $\beta \beta$ decay is the only possibility.

2. Nuclear Double Beta Decay

The nuclear $\beta \beta$ decay is expected to proceed through two modes

(a) $2\nu \beta \beta$ decay

(i) electron emission $(A, Z) \rightarrow (A, Z+2) + 2e^- + 2\nu_e$
(ii) positron emission $(A, Z) \rightarrow (A, Z-2) + 2e^+ + 2\nu_e$
(iii) electron capture $e^- + (A, Z) \rightarrow (A, Z-2) + e^+ + 2\nu_e$
(iv) double electron capture $2e^- + (A, Z) \rightarrow (A, Z-2) + 2\nu_e$

(b) $0\nu \beta \beta$ decay

(i) $(A, Z) \rightarrow (A, Z+2) + 2e^-$
(ii) $(A, Z) \rightarrow (A, Z-2) + 2e^+$
(iii) $e^- + (A, Z) \rightarrow (A, Z-2) + e^+$
(iv) $2e^- + (A, Z) \rightarrow (A, Z-2)^*$

In this talk, we will restrict ourselves to electron emitting $\beta \beta$ decay process only and the word $\beta \beta$ decay will loosely signify to processes given by (i) for both $2\nu \beta \beta$ decay and $0\nu \beta \beta$ decay modes.

The present far reaching implications associated with the study of nuclear $\beta \beta$ decay was pointed out by Doi et al.

The $0\nu \beta \beta$ decay is a convenient tool to test the following important ramifications namely

- lepton number conservation
the mass and charge conjugation properties of $\nu_e$
possible right handed admixtures in the weak leptonic current
vis-à-vis constraints on parameters of various gauge theoretical models beyond SM

2.1 2ν $\beta\beta$ decay

The half-life of the 2ν $\beta\beta$ decay for $0^+ \rightarrow 0^+$ transition in 2n mechanism is given by

$$\left[T^ {2\nu}_{\beta\beta}(0^+ \rightarrow 0^+)\right]^{-1} = G_{2\nu} |R_{2\nu}|^2$$

where the integrated kinematical factor $G_{2\nu}$ can be calculated with good accuracy and the response function $R_{2\nu}$ is defined as

$$R_{2\nu} = \left|M_{2\nu} - \left(\frac{g^2_\nu}{g^2_A}\right)M_F\right|^2$$

The nuclear transition matrix element (NTME) $M_{2\nu}$ is given by

$$M_{2\nu} = \sum_N \frac{\langle 0^+_F | \sigma^+ \tau^+ 1^+_N \rangle \langle 1^+_F | \sigma^+ \tau^+ 0^+_N \rangle}{E_N - (E_J + E_F)/2}$$

In case the $E_N$ of Eq. (3) is replaced by an average $\langle E_N \rangle$, the summation over intermediate states can be completed using the closure approximation and one obtains

$$M_{2\nu} = \frac{2M^{2\nu}_{GT}}{\langle E_N \rangle - (E_J + E_F)/2}$$

where the DGT $M^{2\nu}_{GT}$ is defined as follows

$$M^{2\nu}_{GT} = \frac{1}{2} \left\langle 0^+ \left| \sum_{i,m} \sigma_i \sigma^+_m \tau^+_i \tau^+_m \right| 0^+ \right\rangle$$
• The 2ν ββ decay has been already observed for 10 nuclei out of possible 35 naturally occurring ββ emitters, and for the rest of the isotopes lower limits have been set. The observed half lives of 2ν ββ decay is $10^{19}$ to $10^{24}$ yr. Hence, the values of $M_{2\nu}$ can be extracted directly.

• Consequently, the validity of different models employed for nuclear structure calculations can be tested by calculating the $M_{2\nu}$.

• The $M_{2\nu}$ extracted from the experimental data is of the order of 0.1 to 0.01 (in units of $m_e^{-1}$), which is smaller than the single particle values by 1 to 2 orders of magnitude.

• The main motive of all the theoretical calculations is to understand the physical mechanism responsible for the suppression of the $M_{2\nu}$.

• The nuclear response $R_{2\nu}$ is very sensitive to nuclear spin-isospin interaction as well as spin-isospin correlations.

2.2 0ν ββ decay

In case of 0ν ββ decay, we restrict ourselves to mass mechanism only. The half life $T_{0\nu}^{1/2}$ of the 0ν ββ decay in 2n mechanism for $0^+ \rightarrow 0^+$ transition is given by the following relations.

(a) Exchange of light neutrinos
\[
[T^{(0^+ \rightarrow 0^+)}_{\nu}]^{-1} = \left( \frac{m_\nu}{m_e} \right)^2 C_{mm} = \left( \frac{m_\nu}{m_e} \right)^2 G_{01} R_{0v}^L
\]  \hfill (6)

where
\[
\langle m_\nu \rangle = \sum_i m_i U_{ei}^2
\]  \hfill (7)

The nuclear factor-of-merit \( C_{mm} \) is a product of the acceptance \( G_{01} \) and intrinsic sensitivity \( R_{0v}^L \) given by

\[
R_{0v}^L = (M_{GT}^{(0v)} - M_F^{(0v)})^2
\]  \hfill (8)

The required NTMEs are given by

\[
M_{GT}^{(0v)} = \sigma_1 \sigma_2 H_m(r)
\]  \hfill (9)

\[
M_F^{(0v)} = \left( \frac{g_Y}{g_A} \right)^2 H_m(r)
\]  \hfill (10)

The neutrino potential \( H_m(r) \) is given by

\[
H_m(r) = \frac{4\pi R}{(2\pi)^3} \int \frac{d^3q}{\omega} \frac{e^{iqr}}{(\omega + \overline{A})}
\]

\[
= \frac{R}{r} \phi(\overline{A}r)
\]  \hfill (11)

Here \( \omega = (q^2 + m^2)^{1/2} \) is the neutrino energy and the neutrino mass has been neglected in comparison with the typical neutrino momentum \( q \approx 200m_e \). Further

\[
\overline{A} = \langle E_N \rangle - \frac{1}{2}(E_i + E_f)
\]  \hfill (12)

(b) Exchange of heavy neutrinos
\[
\left[ T_{\beta\beta}^{\nu}(0^+ \rightarrow 0^+) \right]^{-1} = \left( \frac{m_p}{\langle M_N \rangle} \right)^2 G_{01} R_{\nu}^H
\]  \hspace{1cm} (13)

where
\[
\langle M_N \rangle^{-1} = \sum_i m_i U_i^2
\]  \hspace{1cm} (14)

and
\[
R_{\nu}^H = \left( M^{(\nu)}_{GTh} - M^{(\nu)}_{Fh} \right)^2
\]  \hspace{1cm} (15)

The nuclear transition matrix elements \( M^{(\nu)}_{GTh} \) and \( M^{(\nu)}_{Fh} \) are defined as
\[
M^{(\nu)}_{Fh} = 4\pi \left( m_p m_e \right)^{-1} \left( \frac{g_V}{g_A} \right)^2 U_0 (r, \Lambda)
\]  \hspace{1cm} (16)
\[
M^{(\nu)}_{GTh} = 4\pi \left( m_p m_e \right)^{-1} \sigma \cdot \sigma U_0 (r, \Lambda)
\]  \hspace{1cm} (17)

The neutrino potential \( U_0 (r, \Lambda) \) is defined as
\[
U_0 (r, \Lambda) = \frac{R}{(2\pi)^3} \int dk e^{i \mathbf{k} \cdot \mathbf{r}} \left( \frac{\Lambda^2}{\Lambda^2 + k^2} \right)^4
\]
\[
= \frac{RA^3}{64\pi} e^{-\Lambda r} \left[ 1 + \frac{\Lambda r + \frac{1}{3} (\Lambda r)^2}{1 + \Lambda r + \frac{1}{3} (\Lambda r)^2} \right]
\]  \hspace{1cm} (18)

with \( \Lambda = 850 \) MeV and is short ranged. Hence the finite size and short range correlation effects have to be taken care of properly.

- The aim of all the present activities is to observe \( 0\nu \beta\beta \) decay.
- The observed half-lives of \( 0\nu \beta\beta \) decay may be of the order of \( 10^{25} \) yr.
- The \( 0\nu \beta\beta \) decay is a weak process associated with nucleon, meson and isobar currents. Usually the \( 2n \) mechanism is the most dominant one. The mesons and isobars may be significant if the matrix element of \( 2n \) mode is small.
- The nuclear response \( R^L \) as well as \( R^H \) for \( 0\nu \beta\beta \) decay is not only sensitive to nuclear spin-isospin interaction but also momentum dependent. The nuclear responses \( R_{2\nu} \), \( R^L \) and \( R^H \) probe the nucleus at different energy scale.
- In case of \( 0\nu \beta\beta \) decay, experimental nuclear responses have not been measured so far.
• In case of 0ν ββ decay, the models predict the half-lives assuming certain value for the neutrino mass or conversely extract various parameters from the observed limits of the half-lives of 0ν ββ decay.
• The reliability of predictions can be judged a priori only from the success of the nuclear model in explaining various observed physical properties of nuclei.
• The common practice is to calculate the $M_{2n}$ to start with and compare with the experimentally observed value as the two decay modes involve the same set of initial and final nuclear wave functions.

3. Nuclear Models

Over the past few years, several nuclear models have been employed to calculate the 2ν ββ decay transition matrix elements in two-nucleon (2n) mechanism.
• The $M_{2ν}$ is calculated mainly in three types of models.
  ➢ Shell model and its variants
  ➢ QRPA and extensions their of
• Alternative models
• In the below for the sake of completeness, we briefly discuss these nuclear models to clarify the approximations involved in the structure calculations and to estimate the uncertainties associated with them if required.

Four main assumptions of conventional nuclear structure calculations
• The nucleus can be treated as a quantum mechanical many-body system.
• The nucleus consists of nucleons only, which are treated as elementary - infinitely hard and structure less – particles.
• The velocity of a nucleon inside the nucleus is small enough so that the nucleus can be treated as a non-relativistic quantum mechanical many-body system.
The effects of three or many nucleon forces are negligible.

In addition, two other problems remain open.

• the form nucleon-nucleon (NN) interaction
• a numerically solvable many-body theory
• Main philosophy for the former is to analyze the two-nucleon data i.e.
  ➢ phase-shifts obtained from NN-scattering experiments and
  ➢ deuteron ground state properties
• Solution of the second problem is through a perturbative approach with accompanying problems of convergence.

3.1 NN Potential

• The exact form of the nucleon-nucleon interaction is still unknown in spite of the fact that in the last sixty years, maximum number of scientific man-hours have been spent in the study of NN interaction.
• In the light of QCD, the failure can be attributed to the fact that what we are looking for is a residual interaction and the attempt is something like to study the Coulomb interaction from the collision of atoms.
• The past attempts to study the NN interaction can be broadly classified under the following categories.

(1) Phenomenological potential
(2) Meson exchange potentials
  (a) OBEP model – Bonn Potential
  (b) Dispersion theoretical calculations- Paris potential
(3) The QCD motivated NN potential
(4) The Skyrmion model

• intermediate check by calculating binding energy and saturation density of nuclear matter.
3.2 Perturbation Theory

The many-body Schroedinger equation for a nuclear system is written as

\[ H |\psi_n\rangle = E_n |\psi_n\rangle \]  

(19)

where

\[ H = H_0 + H_1 \]  

(20)

and

\[ H_0 = \sum_{i=1}^{A} T_i + \sum_{i=1}^{A} U_i \]  

(21)

\[ H_1 = \frac{1}{2} \sum_{ij=1}^{A} V_{ij} - \sum_{i=1}^{A} U_i \]  

(22)

In the standard perturbative approach, the Schroedinger equation is cast into the following form

\[ \sum_{k=1}^{\infty} H_{nk} a_{nk} = E_n a_{nl} \]  

(23)

where

\[ H_{nk} = E_k \delta_{nk} + \langle \Phi_i | H_i | \Phi_k \rangle \]  

(24)

The solution of the above Eq. (24) is obtained from

\[
\begin{vmatrix}
H_{11} - E_n & H_{12} & \cdots & H_{1k} \\
H_{21} & H_{22} - E_n & \cdots & H_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
H_{k1} & H_{k2} & \cdots & H_{kk} - E_n
\end{vmatrix} = 0
\]  

(25)

The many-body Schroedinger equation is not exactly solvable as the complete set of basis states span an infinite dimensional Hilbert space.

3.3 Shell Model

• In the Shell model, one attempts to solve the many-body Schroedinger equation as exactly as possible.
Hence, some sort of basis truncation is unavoidable.
The full Hilbert space is divided into a model space $M$ and an excluded space $N$ by the use of projection operators $P$ and $Q$ respectively.
These projection operators $P$ and $Q$ are defined as

$$ P = \sum_{i \in M} |\phi_i\rangle \langle \phi_i| $$

$$ Q = \sum_{i \in N} |\phi_i\rangle \langle \phi_i| $$

satisfy the relations

$$ P + Q = 1, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 $$

$$ [P, H_0] = 0, \quad [Q, H_0] = 0 $$

(28)

The eigenvalue problem in the model space $M$ can be written as

$$ PH_{\text{eff}} P |\phi\rangle = E P |\phi\rangle $$

(29)

where

$$ H_{\text{eff}} = H_0 + V_{\text{eff}} $$

(30)

and

$$ V_{\text{eff}} = H_1 + H_1 \frac{Q}{E - H_0 - QH_1 Q} V_{\text{eff}} $$

(31)

Finally the core energy is removed by the use of Brandow’s linked cluster diagram expansion

$$ V_{\text{eff}} = H_1 + H_1 \frac{Q}{E - H_0 - H_{0V}} V_{\text{eff}} $$

(32)

and the energy dependence of the $V_{\text{eff}}$ can be eliminated by Kuo’s folded diagram expansion.

The additional price for this step is that an effective operator $O_{\text{eff}}$ is to be defined such that the same observable is produced in a finite dimensional model space $M$ as reproduced by a bare operator $O$ acting in an infinite dimensional Hilbert space.

This can be formulated mathematically as
The “realistic interactions” derived from the above theoretical procedure are not quite successful in reproducing spectroscopic properties of nuclei.

Hence, “empirical effective interactions” and “schematic effective interactions” are used quite often in reproducing the observed spectroscopic data.

3.4 PHFB Model

The essential idea behind the HFB theory is to transform particle coordinates to quasiparticle coordinates through general Bogoliubov transformation such that the quasiparticles are relatively weakly interacting.

Essentially, the Hamiltonian $H$ is expressed as

$$H = E_0 + H_{qp} + H_{qp-int}$$

(34)

where $E_0$ is the quasiparticle vacuum, $H_{qp}$ is the elementary quasiparticle excitations and $H_{qp-int}$ is a weak interaction between the quasiparticles.

In HFB theory, the interaction between the quasiparticles is usually neglected and the Hamiltonian $H$ is approximated by an independent quasiparticle Hamiltonian.

In time dependent HFB (TDHFB) or the quasiparticle random phase approximation (QRPA), some effects of quasiparticle interaction can be included

The ground state energy $E_{HFB}$ is given by

$$E_{HFB} = \sum_{k=1}^{n} (T_{kk} + \lambda - E_k) v_k^2$$

(35)

The quadrupole moment of the axially symmetric HFB intrinsic state is given by
\[ Q_{HF} = \sum_{k=1}^{n} Q_0^2 \nu_k^2 \]  
\[ where \quad Q_0^2 = \left( \frac{16\pi}{5} \right) r^2 y_0^2 \]  

- The states \( |\psi^{J'}_K \rangle \) of good angular momentum \( J \) are obtained from \( |\Phi_K \rangle \) through the projection technique given by
  \[ |\psi_{M}^{J'} \rangle = \frac{2J + 1}{8\pi a_J} \int d\Omega \Omega D_{MK}^{J'}(\Omega) \hat{R}(\Omega) |\Phi_K \rangle \]  
The energy \( E_J \) of the projected state \( J \) is given by
  \[ E_J = \frac{\langle \Phi_K | H P_{KK}^{J'} | \Phi_K \rangle}{\langle \Phi_K | P_{KK}^{J'} | \Phi_K \rangle} \]

### 3.5 QRPA Model

- The QRPA approach of Baranger was extended by Hableib and Sorensen to include proton-neutron excitations from even-even to odd-odd nuclei for the study of GT \( \beta \)-decays and it has been applied by several workers since then.
- The proton-neutron mode of a nucleus with open shells i.e. QRPA phonons acting on the correlated vacuum are defined by
  \[ Q_{JM}^+(m) = \sum_{pn} \left[ X_{pn}(J^x,m)A^+(pn,JM) - Y_{pn}(J^x,m)a(pn,JM) \right] \]  
The quasiparticle creation operator and the nucleon creation operators are related through the Bogoliubov-Valetin transformation
\[ A^+ (pn, JM) = u_j C_{jm}^+ + v_j \tilde{C}_{jm} \]  

(41)

- The forward and backward going amplitudes \( X_{pn} \) and \( Y_{pn} \) as well as the energy eigenvalue \( \Omega \) are obtained by solving the QRPA equation

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix}
\]  

(42)

- In QRPA, the states representing the excited states of an odd-odd nucleus are obtained by a diagonalization procedure where the corresponding amplitudes are determined from the non-hermitian eigenvalue problem associated with the matrix given by the Eq. (44).

- The sub-matrices A and B are given by the equation

\[
A_{pn,p'n'} = \delta_{pp'} \delta_{nn'}(E_p + E_n) - 2g_{pp'}G(pnp'n',J)(u_p u_n u_p u_n + v_p v_n v_p v_n) \\
- 2g_{ph}F(pnp'n',J)(u_p v_n u_p v_p + v_p u_n v_p u_p)
\]

\[
B_{pn,p'n'} = 2g_{pp'}G(pnp'n',J)(u_p v_n v_p u_n + v_p v_n v_p u_p) \\
- 2g_{ph}F(pnp'n',J)(u_p v_n v_p u_n + v_p u_n v_p u_p)
\]  

(43)

where \( E_p \) and \( E_n \) are the proton and neutron quasiparticle energies and \( G(pnp'n',J) \) and \( F(pnp'n',J) \) are the particle-particle and particle-hole two-body matrix elements.

- The coefficients \( g_{ph} \) and \( g_{pp} \) are overall scaling factors of the two-body matrix elements in the particle-hole and particle-particle channels, respectively.

Usually, the \( g_{ph} \) is fixed by reproducing the excitation energy of the GT giant resonance and \( g_{pp} \) is fixed by \( \beta \) decay or the \( 2\nu \beta \beta \) decay data.

4. Nuclear Structure Aspect

4.1 Brief Review of Existing Calculations
(a) $2\nu\beta\beta$ decay

The nuclear transition matrix element (NTME) $M_{2\nu}$ is given by

$$M_{2\nu} = \sum_N \frac{0^+_r \langle 1^+_N \hat{\sigma} \tau^+ \rangle 1^+_r \langle 1^+_N \hat{\sigma} \tau^+ \rangle 0^+_r}{E_N - (E_i + E_f)/2}$$

(44)

In Eq. (64), the first factor in the numerator represents $\hat{\alpha}^+$ or (n, p) amplitude for the final nucleus and the second factor represents the $\hat{\alpha}^-$ or (p, n) amplitude of the initial nucleus.

- Hence, all the GT amplitudes with their signs for both $\hat{\alpha}^+$ and $\hat{\alpha}^-$ process should be known in principle to evaluate the $2\nu\beta\beta$ decay rate correctly.

- The NTME $M_{2\nu}$ exhausts a small fraction ($10^{-5}$ to $10^{-7}$) of the DGT sum rule. Hence, it is sensitive to details of the nuclear structure.

- Problems:
  - Calculation of $M_{2\nu}$
  - Understanding the suppression mechanism of $M_{2\nu}$

Models Used

(1) Shell model and its variants

- The shell model, which attempts to solve the nuclear many-body problems as exactly as possible, is the best choice for the calculation of the $M_{2\nu}$.

(a) WCSM  
(b) LSSM  
(c) MCSM

In Shell-model

- The quenching of $M_{2\nu}$ can be explained through the configuration mixing or the renormalization of $g_A$ due to nuclear medium effects resulting in $g_V/g_A \approx 1$. 
• The renormalization of Gamow-Teller operator can lead to quenching of about 20% of the $\beta\beta$ decay strength.
• Stringent truncation imposed on number of shell model basis states prevents configurations responsible for the reduction of transition matrix element by destructive interference.
• The omission of spin-orbit partners may also hinder the destructive interference between spin-flip and spin non-flip contributions to $2\nu$ $\beta\beta$ decay matrix elements leading to the suppression of transition matrix element.

(2) QRPA and extensions

• Vogel and Zirnbauer were the first to provide an understanding of the observed suppression of $M_{2\nu}$ in the QRPA model.
• In $\beta\beta$ decay, the initial nucleus decays into the final nucleus through the virtual excitation of all possible states of the intermediate odd-odd nucleus.
• The proton-neutron particle-hole (p-h) or proton-neutron particle-particle (p-p) interaction matrix elements are required to calculate the excited states of the intermediate nucleus.

In QRPA model
• The repulsive p-h and attractive p-p interactions play a crucial role in the concentration of $\beta^-$ strength in the giant GT resonance and the suppression of $\beta^+$ strength and its concentration at low energies.
• The p-p interaction has negligible effect on the strength of the giant GT resonance and was usually neglected.
• It was observed that the quenching of $M_{2\nu}$ can be achieved by a proper inclusion of ground state correlation through the p-p interaction in the $S=1, T=0$ channel and the calculated half lives are in close agreement with all the experimental data.
• The sum over excited states converges very rapidly and a few low lying states exhaust the whole NTME $M_{2\nu}$. 

• Hence it is sufficient to consider the $\beta^-$ and $\beta^+$ strengths of few low lying states and include the rest of contributions in the renormalization of the GT strength.
• The QRPA frequently overestimates the ground state correlations as a result of an increase in the strength of attractive proton-neutron interaction leading to the collapse of QRPA solutions.
• The physical value of this force is usually close to the point at which the QRPA solutions collapse.
• To cure the strong suppression of $M_{\gamma\nu}$ several extensions of QRPA have been proposed.
  ➢ Proton-neutron pairing
  ➢ Full-QRPA
  ➢ RQRPA
  ➢ Full-RQRPA
  ➢ Higher QRPA
  ➢ MCM
  ➢ Particle number projection

(3) Alternative models
  ➢ OEM
  ➢ Broken SU(4) symmetry
  ➢ Pseudo SU(3) symmetry
  ➢ TVRPA
  ➢ SSDH
  ➢ Group theoretical methods
  ➢ PHFB

In SSDH
• The $M_{\gamma\nu}$ is expressed as a product of the successive single GT matrix elements through the single particle-hole state as
\[ M_{2\nu} = \frac{M_{\nu}(S)M_{\nu}(S')}{\Delta S} \] (45)

- The \( M_{2\nu} \) is calculated using the realistic single particle-hole matrix elements from the experimental \( \beta \) decay rates and/or charge exchange hadronic reactions.
- The \( M_{2\nu} \) has been calculated for \(^{100}\)Mo and \(^{116}\)Cd using charge exchange reaction \((^3\text{He}, t)\) and single \( \beta \) decay.
- The agreement between the experimental and theoretical analysis supports SSDH.

(b) \( 0\nu \beta\beta \) decay

Problems:

- **Light Majorana neutrino exchange**: No selection rule on multipoles, role of NN correlations and sensitivity to nuclear models.
- **Heavy Majorana neutrino exchange**: Physics of NN states at short distances.

**Light Majorana neutrino exchange**

In the \( 0\nu \beta\beta \) decay, both the shell model and QRPA calculations show the following features.

- Momentum of exchanged neutrino – 50 to 100 MeV for internucleon distance 2 to 4 fm. Hence, the closure approximation is valid as dependence on energy is weak.
- As \( qR > 1 \), the expansion in multipoles is not convergent.
- In \( 2\nu \beta\beta \) decay, real neutrinos are emitted in s-wave. However in \( 0\nu \beta\beta \) decay, large angular momenta upto \( 5\hbar \) are involved.
- All multipoles have comparable contribution.
- NTMEs are less sensitive to nuclear structure.

In QRPA model, the following features are observed.
The NTMEs are less sensitive to $g_{pp}$ than the $2\nu \beta\beta$ decay as the contribution of the $1^+$ multipole is relatively small.

The short ranged NN repulsion has to be included in spite of the fact that the neutrino potential is long ranged. Spread in calculated matrix elements of $^{76}\text{Ge}$ is a factor of 6-7 resulting in an uncertainty by a factor of 3 in the neutrino mass.

Heavy Majorana neutrino exchange

- In LR model of Mohapatra, the mass of heavy neutrino $M_N$ and mass of right handed vector boson $W_R$ are related.
- In case of heavy neutrino exchange, the potential is of zero range. Hence, finite size and short range NN repulsion has to be incorporated properly.
- Mass of $W_R$ is 1.6 TeV for $^{76}\text{Ge}$.

4.2 PHFB Model

- The $\beta\beta$ decay is not an isolated nuclear process.
- The basic philosophy of nuclear many-body theory is to explain all the observed properties of nuclei in a coherent manner.
- Experimental studies involving in-beam $\gamma$-ray spectroscopy have yielded a vast amount of data concerning the level energies as well as electromagnetic properties over the past years.
- The availability of data permits a rigorous and detailed critique of the ingredients of the microscopic framework that seeks to provide a description of these isotopes, most of the calculations of $\beta\beta$ decay matrix elements performed so far do not satisfy this criterion.
- The structure of nuclei in the mass region $A=90-130$ involving Zr, Mo, Ru, Pd, Cd, Sn, Te and Xe isotopes is quite complex.
- The mass region $A=100$ offers a nice example of shape transition that is the sudden onset of deformation at neutron
number N=60. The nuclei are soft vibrators for neutron number 
N < 60 and quasi-rotors for N > 60.

- The nuclei with neutron number N=60 are transitional nuclei.
- The yrast spectra of Te and Xe isotopes follow an approximate type of systematic.
- All the nuclei undergoing $\beta\beta$ decay are even-even type, in which the pairing degrees of freedom play an important role.
- Moreover, it has been already conjectured that the deformation can play a crucial role in case of $\beta\beta$ decay of $^{100}$Mo and $^{150}$Nd.
- Hence, it is desirable to have a model, which incorporates the pairing and deformation degrees of freedom on equal footing in its formalism.
- For this purpose, the PHFB model is one of the most natural choices.
- However in the PHFB model, the double Gamow-Teller matrix element (DGT) $M_{GT}^{2\nu}$ has to be calculated using the closure approximation.
- The closure approximation, which is used to simplify the numerical calculation avoiding the explicit construction of intermediate states, estimates some average excitation energy $<E_N>$ for the intermediate states.
- The validity of this approximation for $2\nu$ $\beta\beta$ decay is ambiguous while for $0\nu$ $\beta\beta$ decay it is quite good.
- This approximation has been shown to work badly in case the signs of $M_{GT}^{2\nu}$ are predominantly of one sign for some lower $<E_N>$ and of opposite sign for a larger $<E_N>$.
- It is therefore better to avoid the closure approximation whenever possible.
- Hence, there is no apriori justification against the closure approximation.
- The validity of the closure approximation is to be decided a posteriori by comparing the theoretically calculated and experimentally extracted $M_{2\nu}$. 
Over the past fifteen years, extensive studies of shape transition vis-à-vis electromagnetic properties of Zr and Mo isotopes [Sharma et al] have been successfully carried out in the PHFB framework using the PPQQ interaction.

The success of the PHFB model in explaining the observed experimental trends in the mass region A=100 has motivated us to apply the PHFB wave functions to study nuclear $\beta\beta$ decay transitions in the mass range A=90 to 130.

It is well known that the pairing part of the two-body interaction is responsible for the reduction of collectivity where as the QQ interaction enhances the collectivity in the nuclear intrinsic wave functions.

Hence, to examine the explicit role of deformation degrees of freedom vis-à-vis the suppression $M_{GT}^{2\nu}$, the PPQQ will be the most appropriate choice.

To summarize, our aim is to study the $2\nu$ $\beta\beta$ decay not in isolation but together with other observed nuclear phenomena.

Hence as a test of the reliability of the wave functions, we have calculated the

- yrast spectra
- reduced B(E2:0$^+\rightarrow2^+$) transition probabilities
- static quadrupole moments Q(2$^+$)
- g-factors g(2$^+$)
- $M_{2\nu}$ and corresponding $T_{1/2}$
- role of deformation on $M_{2\nu}$ through varying single particle field and strength of the QQ interaction

An overall agreement with the above observed data makes us confident to study the $0\nu$ $\beta\beta$ decay.

5. Conclusions

The validity of nuclear models presently employed to calculate the $M_{2\nu}$ cannot be uniquely established due to uncertainty in $g_A$ as well as large error bars in experimental results.
• The nuclear structure effects are important for $2\nu \beta\beta$ and $0\nu \beta\beta$ decay.
• There is an uncertainty in estimated neutrino mass by a factor of 2-3 due to uncertainty in NTMEs.
• Measurement of nuclear response $R_{\nu\nu}^L$ as well as $R_{\nu\nu}^H$ for $0\nu \beta\beta$ decay.
• Further work is necessary both in the experimental as well as theoretical front to judge the relative applicability, success and failure of various models used so far for the study of double beta decay processes.

In spite of all this progress, a long way is still ahead of us. What is at stake is worth the effort. – A. Morales
Table 1: Variation in excitation energies (in MeV) of $J^\pi = 2^+; 4^+; 6^+$ yrast states of some nuclei in the mass range $A = 94$ to 130.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$A_{np}$</th>
<th>Theory</th>
<th>Experiment</th>
<th>Nucleus</th>
<th>$A_{np}$</th>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{94}$Zr</td>
<td>0.02519</td>
<td>$E_2^+$</td>
<td>0.9182</td>
<td>$E_4^+$</td>
<td>1.9732</td>
<td>$E_6^+$</td>
<td>2.7993</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td>0.01717</td>
<td>$E_2^+$</td>
<td>1.7570</td>
<td>$E_4^+$</td>
<td>3.5269</td>
<td>$E_6^+$</td>
<td>9.7261</td>
</tr>
<tr>
<td>$^{98}$Mo</td>
<td>0.02670</td>
<td>$E_2^+$</td>
<td>0.8715</td>
<td>$E_4^+$</td>
<td>1.4688</td>
<td>$E_6^+$</td>
<td>3.3136</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>0.02557</td>
<td>$E_2^+$</td>
<td>0.7779</td>
<td>$E_4^+$</td>
<td>0.9183</td>
<td>$E_6^+$</td>
<td>1.9732</td>
</tr>
<tr>
<td>$^{102}$Ru</td>
<td>0.01955</td>
<td>$E_2^+$</td>
<td>0.7892</td>
<td>$E_4^+$</td>
<td>1.51013</td>
<td>$E_6^+$</td>
<td>3.3098</td>
</tr>
<tr>
<td>$^{104}$Ru</td>
<td>0.01906</td>
<td>$E_2^+$</td>
<td>0.5356</td>
<td>$E_4^+$</td>
<td>1.1359</td>
<td>$E_6^+$</td>
<td>2.6738</td>
</tr>
<tr>
<td>$^{106}$Ru</td>
<td>0.02110</td>
<td>$E_2^+$</td>
<td>0.3580</td>
<td>$E_4^+$</td>
<td>0.8885</td>
<td>$E_6^+$</td>
<td>2.2280</td>
</tr>
<tr>
<td>$^{108}$Pd</td>
<td>0.01417</td>
<td>$E_2^+$</td>
<td>0.3737</td>
<td>$E_4^+$</td>
<td>0.9208</td>
<td>$E_6^+$</td>
<td>2.2254</td>
</tr>
<tr>
<td>$^{110}$Pd</td>
<td>0.02576</td>
<td>$E_2^+$</td>
<td>0.7433</td>
<td>$E_4^+$</td>
<td>1.4971</td>
<td>$E_6^+$</td>
<td>3.7840</td>
</tr>
<tr>
<td>$^{112}$Te</td>
<td>0.01491</td>
<td>$E_2^+$</td>
<td>0.8387</td>
<td>$E_4^+$</td>
<td>1.6325</td>
<td>$E_6^+$</td>
<td>3.1056</td>
</tr>
</tbody>
</table>

Table 2: Comparison of calculated and experimentally observed reduced transition probabilities $B(E2; 0^+ \rightarrow 2^+)$, static quadrupole moments $Q(2^+)$ and $g$ factors $g(2^+)$. Here $B(E2)$ and $Q(2^+)$ are calculated in units of $e^2$ and $eb$ respectively, for effective charge $e_{\text{eff}} = e + e_{\text{ne}}$ and $e_{\text{eff}} = e_{\text{ne}}$. The $g(2^+)$ has been calculated in units of nuclear magnetons for $g_n^0 = 1.0$ and $g_p^0 = 0.60$.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$B(E2; 0^+ \rightarrow 2^+)$</th>
<th>$Q(2^+)$</th>
<th>$g(2^+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
<td>Theory</td>
</tr>
<tr>
<td></td>
<td>$e_{\text{eff}}$</td>
<td>0.50</td>
<td>0.60</td>
</tr>
<tr>
<td>$^{94}$Zr</td>
<td>0.046</td>
<td>0.062</td>
<td>0.081</td>
</tr>
<tr>
<td>$^{94}$Mo</td>
<td>0.148</td>
<td>0.188</td>
<td>0.232</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td>0.044</td>
<td>0.060</td>
<td>0.078</td>
</tr>
<tr>
<td>$^{96}$Mo</td>
<td>0.265</td>
<td>0.335</td>
<td>0.413</td>
</tr>
<tr>
<td>$^{98}$Mo</td>
<td>0.234</td>
<td>0.302</td>
<td>0.378</td>
</tr>
<tr>
<td>$^{98}$Ru</td>
<td>0.433</td>
<td>0.543</td>
<td>0.665</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>0.320</td>
<td>0.412</td>
<td>0.515</td>
</tr>
<tr>
<td>$^{100}$Ru</td>
<td>0.308</td>
<td>0.393</td>
<td>0.488</td>
</tr>
<tr>
<td>$^{104}$Ru</td>
<td>0.572</td>
<td>0.732</td>
<td>0.912</td>
</tr>
<tr>
<td>$^{104}$Pd</td>
<td>0.361</td>
<td>0.460</td>
<td>0.571</td>
</tr>
<tr>
<td>$^{110}$Pd</td>
<td>0.479</td>
<td>0.614</td>
<td>0.766</td>
</tr>
<tr>
<td>$^{110}$Cd</td>
<td>0.427</td>
<td>0.548</td>
<td>0.685</td>
</tr>
<tr>
<td>$^{128}$Te</td>
<td>0.292</td>
<td>0.373</td>
<td>0.465</td>
</tr>
<tr>
<td>$^{128}$Xe</td>
<td>0.535</td>
<td>0.681</td>
<td>0.844</td>
</tr>
<tr>
<td>$^{130}$Te</td>
<td>0.220</td>
<td>0.276</td>
<td>0.338</td>
</tr>
<tr>
<td>$^{130}$Xe</td>
<td>0.464</td>
<td>0.587</td>
<td>0.724</td>
</tr>
</tbody>
</table>

$^{[2]}$ $^{[3]}$
Table 3: Experimental half-lives $T_{1/2}$ and corresponding matrix element $M_{2\gamma}$ for $0^+ \rightarrow 0^+$ transition of $^{94;96}$Zr, $^{98;100}$Mo, $^{104}$Ru, $^{110}$Pd and $^{128;130}$Te along with the theoretically calculated $M_{2\gamma}$ in different models. The numbers corresponding to (a) and (b) are calculated for $g_A = 1.25$ and 1.0 respectively.

<table>
<thead>
<tr>
<th>Nuclei</th>
<th>Ref.</th>
<th>Projects</th>
<th>Experiment $T_{1/2}$ (yrs)</th>
<th>$jM_{2\gamma}$</th>
<th>Ref. Models</th>
<th>Theory $jM_{2\gamma}$</th>
<th>$T_{1/2}$ (yrs)</th>
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<tr>
<td>$^{94}$Zr</td>
<td>[9]</td>
<td>NEMO</td>
<td>$&gt;1.1\times 10^{-5}$</td>
<td>(a) $&lt;62.815$</td>
<td>*</td>
<td>PHFB</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) $&lt;98.148$</td>
<td></td>
<td></td>
<td>(b) 9.61</td>
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<tr>
<td></td>
<td>[10]</td>
<td>SRQRPA</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td>3.08-659</td>
</tr>
<tr>
<td></td>
<td>[12]</td>
<td>Q R PA</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td>6.93</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td>[13]</td>
<td>$^g$ch.</td>
<td>0.943 $\pm$ 0.32</td>
<td>(a) 0.074 $^{+0.017}_{-0.010}$</td>
<td>*</td>
<td>PHFB</td>
<td>0.080</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.116 $^{+0.027}_{-0.018}$</td>
<td></td>
<td></td>
<td>(b) 1.97</td>
</tr>
<tr>
<td></td>
<td>[9]</td>
<td>NEMO</td>
<td>2.1 $^{+0.8}_{-0.4}$ $\pm$ 0.2</td>
<td>(a) 0.050 $^{+0.009}_{-0.014}$</td>
<td>[10]</td>
<td>SRQRPA</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.078 $^{+0.014}_{-0.021}$</td>
<td>[11]</td>
<td>OEM</td>
<td>-</td>
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<tr>
<td></td>
<td>[14]</td>
<td>NEMO</td>
<td>2.0 $^{+0.9}_{-0.5}$ $\pm$ 0.5</td>
<td>(a) 0.051 $^{+0.012}_{-0.010}$</td>
<td>[12]</td>
<td>Q R PA</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.080 $^{+0.033}_{-0.019}$</td>
<td>[13]</td>
<td>SU(4)</td>
<td>0.0678</td>
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<tr>
<td></td>
<td>[15]</td>
<td>$^g$ch.</td>
<td>3.9 $^{+0.9}_{-0.4}$</td>
<td>(a) 0.036 $^{+0.015}_{-0.009}$</td>
<td>[14]</td>
<td>RQRPA</td>
<td>-</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.057 $^{+0.008}_{-0.006}$</td>
<td>[18]</td>
<td>RQRPA</td>
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<td>[4]</td>
<td>Average</td>
<td>1.4 $^{+3.5}_{-0.5}$</td>
<td>(a) 0.061 $^{+0.028}_{-0.023}$</td>
<td>[19]</td>
<td>Q RPA(AWS)</td>
<td>0.12-0.31</td>
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<td></td>
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<td>(b) 0.095 $^{+0.054}_{-0.024}$</td>
<td>[20]</td>
<td>SRPA(AWS)</td>
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<td>Recommended Value</td>
<td>2.1 $^{+0.8}_{-0.4}$</td>
<td>(a) 0.050 $^{+0.006}_{-0.007}$</td>
<td>[11]</td>
<td>OEM</td>
<td>-</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.076 $^{+0.009}_{-0.012}$</td>
<td>[12]</td>
<td>Q R PA</td>
<td>-</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
<td>[6]</td>
<td>Q R PA</td>
<td>0.124</td>
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<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b) 0.82</td>
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<tr>
<td>$^{98}$Mo</td>
<td>[10]</td>
<td>Average</td>
<td>*</td>
<td>PHFB</td>
<td>0.182</td>
<td>(a) 3.10</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td>(b) 7.56</td>
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<td></td>
<td>[12]</td>
<td>Q R PA</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td>61.6</td>
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<td></td>
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<td></td>
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<td>29.6</td>
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continued..
<table>
<thead>
<tr>
<th>Reference</th>
<th>Value</th>
<th>Error (a)</th>
<th>Error (b)</th>
<th>Method</th>
<th>Error (a)</th>
<th>Error (b)</th>
<th>Model</th>
<th>Error (a)</th>
<th>Error (b)</th>
<th>Time (a)</th>
<th>Error (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[21] ITEP + INFN</td>
<td>7.2 ± 0.9 ± 1.8</td>
<td>(a) 0.121 ± 0.026</td>
<td>(b) 0.190 ± 0.036</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 11.15</td>
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<tr>
<td>[22] ITEP</td>
<td>8.5</td>
<td>(a) 0.112</td>
<td>(b) 0.174</td>
<td>SSDH</td>
<td>-</td>
<td>(a) 7.15-8.97</td>
<td>(b) 5.04-16800</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[23] UC Irvine</td>
<td>6.82 ± 0.38 ± 0.68</td>
<td>(a) 0.125 ± 0.013</td>
<td>(b) 0.195 ± 0.020</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 10.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[24] LBL+MHC+ UNM+INEL</td>
<td>7.6 ± 2.2 ± 1.4</td>
<td>(a) 0.118 ± 0.014</td>
<td>(b) 0.185 ± 0.022</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 7.99</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[25] NEMO</td>
<td>9.5 ± 0.4 ± 0.9</td>
<td>(a) 0.106 ± 0.010</td>
<td>(b) 0.165 ± 0.013</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 30.45</td>
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<tr>
<td>[26] LBL</td>
<td>9.7 ± 4.9</td>
<td>(a) 0.163 ± 0.030</td>
<td>(b) 0.163 ± 0.030</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 74.34</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>[27] ELEGANTS V</td>
<td>11.5 ± 3.0 ± 2.0</td>
<td>(a) 0.150 ± 0.011</td>
<td>(b) 0.150 ± 0.011</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 11.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[28] UC Irvine</td>
<td>11.6 ± 3.4 ± 0.8</td>
<td>(a) 0.149 ± 0.010</td>
<td>(b) 0.149 ± 0.010</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 22.19</td>
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<tr>
<td>[29] INS Baksan</td>
<td>3.3 ± 2.0 ± 1.0</td>
<td>(a) 0.179 ± 0.036</td>
<td>(b) 0.280 ± 0.035</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
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<td>[30] SSDH</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>[31] MCM</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[32] SRPA(WS)</td>
<td>0.059</td>
<td>(a) 0.106</td>
<td>(b) 0.165</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[33] SU(3)(DEF)</td>
<td>1.088</td>
<td>(a) 0.106</td>
<td>(b) 0.165</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[34] Average</td>
<td>8.0 ± 0.6</td>
<td>(a) 0.115 ± 0.004</td>
<td>(b) 0.180 ± 0.007</td>
<td>PHFB</td>
<td>0.152</td>
<td>(a) 4.57</td>
<td>(b) 35.8</td>
<td></td>
<td></td>
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<td>(a) 4.57</td>
<td>(b) 35.8</td>
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Ygch. denotes geochemical experiment.
*Present work
Table 4: Table of $M_F, M_{GT}, M_{Fh}$ and $M_{GT h}$ for various nuclei considered.

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<th>$M_{Fh}$</th>
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<th>$M_F$</th>
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<th>$m_i$ (eV)</th>
<th>$M_{Fh}$</th>
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<th>$M_N i$ (GeV)</th>
<th>$T_{1=2}^{\text{op}} A$ (yr)</th>
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Table 6: Ratio $D_k$ is defined as $D_k = \frac{M_k(\hat{A}_{qq} = 0)}{M_k(\hat{A}_{qq} = 1)}$, where $k = 2^\circ, F, GT, Fh$ and $GTh$.

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<th>$^{98}\text{Mo}$</th>
<th>$^{100}\text{Mo}$</th>
<th>$^{106}\text{Ru}$</th>
<th>$^{110}\text{Pd}$</th>
<th>$^{128}\text{Te}$</th>
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<td>3.15</td>
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<td>3.18</td>
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<td>2.95</td>
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<td>$D_{GT h}$</td>
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<td>2.81</td>
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References

Fig. 1 The dependence of $M_{2\nu}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{94}\text{Zr} \rightarrow ^{94}\text{Mo}$ and $^{96}\text{Zr} \rightarrow ^{96}\text{Mo}$.

Fig. 2 The dependence of $M_{2\nu}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{98}\text{Mo} \rightarrow ^{98}\text{Ru}$ and $^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$. 
Fig. 3 The dependence of $M_{2\nu}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{104}\text{Ru} \rightarrow ^{104}\text{Pd}$ and $^{110}\text{Pd} \rightarrow ^{110}\text{Cd}$.

Fig. 4 The dependence of $M_{2\nu}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{128}\text{Te} \rightarrow ^{128}\text{Xe}$ and $^{130}\text{Te} \rightarrow ^{130}\text{Xe}$. 
**Fig. 5** The dependence of $M_F$ and $M_{GT}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{96}\text{Zr} \rightarrow ^{96}\text{Mo}$.

**Fig. 6** The dependence of $M_F$ and $M_{GT}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$.
Fig. 7 The dependence of $M_{Fh}$ and $M_{GTh}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{96}\text{Zr} \rightarrow ^{96}\text{Mo}$.

Fig. 8 The dependence of $M_{Fh}$ and $M_{GTh}$ on the strength of the quadrupole-quadrupole interaction $\chi_{qq}$ for $^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$. 