FUF: 公式集

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ウィグナー-エッカルトの定理と換算行列要素の計算

$$\langle JM |T_{\lambda\mu}| J'M'\rangle = (-)^{J-M}\begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix}\langle J \|T_{\lambda}\| J'\rangle = \frac{1}{\hat{J}}\left(J'M'\lambda\mu|JM\right)\langle J \|T_{\lambda}\| J'\rangle .$$

$$\langle JM |T_{00}| J'M'\rangle = \frac{1}{\hat{J}}\langle J \|T_{0}\| J\rangle \delta_{JJ'}\delta_{MM'}.$$

$$\left\langle j_1 j_2 J \left\| \left[T^{k_1}(1)\otimes T^{k_2}(2)\right]_K \right\| j_1' j_2' J'\right\rangle = \hat{J}\hat{J}'\hat{K}\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ k_1 & k_2 & K \end{Bmatrix}\left\langle j_1 \left\|T^{k_1}\right\| j_1'\right\rangle\left\langle j_2 \left\|T^{k_2}\right\| j_2'\right\rangle .$$

$$\begin{aligned}\left\langle j_1 j_2 J \left\| \left[T^{k}(1)\otimes T^{k}(2)\right]_0 \right\| j_1' j_2' J\right\rangle &= \hat{J}^2\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J \\ k & k & 0 \end{Bmatrix}\left\langle j_1 \left\|T^{k}\right\| j_1'\right\rangle\left\langle j_2 \left\|T^{k}\right\| j_2'\right\rangle \\ &= \hat{J}\frac{(-)^{j_2+J+j_1'+k}}{\hat{k}}\begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & k \end{Bmatrix}\left\langle j_1 \left\|T^{k}\right\| j_1'\right\rangle\left\langle j_2 \left\|T^{k}\right\| j_2'\right\rangle .\end{aligned}$$

$$\left\langle j_1 j_2 J \left\| \left(\mathbf{T}^{k}(1)\cdot \mathbf{T}^{k}(2)\right) \right\| j_1' j_2' J\right\rangle = (-)^{j_2+J+j_1'}\,\hat{J}\begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & k \end{Bmatrix}\left\langle j_1 \left\|T^{k}\right\| j_1'\right\rangle\left\langle j_2 \left\|T^{k}\right\| j_2'\right\rangle .$$

$$\left\langle j_1 j_2 J \left\| T^{k}(1) \right\| j_1' j_2' J'\right\rangle = (-)^{j_1+j_2+J'+k}\,\hat{J}\hat{J}'\begin{Bmatrix} j_1 & J & j_2 \\ J' & j_1' & k \end{Bmatrix}\left\langle j_1 \left\|T^{k}\right\| j_1'\right\rangle\delta_{j_2 j_2'}.$$

$$\left\langle j_1 j_2 J \left\| T^{k}(2) \right\| j_1' j_2' J'\right\rangle = (-)^{j_1+j_2'+J+k}\,\hat{J}\hat{J}'\begin{Bmatrix} j_2 & J & j_1 \\ J' & j_2' & k \end{Bmatrix}\left\langle j_2 \left\|T^{k}\right\| j_2'\right\rangle\delta_{j_1 j_1'}.$$

$$\left\langle j \left\| \left[T^{k_1}(1)\otimes T^{k_2}(1)\right]_k \right\| j'\right\rangle = (-)^{j+k+j'}\,\hat{k}\sum_{j''}\left\langle j \left\|T^{k_1}(1)\right\| j''\right\rangle\left\langle j'' \left\|T^{k_2}(1)\right\| j'\right\rangle\begin{Bmatrix} k_1 & k_2 & k \\ j' & j & j'' \end{Bmatrix}.$$

$$\langle j \|1\| j'\rangle = \hat{j}\delta_{jj'}.$$

$$\left\langle S' \left\|\hat{\mathbf{S}}\right\| S\right\rangle = \sqrt{S(S+1)(2S+1)}\delta_{S'S}.$$

$$\langle Y_{l_1} \|Y_{\lambda}\| Y_{l_2}\rangle = (-)^{\lambda}\frac{\hat{l}_1\hat{\lambda}}{\sqrt{4\pi}}\left(l_1 0 \lambda 0|l_2 0\right).$$

$$\left\langle Y_{L'}\left(\hat{\mathbf{R}}\right) \|\nabla_{\mathbf{R}}\| Y_L\left(\hat{\mathbf{R}}\right)\right\rangle = \hat{L}\left(L010|L'0\right)\left[\left(\frac{\partial}{\partial R}-\frac{L}{R}\right)\delta_{L',L+1}+\left(\frac{\partial}{\partial R}+\frac{L+1}{R}\right)\delta_{L',L-1}\right].$$

$$\left\langle\left[\eta_{1/2}\otimes Y_l\right]_j \|Y_k\| \left[\eta_{1/2}\otimes Y_{l'}\right]_{j'}\right\rangle = (-)^{j+j'+1}\frac{\hat{j}\hat{k}}{\sqrt{4\pi}}\left(j1/2k0|j'1/2\right)\frac{1}{2}\left[1+(-)^{l+k+l'}\right].$$

$$\left\{\begin{array}{c} \langle SM' |S_{11}| SM\rangle = -\delta_{M',M+1}\sqrt{\frac{(S-M)(S+M+1)}{2}} \\ \langle SM' |S_{1-1}| SM\rangle = \delta_{M',M-1}\sqrt{\frac{(S+M)(S-M+1)}{2}} \\ \langle SM' |S_{10}| SM\rangle = \delta_{M',M}M \end{array}\right. .$$

規約テンソルの作成

$$\mathbf{C}^{(\lambda)}\left(\hat{\mathbf{R}}_1\right)\cdot\mathbf{C}^{(\lambda)}\left(\hat{\mathbf{R}}_2\right)=\hat{\lambda}\,(-)^{\lambda}\left[C^{(\lambda)}\left(\hat{\mathbf{R}}_1\right)\otimes C^{(\lambda)}\left(\hat{\mathbf{R}}_2\right)\right]_{00}.$$

特に $\lambda=1$ のとき、

$$\mathbf{R}_1\cdot\mathbf{R}_2=-\sqrt{3}\left[R_1^{(1)}\left(\hat{\mathbf{R}}_1\right)\otimes R_2^{(1)}\left(\hat{\mathbf{R}}_2\right)\right]_{00},$$

$$R_m^{(1)}=\sqrt{\frac{4\pi}{3}}RY_{1m}\left(\hat{\mathbf{R}}\right).$$

3 次元ベクトルの規約テンソル表示 (具体形)

$$R_{+1}^{(1)}=\sqrt{\frac{4\pi}{3}}RY_{11}\left(\hat{\mathbf{R}}\right)=-\sqrt{\frac{1}{2}}R\sin\theta e^{i\varphi}=\frac{1}{\sqrt{2}}\left(-x-iy\right),$$

$$R_{-1}^{(1)}=\sqrt{\frac{4\pi}{3}}RY_{1-1}\left(\hat{\mathbf{R}}\right)=\sqrt{\frac{1}{2}}R\sin\theta e^{-i\varphi}=\frac{1}{\sqrt{2}}\left(x-iy\right),$$

$$R_0^{(1)}=\sqrt{\frac{4\pi}{3}}RY_{10}\left(\hat{\mathbf{R}}\right)=R\cos\theta=z.$$

3 次元ベクトルの外積の規約テンソル表示[1]

$$\left(\mathbf{s}\times\mathbf{t}\right)_m^{(1)}=-i\sqrt{2}\left[s^{(1)}\otimes t^{(1)}\right]_{1m}.$$

角運動量の組み替え

$$\left[\left[j_1\otimes j_2\right]_{J_{12}}\otimes\left[j_3\otimes j_4\right]_{J_{34}}\right]_{JM}=\sum_{J_{13}J_{24}}\hat{J}_{12}\hat{J}_{34}\hat{J}_{13}\hat{J}_{24}\begin{Bmatrix} j_1 & j_2 & J_{12}\\ j_3 & j_4 & J_{34}\\ J_{13} & J_{24} & J\end{Bmatrix}\left[\left[j_1\otimes j_3\right]_{J_{13}}\otimes\left[j_2\otimes j_4\right]_{J_{24}}\right]_{JM}.$$

$$\left[\left[j_1\otimes j_2\right]_{J_{12}}\otimes j_3\right]_{JM}=\sum_{J_{13}}(-)^{j_2+j_3+J_{12}+J_{13}}\,\hat{J}_{12}\hat{J}_{13}\begin{Bmatrix} j_1 & j_3 & J_{13}\\ J & j_2 & J_{12}\end{Bmatrix}\left[\left[j_1\otimes j_3\right]_{J_{13}}\otimes j_2\right]_{JM}.$$

$$\begin{aligned}\left[Y_l\left(\Omega_1\right)\otimes Y_l\left(\Omega_2\right)\right]_{00}\left[Y_L\left(\Omega_1\right)\otimes Y_L\left(\Omega_2\right)\right]_{00}&=\frac{1}{4\pi}\sum_j\frac{\hat{l}\hat{L}}{\hat{j}}\left(l0L0|j0\right)^2\left[Y_j\left(\Omega_1\right)\otimes Y_j\left(\Omega_2\right)\right]_{00}\\&=\sum_{jj_z}\frac{(-)^j}{4\pi}\frac{\hat{l}\hat{L}}{\hat{j}^2}\left(l0L0|j0\right)^2Y_{jj_z}^*\left(\Omega_1\right)Y_{jj_z}\left(\Omega_2\right)\\&=\sum_j\frac{(-)^j}{(4\pi)^2}\hat{l}\hat{L}\left(l0L0|j0\right)^2P_j\left(\cos\theta_{\Omega_1\Omega_2}\right).\end{aligned}$$

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[1] この表式は、以下のようにして確認することが出来る。

$$\begin{aligned}\left[s^{(1)}\otimes t^{(1)}\right]_{11}&=\sum_{m'm''}\left(1m'1m''|11\right)s_{1m'}t_{1m''}=\left(1011|11\right)s_{10}t_{11}+\left(1110|11\right)s_{11}t_{10}=-\sqrt{\frac{2}{4}}s_z\frac{1}{\sqrt{2}}\left(-t_x-it_y\right)+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(-s_x-is_y\right)t_z\\&=\frac{1}{2}\left\{s_zt_x-s_xt_z+i\left(s_zt_y-s_yt_z\right)\right\}=\frac{1}{2}\left\{\left(\mathbf{s}\times\mathbf{t}\right)_y-i\left(\mathbf{s}\times\mathbf{t}\right)_x\right\}=\frac{i}{\sqrt{2}}\left(\mathbf{s}\times\mathbf{t}\right)_1^{(1)},\end{aligned}$$

$$\begin{aligned}\left[s^{(1)}\otimes t^{(1)}\right]_{1-1}&=\sum_{m'm''}\left(1m'1m''|1,-1\right)s_{1m'}t_{1m''}=\left(1,-1,10|1,-1\right)s_{1-1}t_{10}+\left(101,-1|1,-1\right)s_{10}t_{1-1}\\&=-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(s_x-is_y\right)t_z+\sqrt{\frac{2}{4}}s_z\frac{1}{\sqrt{2}}\left(t_x-it_y\right)\\&=\frac{1}{2}\left\{-s_xt_z+s_zt_x+i\left(s_yt_z-s_zt_y\right)\right\}=\frac{1}{2}\left\{-\left(\mathbf{s}\times\mathbf{t}\right)_y+i\left(\mathbf{s}\times\mathbf{t}\right)_x\right\}=\frac{i}{\sqrt{2}}\left(\mathbf{s}\times\mathbf{t}\right)_{-1}^{(1)},\end{aligned}$$

$$\begin{aligned}\left[s^{(1)}\otimes t^{(1)}\right]_{10}&=\sum_{m'm''}\left(1m'1m''|10\right)s_{1m'}t_{1m''}=\left(1,-1,11|10\right)s_{1-1}t_{11}+\left(111,-1|10\right)s_{11}t_{1-1}\\&=-\sqrt{\frac{2}{4}}\frac{1}{\sqrt{2}}\left(s_x-is_y\right)\frac{1}{\sqrt{2}}\left(-t_x-it_y\right)+\sqrt{\frac{2}{4}}\frac{1}{\sqrt{2}}\left(-s_x-is_y\right)\frac{1}{\sqrt{2}}\left(t_x-it_y\right)\\&=\frac{1}{2\sqrt{2}}\left\{s_xt_x+s_yt_y-s_xt_x-s_yt_y+i\left(-s_yt_x+s_xt_y-s_yt_x+s_xt_y\right)\right\}=\frac{i}{\sqrt{2}}\left(\mathbf{s}\times\mathbf{t}\right)_z=\frac{i}{\sqrt{2}}\left(\mathbf{s}\times\mathbf{t}\right)_0^{(1)}.\end{aligned}$$

クレブシュ・ゴルダン係数

$$\sum_{m_1 m_2} (j_1 m_1 j_2 m_2 | jm)\,(j_1 m_1 j_2 m_2 | j' m') = \delta_{jj'}\delta_{mm'}.$$

$$\sum_{jm} (j_1 m_1 j_2 m_2 | jm)\,(j_1 m_1' j_2 m_2' | jm) = \delta_{m_1 m_1'}\delta_{m_2 m_2'}.$$

$$\sum_{m} (j_1 0 jm | jm) = \sum_{m} (jm j_1 0 | jm) = \hat{j}^2 \delta_{j_1 0}.$$

$$\begin{aligned}(j_1 m_1 j_2 m_2 | jm) &= (-)^{j_1+j_2-j}\,(j_2 m_2 j_1 m_1 | jm)\\ &= (-)^{j_1+j_2-j}\,(j_1, -m_1 j_2, -m_2 | j, -m)\\ &= (-)^{j_1-m_1}\,\frac{\hat{j}}{\hat{j}_2}\,(j_1 m_1 j, -m | j_2, -m_2)\\ &= (-)^{j_2-m_2}\,\frac{\hat{j}}{\hat{j}_1}\,(j, -m j_2 m_2 | j_1, -m_1)\,.\end{aligned}$$

$$(j_1 m_1 0 0 | jm) = \delta_{j_1 j}\delta_{m_1 m}.$$

$$(j_1 m_1 j_2 m_2 | 00) = \delta_{j_1 j_2}\delta_{m_1, -m_2}\frac{(-)^{j_1-m_1}}{\hat{j}_1}.$$

$$(j_1 0 j_2 0 | j0) = \begin{cases} (-)^{j+g}\,\hat{j}\sqrt{\dfrac{(j_1+j_2-j)!\,(j_1+j-j_2)!\,(j_2+j-j_1)!}{(j_1+j_2+j+1)!}}\,\dfrac{g!}{(g-j_1)!\,(g-j_2)!\,(g-j)!} \\ \qquad\qquad (\text{if } j_1+j_2+j \equiv 2g = \text{偶数}), \\ 0 \quad (\text{if } j_1+j_2+j = \text{奇数}). \end{cases}$$

漸近形

$$(lmk0|lm) \to \frac{(-)^k}{\hat{l}} P_k\left(\frac{m}{l}\right) \qquad \text{for } l\to\infty \quad (k\text{ は有限},\ m \text{ は } 0 \text{ または正の整数})\,.$$

$3j$ 記号との関係

$$(j_1 m_1 j_2 m_2 | j_3 m_3) = (-)^{-j_1+j_2-m_3}\,\hat{j}_3 \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}.$$

簡単な場合の値

$(j_1 m_1 1/2 m_2 | jm)$:

$j \setminus m_2$	$\frac{1}{2}$	$-\frac{1}{2}$
$j_1+\frac{1}{2}$	$\sqrt{\frac{j_1+m+1/2}{2j_1+1}}$	$\sqrt{\frac{j_1-m+1/2}{2j_1+1}}$
$j_1-\frac{1}{2}$	$-\sqrt{\frac{j_1-m+1/2}{2j_1+1}}$	$\sqrt{\frac{j_1+m+1/2}{2j_1+1}}$

$(j_1 m_1 1 m_2 | jm)$:

$j \setminus m_2$	1	0	-1
j_1+1	$\sqrt{\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}}$
j_1	$-\sqrt{\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}}$
j_1-1	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

$6j$ と$9j$ の関係およびそれらの性質

$$\left\{\begin{matrix} j_1 & j_2 & J \\ j_3 & j_4 & J \\ k & k & 0 \end{matrix}\right\} = \frac{(-)^{-j_2-J-j_3-k}}{\hat{J}\hat{k}} \left\{\begin{matrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{matrix}\right\}.$$

$$\left\{\begin{matrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{matrix}\right\} = (-)^{j_1+j_2+j_3+j_4} W\left(j_1 j_2 j_3 j_4; Jk\right).$$

$$\left\{\begin{matrix} j_1 & j_1' & 0 \\ j_2 & j_2' & j_3 \end{matrix}\right\} = \frac{(-)^{j_1+j_2+j_3}}{\hat{j}_1\hat{j}_2}\delta_{j_1 j_1'}\delta_{j_2 j_2'}.$$

$$\left\{\begin{matrix} j_1 & j_2 & j_3 \\ j_2 & j_1 & 0 \end{matrix}\right\} = \frac{(-)^{j_1+j_2+j_3}}{\hat{j}_1\hat{j}_2}.$$

$$\left\{\begin{matrix} j_1 & j_1 & 0 \\ j_2 & j_2 & 0 \\ j_3 & j_3 & 0 \end{matrix}\right\} = \frac{1}{\hat{j}_1\hat{j}_2\hat{j}_3}.$$

$$\left\{\begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix}\right\} = \left\{\begin{matrix} j_2 & j_3 & j_1 \\ l_2 & l_3 & l_1 \end{matrix}\right\} = \left\{\begin{matrix} j_3 & j_1 & j_2 \\ l_3 & l_1 & l_2 \end{matrix}\right\} = \left\{\begin{matrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \end{matrix}\right\} = \left\{\begin{matrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{matrix}\right\}.$$

$$\sum_{j_3} \hat{j}_3^2 \left\{\begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix}\right\} \left\{\begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3' \end{matrix}\right\} = \frac{\delta_{l_3 l_3'}}{\hat{l}_3^2}.$$

$$\sum_{j} (-)^{j_1+j_2+j} \hat{j}^2 \left\{\begin{matrix} j_1 & j_1 & j' \\ j_2 & j_2 & j \end{matrix}\right\} = \hat{j}_1\hat{j}_2\delta_{j'0}.$$

$$\sum_{j} (-)^{j+j'+j''} \hat{j}^2 \left\{\begin{matrix} j_1 & j_2 & j' \\ j_3 & j_4 & j \end{matrix}\right\} \left\{\begin{matrix} j_1 & j_3 & j'' \\ j_2 & j_4 & j \end{matrix}\right\} = \left\{\begin{matrix} j_1 & j_2 & j' \\ j_4 & j_3 & j'' \end{matrix}\right\}.$$

$$\sum_{j} \hat{j}^2 \left\{\begin{matrix} j_1 & j_2 & j \\ j_1 & j_2 & j' \end{matrix}\right\} = (-)^{2(j_1+j_2)}.$$

$$\begin{aligned} &\sum_{k} (-)^{k_1+k_2+k} \hat{k}^2 \left\{\begin{matrix} j_1 & j_1' & k \\ j_2' & j_2 & j \end{matrix}\right\} \left\{\begin{matrix} k_1 & k_2 & k \\ j_1' & j_1 & j_2'' \end{matrix}\right\} \left\{\begin{matrix} k_1 & k_2 & k \\ j_2' & j_2 & j_1'' \end{matrix}\right\} \\ &= (-)^{j_1+j_2+j_1'+j_2'+j_1''+j_2''+j} \left\{\begin{matrix} j_1 & j_2 & j \\ j_1'' & j_2'' & k_1 \end{matrix}\right\} \left\{\begin{matrix} j_1' & j_2' & j \\ j_1'' & j_2'' & k_2 \end{matrix}\right\}. \end{aligned}$$

$$\left\{\begin{matrix} s_1 & s_2 & s_3 \\ l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{matrix}\right\} = \left\{\begin{matrix} s_1 & l_1 & j_1 \\ s_2 & l_2 & j_2 \\ s_3 & l_3 & j_3 \end{matrix}\right\} = \left\{\begin{matrix} j_1 & j_2 & j_3 \\ s_1 & s_2 & s_3 \\ l_1 & l_2 & l_3 \end{matrix}\right\} = \left\{\begin{matrix} l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \\ s_1 & s_2 & s_3 \end{matrix}\right\}.$$

$$\left\{\begin{matrix} s_1 & s_2 & s_3 \\ l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{matrix}\right\} = (-)^{\Sigma} \left\{\begin{matrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \\ j_1 & j_2 & j_3 \end{matrix}\right\},$$

$$\Sigma = l_1 + l_2 + l_3 + s_1 + s_2 + s_3 + j_1 + j_2 + j_3.$$

$$\left\{\begin{matrix} l_1 & l_2 & l_3 \\ l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{matrix}\right\} = 0, \qquad 2\left(l_1 + l_2 + l_3\right) + j_1 + j_2 + j_3 \text{が奇数のとき}.$$

$$\sum_{J_{13}J_{24}} \hat{J}_{13}^2 \hat{J}_{24}^2 \left\{\begin{matrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{matrix}\right\} \left\{\begin{matrix} j_1 & j_2 & J_{12}' \\ j_3 & j_4 & J_{34}' \\ J_{13} & J_{24} & J \end{matrix}\right\} = \frac{\delta_{J_{12}J_{12}'}\delta_{J_{34}J_{34}'}}{\hat{J}_{12}^2 \hat{J}_{34}^2}.$$

$$\sum_{J_{13}J_{24}} (-)^{j_2+j_4+J_{24}} (-)^{j_2+j_3+J_{23}} \hat{J}_{13}^2 \hat{J}_{24}^2 \begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{Bmatrix} \begin{Bmatrix} j_1 & j_4 & J_{14} \\ j_3 & j_2 & J_{23} \\ J_{13} & J_{24} & J \end{Bmatrix} = (-)^{j_3+j_4+J_{34}} \begin{Bmatrix} j_1 & j_4 & J_{14} \\ j_2 & j_3 & J_{23} \\ J_{12} & J_{34} & J \end{Bmatrix}.$$

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_{\substack{m_1m_2m_3m_4 \\ M_{12}M_{34}M_{13}M_{24}M}} \frac{1}{\hat{J}_{12}\hat{J}_{34}\hat{J}_{13}\hat{J}_{24}\hat{J}^2} (j_1m_1j_2m_2|J_{12}M_{12}) (j_3m_3j_4m_4|J_{34}M_{34}) (j_1m_1j_3m_3|J_{13}M_{13})$$
$$\times (j_2m_2j_4m_4|J_{24}M_{24}) (J_{13}M_{13}J_{24}M_{24}|JM) (J_{12}M_{12}J_{34}M_{34}|JM)$$

$$(J_{12}M_{12}J_{34}M_{34}|JM) \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_{\substack{m_1m_2m_3m_4 \\ M_{13}M_{24}}} \frac{1}{\hat{J}_{12}\hat{J}_{34}\hat{J}_{13}\hat{J}_{24}} (j_1m_1j_2m_2|J_{12}M_{12}) (j_3m_3j_4m_4|J_{34}M_{34})$$
$$\times (j_1m_1j_3m_3|J_{13}M_{13}) (j_2m_2j_4m_4|J_{24}M_{24}) (J_{13}M_{13}J_{24}M_{24}|JM).$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_{m_1'm_2'm_3'} (-)^{l_1+l_2+l_3+m_1'+m_2'+m_3'} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & m_2' & -m_3' \end{pmatrix}$$
$$\times \begin{pmatrix} l_1 & j_2 & l_3 \\ -m_1' & m_2 & m_3' \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ m_1' & -m_2' & m_3 \end{pmatrix}.$$

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} &= \sum_{J'} (-)^{2J'} \hat{J}'^2 \begin{Bmatrix} j_1 & j_3 & J_{13} \\ J_{24} & J & J' \end{Bmatrix} \begin{Bmatrix} j_2 & j_4 & J_{24} \\ j_3 & J' & J_{34} \end{Bmatrix} \begin{Bmatrix} J_{12} & J_{34} & J \\ J' & j_1 & j_2 \end{Bmatrix} \\ &= \sum_{J'} (-)^{2J'} \hat{J}'^2 \begin{Bmatrix} J_{12} & J_{34} & J \\ J_{13} & J_{24} & J' \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ J_{34} & J' & j_4 \end{Bmatrix} \begin{Bmatrix} j_2 & j_4 & J_{24} \\ J' & J_{12} & j_1 \end{Bmatrix} \\ &= \sum_{J'} (-)^{2J'} \hat{J}'^2 \begin{Bmatrix} j_2 & j_4 & J_{24} \\ J & J_{13} & J' \end{Bmatrix} \begin{Bmatrix} J_{12} & J_{34} & J \\ j_4 & J' & j_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ J' & j_2 & J_{12} \end{Bmatrix}. \end{aligned}$$

$$\sum_J \hat{J}^2 \begin{Bmatrix} J_{12} & J_{34} & J \\ J_{13} & J_{24} & J' \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = (-)^{2J'} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ J_{34} & J' & j_4 \end{Bmatrix} \begin{Bmatrix} j_2 & j_4 & J_{24} \\ J' & J_{12} & j_1 \end{Bmatrix}.$$

位相因子

角運動量の合成 $\boldsymbol{j}_1 + \boldsymbol{j}_2 = \boldsymbol{j}_3$ (z 成分: $m_1 + m_2 = m_3$) について

$$\pm (j_1 \text{ or } m_1) \pm (j_2 \text{ or } m_2) \pm (j_3 \text{ or } m_3) \in \mathbb{Z} \quad (\pm \text{ は任意の組み合わせ}).$$

$$j_i \pm m_i \in \mathbb{Z}, \quad (i = 1, 2, 3).$$

$3j$ 記号

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-)^{j_1-j_2-m_3}}{\hat{j}_3} \left(j_1 m_1 j_2 m_2 | j_3, -m_3\right).$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}.$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \varepsilon\left(j_1 j_2 j_3\right) \frac{1}{\hat{j}_3^2} \delta_{j_3 j_3'} \delta_{m_3 m_3'}.$$

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \varepsilon\left(j_1 j_2 j_3\right).$$

$$\sum_{j_3 m_3} \hat{j}_3^2 \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \varepsilon\left(j_1 j_2 j_3\right) \delta_{m_1 m_1'} \delta_{m_2 m_2'}.$$

$$\varepsilon\left(j_1 j_2 j_3\right) = \begin{cases} 1 & \text{if } j_1 + j_2 \geq j_3 \geq |j_1 - j_2|, \\ 0 & \text{otherwise.} \end{cases}$$

$6j$ と $9j$ の具体的表式

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 + \frac{1}{2} & j_1 + \frac{1}{2} & \frac{1}{2} \end{Bmatrix} = (-)^{j_1+j_2+j_3+1} \left[\frac{\left(j_1+j_2+j_3+2\right)\left(j_1+j_2-j_3+1\right)}{\left(2j_1+1\right)\left(2j_1+2\right)\left(2j_2+1\right)\left(2j_2+2\right)}\right]^{1/2}.$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 + \frac{1}{2} & j_1 - \frac{1}{2} & \frac{1}{2} \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{\left(j_3+j_1-j_2\right)\left(j_2+j_3-j_1+1\right)}{2j_1\left(2j_1+1\right)\left(2j_2+1\right)\left(2j_2+2\right)}\right]^{1/2}.$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 - 1 & j_1 - 1 & 1 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{\left(j_1+j_2+j_3\right)\left(j_1+j_2+j_3+1\right)\left(j_1+j_2-j_3\right)\left(j_1+j_2-j_3-1\right)}{\left(2j_1-1\right)2j_1\left(2j_1+1\right)\left(2j_2-1\right)2j_2\left(2j_2+1\right)}\right]^{1/2}.$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 - 1 & j_1 & 1 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{2\left(j_1+j_2+j_3+1\right)\left(j_1+j_2-j_3\right)\left(j_2+j_3-j_1\right)\left(j_1-j_2+j_3+1\right)}{2j_1\left(2j_1+1\right)\left(2j_1+2\right)\left(2j_2-1\right)2j_2\left(2j_2+1\right)}\right]^{1/2}.$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 + 1 & j_1 - 1 & 1 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{\left(j_1-j_2+j_3-1\right)\left(j_1-j_2+j_3\right)\left(j_2+j_3-j_1+1\right)\left(j_2+j_3-j_1+2\right)}{\left(2j_2+1\right)\left(2j_2+2\right)\left(2j_2+3\right)\left(2j_1-1\right)2j_1\left(2j_1+1\right)}\right]^{1/2}.$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 & j_1 & 1 \end{Bmatrix} = (-)^{j_1+j_2+j_3+1} \frac{1}{2} \frac{j_1\left(j_1+1\right)+j_2\left(j_2+1\right)-j_3\left(j_3+1\right)}{\sqrt{j_1\left(j_1+1\right)\left(2j_1+1\right)j_2\left(j_2+1\right)\left(2j_2+1\right)}}.$$

$$\begin{Bmatrix} S & S & 1 \\ l_1 & l_2 & L \\ j_1 & j_2 & L \end{Bmatrix} = (-)^{l_1+j_2+S+L} \frac{\left[l_1\left(l_1+1\right)-l_2\left(l_2+1\right)\right]-\left[j_1\left(j_1+1\right)-j_2\left(j_2+1\right)\right]}{2\left(2S+1\right)\left(2L+1\right)\sqrt{S\left(S+1\right)L\left(L+1\right)}} \begin{Bmatrix} l_1 & l_2 & L \\ j_2 & j_1 & S \end{Bmatrix}.$$

ベクトル球面調和関数

スピン 1 の波動関数 ($\boldsymbol{e}_i$, i =1–3 はデカルト座標の単位ベクトル)

$$\begin{aligned}\boldsymbol{\xi}_1 &= -\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_1 + i\boldsymbol{e}_2\right),\\ \boldsymbol{\xi}_0 &= \boldsymbol{e}_3,\\ \boldsymbol{\xi}_{-1} &= \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_1 - i\boldsymbol{e}_2\right).\end{aligned}$$

$$\boldsymbol{s}^2\boldsymbol{\xi}_\kappa = 2\boldsymbol{\xi}_\kappa, \quad s_z\boldsymbol{\xi}_\kappa = \kappa\boldsymbol{\xi}_\kappa, \quad \kappa = 1\text{–}3.$$

規格直交性

$$\boldsymbol{\xi}^*_\mu \cdot \boldsymbol{\xi}_\nu = (-)^\mu\, \boldsymbol{\xi}_{-\mu} \cdot \boldsymbol{\xi}_\nu = \delta_{\mu\nu}.$$

任意のベクトルの成分

$$\begin{aligned}\boldsymbol{A} &= \sum_\mu A_\mu \boldsymbol{\xi}^*_\mu = \sum_\mu A_\mu (-)^\mu\, \boldsymbol{\xi}_{-\mu},\\ A_\mu &= \boldsymbol{A}\cdot\boldsymbol{\xi}_\mu = (-)^\mu\, \boldsymbol{A}\cdot\boldsymbol{\xi}^*_\mu\end{aligned}$$

2 つのベクトルの内積

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_\mu A_\mu (-)^\mu\, \boldsymbol{\xi}_{-\mu} \cdot \sum_\nu B_\nu \boldsymbol{\xi}^*_\nu = \sum_\mu (-)^\mu A_\mu B_{-\mu}.$$

ベクトル球面調和関数の定義と性質

$$\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}) \equiv \left[Y_l(\hat{\boldsymbol{r}}) \otimes \boldsymbol{\xi}\right]_{jm} = \sum_{\lambda\kappa} (l\lambda 1\kappa|jm)\, Y_{l\lambda}(\hat{\boldsymbol{r}})\, \boldsymbol{\xi}_\kappa.$$

$$\begin{aligned}\boldsymbol{s}^2\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}) &= 2\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}), \quad \boldsymbol{l}^2\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}) = l(l+1)\,\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}),\\ \boldsymbol{j}^2\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}) &= j(j+1)\,\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}), \quad j_z\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}) = m\boldsymbol{Y}^{j(l1)}_m(\hat{\boldsymbol{r}}).\\ \boldsymbol{Y}^{j(j1)}_m(\hat{\boldsymbol{r}}) &= \frac{1}{\sqrt{j(j+1)}}\boldsymbol{l}Y_{jm}(\hat{\boldsymbol{r}}).\end{aligned}$$

勾配公式

$$\begin{aligned}\boldsymbol{\nabla} f(r)\, Y_{lm}(\hat{\boldsymbol{r}}) &= -\left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{df}{dr} - l\frac{f}{r}\right)\boldsymbol{Y}^{l(l+1,1)}_m(\hat{\boldsymbol{r}})\\ &= -\left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{df}{dr} - l\frac{f}{r}\right)\sum_\mu (l+1, m-\mu, 1\mu|lm)\, Y_{l+1,m-\mu}(\hat{\boldsymbol{r}})\,\boldsymbol{\xi}_\mu.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla} r^l Y_{lm}(\hat{\boldsymbol{r}}) &= \sqrt{l(2l+1)}\, r^{l-1}\boldsymbol{Y}^{l(l-1,1)}_m(\hat{\boldsymbol{r}})\\ &= \sqrt{l(2l+1)}\, r^{l-1}\sum_\mu (l-1, m-\mu, 1\mu|lm)\, Y_{l-1,m-\mu}(\hat{\boldsymbol{r}})\,\boldsymbol{\xi}_\mu.\end{aligned}$$

$$\left(\boldsymbol{\nabla} r^l Y_{lm}(\hat{\boldsymbol{r}})\right)\cdot\boldsymbol{v} = \sqrt{l(2l+1)}\, r^{l-1}\left[Y_{l-1}(\hat{\boldsymbol{r}}) \otimes v\right]_{lm}, \quad (\boldsymbol{v}\text{は任意のベクトル}).$$

球面調和関数

$$\int Y_{lm}^{*}(\Omega)\, Y_{l'm'}(\Omega)\, d\Omega = \delta_{ll'}\delta_{mm'}.$$

$$\sum_{lm} Y_{lm}^{*}(\Omega')\, Y_{lm}(\Omega) = \delta(\Omega' - \Omega).$$

$$Y_{l0}(\Omega) = \frac{\hat{l}}{\sqrt{4\pi}} P_l(\cos\theta).$$

$$\int_{-1}^{1} P_l(\cos\theta)\, P_{l'}(\cos\theta)\, d(\cos\theta) = \frac{2}{\hat{l}^2}\delta_{ll'}.$$

$$\sum_{l} \frac{\hat{l}^2}{2} P_l(\cos\theta)\, P_l(\cos\theta') = \delta(\cos\theta - \cos\theta').$$

$$\int_{-1}^{1} x^n P_l(x)\, dx = 0, \quad \text{for } n = 0,\ 1,\ ...,\ l-1.$$

$$P_l(\cos\theta_{\mathbf{r}_1\mathbf{r}_2}) = \frac{4\pi}{\hat{l}^2}\sum_m Y_{lm}^{*}(\hat{\mathbf{r}}_1)\, Y_{lm}(\hat{\mathbf{r}}_2) = \frac{4\pi}{\hat{l}^2}\sum_m Y_{lm}(\hat{\mathbf{r}}_1)\, Y_{lm}^{*}(\hat{\mathbf{r}}_2) = \frac{4\pi}{\hat{l}}(-)^l \left[Y_l(\hat{\mathbf{r}}_1)\otimes Y_l(\hat{\mathbf{r}}_2)\right]_{00}.$$

$$Y_{lm}^{*}(\Omega) = (-)^m Y_{l,-m}(\Omega).$$

$$\int Y_{l_1m_1}(\Omega)\, Y_{l_2m_2}(\Omega)\, Y_{l_3m_3}^{*}(\Omega)\, d\Omega = \frac{1}{\sqrt{4\pi}}\frac{\hat{l}_1\hat{l}_2}{\hat{l}_3}(l_1m_1l_2m_2|l_3m_3)\,(l_10l_20|l_30).$$

$$\left[Y_{l_1}(\Omega)\otimes Y_{l_2}(\Omega)\right]_{LM} = \frac{1}{\sqrt{4\pi}}\frac{\hat{l}_1\hat{l}_2}{\hat{L}}(l_10l_20|L0)\, Y_{LM}(\Omega).$$

$$\int \left[Y_{l_1}(\Omega)\otimes Y_{l_2}(\Omega)\right]_{LM} d\Omega = \hat{l}_1 (-)^{l_1}\delta_{L0}\delta_{M0}\delta_{l_1l_2}.$$

$$Y_{lm}(\Omega)\, Y_{l'm'}(\Omega) = \frac{\hat{l}\hat{l}'}{\sqrt{4\pi}}\sum_K (l'0l0|K0)\,(l'm'lm|K, m'+m)\,\frac{1}{\hat{K}} Y_{K,m'+m}(\Omega).$$

$$Y_{ll}(\Omega) = (-)^l \sqrt{\frac{(2l+1)!}{2}}\frac{1}{2^l l!}\sin^l\theta\,\frac{1}{\sqrt{2\pi}}e^{il\phi}.$$

$$Y_{lm}(\pi-\theta, \pi+\phi) = (-)^l Y_{lm}(\theta,\phi).$$

$$Y_{lm}(0,\phi) = \frac{\hat{l}}{\sqrt{4\pi}}\delta_{m0}.$$

$$z^l Y_{lm}(\hat{\mathbf{z}}) = \sum_{\lambda=0}^{l}\frac{\sqrt{4\pi}}{\hat{\lambda}}\sqrt{{}_{2l+1}C_{2\lambda}}\,(ax)^{l-\lambda}(by)^{\lambda}\left[Y_{l-\lambda}(\hat{\mathbf{x}})\otimes Y_{\lambda}(\hat{\mathbf{y}})\right]_{lm},$$

$$\mathbf{z} = a\mathbf{x} + b\mathbf{y},$$

$${}_{2l+1}C_{2\lambda} = \frac{(2l+1)!}{(2l+1-2\lambda)!\,(2\lambda)!}.$$

ベッセル関数との近似的関係

$$Y_{lm}(\theta,0) \sim \sqrt{\frac{2l+1}{4\pi}}\frac{(-)^m}{(l+1/2)^m}P_{lm}(\cos\theta) \sim \sqrt{\frac{2l+1}{4\pi}}(-)^m J_m((l+1/2)\theta), \quad (m\geq 0, \quad l\gg 1, \quad \theta\ll 1).$$

ルジャンドルの陪関数との関係[2]

$$Y_{lm}(\theta,\phi)=(-)^{(m+|m|)/2}\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_{lm}(\cos\theta)\,e^{im\phi}.$$

$$P_{lm}(w)=\left(1-w^2\right)^{|m|/2}\frac{d^{|m|}}{dw^{|m|}}P_l(w).$$

$$P_{l,-m}(w)=P_{lm}(w).$$

$$P_{l0}(w)=P_l(w).$$

$$\int_{-1}^{1}P_{lm}(w)\,P_{l'm}(w)\,dw=\frac{2}{2l+1}\frac{(l-|m|)!}{(l+|m|)!}\delta_{ll'}.$$

デルタ関数

$$\delta(x)=\delta(-x).$$

$$\delta'(x)=-\delta'(-x).$$

$$x\delta(x)=0.$$

$$x\delta'(x)=-\delta(x).$$

$$\delta(ax)=\frac{1}{a}\delta(x),\quad(a>0).$$

$$\delta\left(x^2-a^2\right)=\frac{1}{2a}\left\{\delta(x-a)+\delta(x+a)\right\},\quad(a>0).$$

$$\delta(f(x))=\sum_i\frac{1}{|f'(x_i)|}\delta(x-x_i),\quad f(x_i)=0,\quad f'(x_i)\neq0.$$

$$\int\delta(a-x)\,\delta(x-b)\,dx=\delta(a-b).$$

$$f(x)\,\delta(x-a)=f(a)\,\delta(x-a).$$

$$\delta(\mathbf{r})\equiv\delta(x)\,\delta(y)\,\delta(z).$$

$$\delta(\mathbf{r}'-\mathbf{r})=\delta(x'-x)\,\delta(y'-y)\,\delta(z'-z)=\frac{\delta(r'-r)}{r^2}\delta(\hat{\mathbf{r}}'-\hat{\mathbf{r}}).$$

$$\mathbf{r}\delta(\mathbf{r}-\mathbf{r}')=\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}').$$

$$\delta(x)=\frac{d}{dx}\theta(x),\quad\theta(x):\ \text{階段関数}.$$

いろいろな表式

$$\delta(x)=\lim_{a\to0}\frac{1}{\sqrt{2\pi}a}\exp\left(-\frac{x^2}{2a^2}\right).$$

$$\delta(x)=\lim_{L\to\infty}\frac{\sin Lx}{\pi x}.$$

$$\delta(x)=\frac{1}{\pi}\lim_{\varepsilon\to0}\frac{\varepsilon}{x^2+\varepsilon^2}.$$

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[2]教科書によっては、因子 $(-)^{(m+|m|)/2}$ をルジャンドルの陪関数に押し込めて、

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\mathcal{P}_{lm}(\cos\theta)\,e^{im\phi}$$

と定義している (例えば Numerical Recipes など)。この場合、

$$\mathcal{P}_{l,-m}(w)=(-)^m\,\mathcal{P}_{lm}(w)$$

となり、また $m\geq0$ に対して、

$$\mathcal{P}_{lm}(w)=(-)^m\,P_{lm}(w)$$

が成り立つ。

調和振動子波動関数[3]

$$\varphi_{nl}(r) = N_{nl} r^l \exp\left(-\frac{1}{2}\nu^2 r^2\right) L_n^{l+1/2}\left(\nu^2 r^2\right), \quad n\text{: 主量子数 } (0,1,2,...)$$

$$\nu = \sqrt{\frac{m\omega}{\hbar}},$$

$$N_{nl} = \sqrt{\frac{2\nu^{2l+3} n!}{\Gamma(n+l+3/2)}}.$$

特に

$$\varphi_{00}(r) = \sqrt{\frac{4\nu^3}{\sqrt{\pi}}} \exp\left(-\frac{1}{2}\nu^2 r^2\right),$$

$$\varphi_{10}(r) = \sqrt{\frac{8\nu^3}{3\sqrt{\pi}}} \left(\frac{3}{2} - \nu^2 r^2\right) \exp\left(-\frac{1}{2}\nu^2 r^2\right).$$

ただし $L_q^p(x)$ はラゲールの陪多項式。本稿で使用する $L_q^p(x)$ は、岩波の数学公式集の定義 (Mathematica も同) に従うものとする。高田・池田の『原子核構造論』におけるラゲールの陪多項式は $\Gamma(n+l+3/2)\, L_q^p$ で、de-Shalit and Talmi のそれは $\Gamma(n+l+3/2)\,(-)^p\, L_{q-p}^p$ で、各々表される。

球ベッセル関数他

$$j_l(z) \equiv \left(\frac{\pi}{2z}\right)^{1/2} J_{l+1/2}(z), \quad n_l(z) \equiv (-)^{l+1}\left(\frac{\pi}{2z}\right)^{1/2} J_{-l-1/2}(z).$$

$$j_l(z) \longrightarrow \frac{z^l}{(2l+1)!!}, \quad n_l(z) \longrightarrow -\frac{(2l-1)!!}{z^{l+1}} \qquad (z \sim 0).$$

$$j_l(z) \longrightarrow \frac{1}{z}\sin\left(z - \frac{1}{2}l\pi\right), \quad n_l(z) \longrightarrow -\frac{1}{z}\cos\left(z - \frac{1}{2}l\pi\right) \qquad (z \to \infty).$$

$$h_l^{(\pm)}(z) \equiv -n_l(z) \pm i j_l(z).$$

$$h_l^{(\pm)}(z) \to \frac{1}{z} e^{\pm i(z - l\pi/2)} \qquad (z \to \infty).$$

$$j_l(z) = \frac{h_l^{(+)}(z) - h_l^{(-)}(z)}{2i}, \quad n_l(z) = -\frac{h_l^{(+)}(z) + h_l^{(-)}(z)}{2}.$$

$$\begin{aligned} h_l^{(-)}(z) - S_l h_l^{(+)}(z) &= -n_l(z) - i j_l(z) - S_l h_l^{(+)}(z) = h_l^{(+)}(z) - 2i j_l(z) - S_l h_l^{(+)}(z) \\ &= -2i\left\{j_l(z) - \frac{1}{2i}(1 - S_l)\, h_l^{(+)}(z)\right\} = \frac{2}{i}\left\{j_l(z) + \frac{i}{2}(1 - S_l)\, h_l^{(+)}(z)\right\}. \end{aligned}$$

クーロン力がある場合 (詳細はクーロン散乱の項を参照)

$$z j_l(z) \to F_l^{\mathrm{C}}(\eta, z), \quad -z n_l(z) \to G_l^{\mathrm{C}}(\eta, z), \quad z h_l^{(\pm)}(z) \to H_l^{\mathrm{C}(\pm)}(\eta, z).$$

[3] ここに示す φ は、動径方向の波動関数であり、r に関するシュレディンガー方程式

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + \frac{1}{2}m\omega^2 r^2\right]\psi = E\psi$$

を満たす ψ は、$\varphi \times r$ であることに注意。なお、エネルギー固有値は

$$E = \left(2n + l + \frac{3}{2}\right)\hbar\omega$$

である。

ベッセル関数と変形ベッセル関数

以下、ν は半整数、L は 0 または正の整数とする。

$$J_\nu(z) \equiv J_{L+1/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} j_L(z), \quad (J_\nu(z)\text{: 第 1 種ベッセル関数})$$

$$N_\nu(z) \equiv N_{L+1/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} n_L(z), \quad (N_\nu(z)\text{: 第 2 種ベッセル関数})$$

$$I_\nu(z) \equiv I_{L+1/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} i_L(z), \quad (I_\nu(z)\text{: 第 1 種変形ベッセル関数})$$

$$I_{L+1/2}(z) = e^{-i\pi(L+1/2)/2} J_{L+1/2}(iz),$$

$$i_L(z) \equiv (-i)^L j_L(iz), \quad (i_L(z)\text{: 第 1 種変形球ベッセル関数})$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}, \quad (\text{第 2 種変形ベッセル関数})$$

$$k_\nu(z) = \sqrt{\frac{\pi}{2z}} K_{\nu+1/2}(z). \quad (\text{第 2 種変形球ベッセル関数})$$

定義の確認 (岩波公式集 III の p.172)[4]

$$I_{L+1/2}(z) = \left(\frac{1}{i}\right)^{1/2} (-i)^L J_{L+1/2}(iz) = (-i)^{L+1/2} J_{L+1/2}(iz) = e^{-i\pi(L+1/2)/2} J_{L+1/2}(iz).$$

$$I_{1/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} \frac{1}{iz}\frac{e^{-z} - e^z}{2i} = \left(\frac{2}{\pi z}\right)^{1/2} \frac{e^z - e^{-z}}{2} = \left(\frac{2}{\pi z}\right)^{1/2} \sinh z,$$

$$I_{3/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} (-i)\left(\frac{1}{-z^2}\frac{e^{-z} - e^z}{2i} - \frac{1}{iz}\frac{e^{-z} + e^z}{2}\right) = \left(\frac{2}{\pi z}\right)^{1/2}\left(\cosh z - \frac{\sinh z}{z}\right).$$

ベッセル関数同士の関係

$$J_{-L}(z) = (-)^L J_L(z), \quad N_{-L}(z) = (-)^L N_L(z),$$

$$N_{L+1/2}(z) = (-)^{n+1} J_{-L-1/2}(z), \quad N_{-L-1/2}(z) = (-)^n J_{L+1/2}(z),$$

$$J_\nu(z) = (\csc\nu\pi) N_{-\nu}(z) - (\cot\nu\pi) N_\nu(z), \quad (\text{任意の実数}\nu\text{について成立})$$

$$J_{-\nu}(z) = (\cot\nu\pi) N_{-\nu}(z) - (\csc\nu\pi) N_\nu(z). \quad (\text{任意の実数}\nu\text{について成立})$$

漸近形

$$J_L(z) \sim \frac{1}{L!}\left(\frac{z}{2}\right)^L \quad (z \to 0),$$

$$N_L(z) \sim \begin{cases} \dfrac{2}{\pi}\ln\dfrac{z}{2} & (L=0) \\ -\dfrac{(L-1)!}{\pi}\left(\dfrac{z}{2}\right)^{-L} & (L \geq 0) \end{cases} \quad (z \to 0),$$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right), \quad N_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{(2\nu+1)\pi}{4}\right) \sim J_{\nu+1}(z) \quad (|z| \to \infty),$$

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\left[e^z + e^{-z+i(\nu+1/2)\pi}\right], \quad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad (|z| \to \infty).$$

[4] この定義は、岡部氏の『量子論』のそれとも一致しているが、同書に記載の純虚数を引数とする球ベッセル関数を求めるコンピュータコード BESSI では、上記の $i_L(z)$ ではなく、$(-)^L i_L(z) e^{-z}$ が得られることに注意すること。

多重極展開

一般形 (定義)

$$g\left(\left|\mathbf{r}_1-\mathbf{r}_0\right|\right)=\sum_\lambda 4\pi\frac{(-)^\lambda}{\hat{\lambda}}g_\lambda\left(r_1,r_0\right)\left[Y_\lambda\left(\hat{\mathbf{r}}_1\right)\otimes Y_\lambda\left(\hat{\mathbf{r}}_0\right)\right]_{00}$$
$$=\sum_{\lambda\mu}\frac{4\pi}{\hat{\lambda}^2}g_\lambda\left(r_1,r_0\right)Y_{\lambda\mu}\left(\hat{\mathbf{r}}_1\right)Y^*_{\lambda\mu}\left(\hat{\mathbf{r}}_0\right)=\sum_\lambda g_\lambda\left(r_1,r_0\right)P_\lambda\left(\cos\theta_{\mathbf{r}_1\mathbf{r}_0}\right),$$

$$g_\lambda\left(r_1,r_0\right)=\frac{\hat{\lambda}^2}{2}\int_{-1}^{1}g\left(\left|\mathbf{r}_1-\mathbf{r}_0\right|\right)P_\lambda\left(\cos\theta_{r_1r_0}\right)d\left(\cos\theta_{r_1r_0}\right).$$

散乱波 (漸近波数 $\mathbf{K}$)

$$\chi\left(\mathbf{K},\mathbf{r}\right)=\frac{4\pi}{Kr}\sum_{LM}i^L\chi_L\left(K,r\right)Y^*_{LM}\left(\hat{\mathbf{K}}\right)Y_{LM}\left(\hat{\mathbf{r}}\right)=\frac{1}{Kr}\sum_L\hat{L}^2i^L\chi_L\left(K,r\right)P_L\left(\cos\theta_{\mathbf{rK}}\right).$$

平面波等 (レイリーの公式)

$$e^{ia\mathbf{X}\cdot\mathbf{Y}}=4\pi\sum_L i^L\hat{L}\left(-\right)^Lj_L\left(aXY\right)\left[Y_L\left(\hat{\mathbf{X}}\right)\otimes Y_L\left(\hat{\mathbf{Y}}\right)\right]_{00}$$
$$=4\pi\sum_{LM}i^Lj_L\left(aXY\right)Y^*_{LM}(\hat{\mathbf{X}})Y_{LM}\left(\hat{\mathbf{Y}}\right)=\sum_L\hat{L}^2i^Lj_L\left(aXY\right)P_L\left(\cos\theta_{\mathbf{XY}}\right),$$

$$e^{-ia\mathbf{X}\cdot\mathbf{Y}}=4\pi\sum_L i^L\hat{L}j_L\left(aXY\right)\left[Y_L\left(\hat{\mathbf{X}}\right)\otimes Y_L\left(\hat{\mathbf{Y}}\right)\right]_{00}$$
$$=4\pi\sum_{LM}\left(-\right)^Li^Lj_L\left(aXY\right)Y^*_{LM}(\hat{\mathbf{X}})Y_{LM}\left(\hat{\mathbf{Y}}\right)=\sum_L\left(-\right)^L\hat{L}^2i^Lj_L\left(aXY\right)P_L\left(\cos\theta_{\mathbf{XY}}\right),$$

$$e^{a\mathbf{X}\cdot\mathbf{Y}}=e^{-i(ia\mathbf{X}\cdot\mathbf{Y})}=4\pi\sum_L\left(-\right)^L\hat{L}i_L\left(aXY\right)\left[Y_L\left(\hat{\mathbf{X}}\right)\otimes Y_L\left(\hat{\mathbf{Y}}\right)\right]_{00}$$
$$=4\pi\sum_{LM}i_L\left(aXY\right)Y^*_{LM}(\hat{\mathbf{X}})Y_{LM}\left(\hat{\mathbf{Y}}\right)=\sum_L\hat{L}^2i_L\left(aXY\right)P_L\left(\cos\theta_{\mathbf{XY}}\right),$$

$$e^{-a\mathbf{X}\cdot\mathbf{Y}}=e^{i(ia\mathbf{X}\cdot\mathbf{Y})}=4\pi\sum_L\hat{L}i_L\left(aXY\right)\left[Y_L\left(\hat{\mathbf{X}}\right)\otimes Y_L\left(\hat{\mathbf{Y}}\right)\right]_{00}$$
$$=4\pi\sum_{LM}\left(-\right)^Li_L\left(aXY\right)Y^*_{LM}(\hat{\mathbf{X}})Y_{LM}\left(\hat{\mathbf{Y}}\right)=\sum_L\left(-\right)^L\hat{L}^2i_L\left(aXY\right)P_L\left(\cos\theta_{\mathbf{XY}}\right).$$

ガウス関数

$$e^{-\mu(\mathbf{X}-\mathbf{Y})^2}=e^{-\mu\left(X^2+Y^2\right)}\sum_\lambda\hat{\lambda}^2i_\lambda\left(2\mu XY\right)P_\lambda\left(\cos\theta_{\mathbf{XY}}\right)=\sum_\lambda g^{\mathrm{G}}_\lambda\left(X,Y\right)P_\lambda\left(\cos\theta_{\mathbf{XY}}\right),$$

$$g^{\mathrm{G}}_\lambda\left(\mu XY\right)\equiv e^{-\mu\left(X^2+Y^2\right)}\hat{\lambda}^2i_\lambda\left(2\mu XY\right).$$

デルタ関数

$$\delta\left(\mathbf{X}-\mathbf{Y}\right)=\sum_\lambda g^{\mathrm{D}}_\lambda\left(X,Y\right)P_\lambda\left(\cos\theta_{\mathbf{XY}}\right),$$

$$g^{\mathrm{D}}_\lambda\left(X,Y\right)=\frac{\hat{\lambda}^2}{4\pi}\frac{\delta\left(X-Y\right)}{XY}.$$

湯川ポテンシャル

$$\frac{e^{-\mu|\mathbf{X}-\mathbf{Y}|}}{|\mathbf{X}-\mathbf{Y}|}=\sum_\lambda g^{\mathrm{Y}}_\lambda\left(X,Y\right)P_\lambda\left(\cos\theta_{\mathbf{XY}}\right),$$

$$g^{\mathrm{Y}}_\lambda\left(X,Y\right)=\frac{2}{\pi}\hat{\lambda}\mu\begin{cases}i_\lambda\left(\mu r'\right)k_\lambda\left(\mu r\right) & \left(r>r'\right)\\ i_\lambda\left(\mu r\right)k_\lambda\left(\mu r'\right) & \left(r<r'\right)\end{cases}.$$

クーロンポテンシャル

$$\frac{1}{|\mathbf{X}-\mathbf{Y}|}=\sum_{\lambda} g_{\lambda}^{\mathrm{C}}(X,Y)\,P_{\lambda}(\cos\theta_{\mathbf{XY}}),$$

$$g_{\lambda}^{\mathrm{C}}(X,Y)=\frac{r_{<}^{l}}{r_{>}^{l+1}}.$$

ポテンシャル中のグリーン関数

$$G^{(\pm)}(\mathbf{r},\mathbf{r}')=-\frac{2m}{\hbar^2}\frac{1}{krr'}\sum_{lm} e^{\pm i\delta_l}Y_{lm}^{*}(\hat{\mathbf{r}})\,Y_{lm}(\hat{\mathbf{r}}')\,F_l(kr_<)\,u_l^{(\pm)}(kr_>),$$

δ_l: 位相差, $F_l(kr)$: 原点で正則な解, $u_l^{(\pm)}(kr)$: 外向き進行波の漸近形を持つ解.

クーロン散乱

クーロン波動関数

$$\varphi^{\mathrm{C}(+)}(\mathbf{k},\mathbf{r})=\frac{1}{(2\pi)^{3/2}}e^{-\pi\eta/2}\Gamma(1+i\eta)\,e^{i\mathbf{k}\cdot\mathbf{r}}F(-i\eta,1,i(kr-\mathbf{k}\cdot\mathbf{r})),$$

$$\eta=\frac{Z_1Z_2e^2}{\hbar v}=\frac{Z_1Z_2e^2}{\hbar^2k/\mu}=\frac{Z_1Z_2e^2}{\hbar\sqrt{2\mu E^{\mathrm{cm}}}/\mu}=\sqrt{\frac{m_1c^2}{E_1^{\mathrm{lab}}}}\frac{\alpha}{\sqrt{2}}Z_1Z_2\sim 0.16Z_1Z_2\sqrt{\frac{A_1}{E_1^{\mathrm{lab}}}}\sim\frac{Z_1Z_2}{40}\sqrt{\frac{A_1}{E_1^{\mathrm{lab}}}},$$

Z_i: 粒子 i の原子番号, v: 相対速度, k: 相対波数, μ: 換算質量, E_1^{cm}: 重心系のエネルギー,
α: 微細構造定数, A_1: 入射粒子の質量数, E_1^{lab}: 入射エネルギー (実験室系).

$$F(a,b,z)=1+\frac{a}{b\cdot 1}z+\frac{a(a+1)}{b(b+1)\cdot 2!}z^2+...,$$

$$\varphi^{\mathrm{C}(-)}(\mathbf{k},\mathbf{r})=\varphi^{\mathrm{C}(+)*}(-\mathbf{k},\mathbf{r}).$$

漸近形と断面積

$$\varphi^{\mathrm{C}(+)}(\mathbf{k},\mathbf{r})\sim\varphi_{\mathrm{in}}^{\mathrm{C}(+)}(\mathbf{k},\mathbf{r};\eta)+\frac{1}{(2\pi)^{3/2}}\frac{1}{r}f^{\mathrm{C}}(\Omega)\,e^{i(kr-\eta\ln 2kr)},$$

$$\varphi_{\mathrm{in}}^{\mathrm{C}(+)}(\mathbf{k},\mathbf{r};\eta)=\frac{1}{(2\pi)^{3/2}}\left(1+\frac{\eta^2}{i(kr-\mathbf{k}\cdot\mathbf{r})}+...\right)\exp\left[i\{(\mathbf{k}\cdot\mathbf{r})+\eta\ln(kr-\mathbf{k}\cdot\mathbf{r})\}\right],$$

$$f_C(\theta)=-\frac{\eta}{2k\sin^2(\theta/2)}\exp\left[-i\eta\ln\left(\sin^2\frac{\theta}{2}\right)+2i\sigma_0^{(\eta)}\right],\quad \text{(ラザフォード散乱振幅)}$$

$$\sigma_0^{(\eta)}=\frac{1}{2i}\ln\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}=\arg\Gamma(1+i\eta).$$

$$\frac{d\sigma_{\mathrm{Ruth}}}{d\Omega}=|f_C(\theta)|^2=\frac{\eta^2}{4k^2\sin^4(\theta/2)}=\left(\frac{Z_1Z^2e^2}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}.\quad\text{(ラザフォード断面積)}$$

ラザフォード散乱振幅を与えるクーロンアイコナール S 行列

$$S_{\mathrm{C}}^{\mathrm{EK}}(b)=\exp\left[2i\eta\ln(Kb)\right],\quad b\text{: 衝突径数, } K\text{: 入射波数.}$$

規格直交性と完全性

$$\left\langle\varphi^{\mathrm{C}(\pm)}(\mathbf{k}',\mathbf{r})\,|\varphi^{\mathrm{C}(\pm)}(\mathbf{k},\mathbf{r})\right\rangle=\delta(\mathbf{k}'-\mathbf{k}),$$

$$\int d\mathbf{k}\varphi^{\mathrm{C}(\pm)}(\mathbf{k},\mathbf{r})\,\varphi^{\mathrm{C}(\pm)*}(\mathbf{k},\mathbf{r}')=\delta(\mathbf{r}'-\mathbf{r})-\sum_{\nu}b_{\nu}^{*}(\mathbf{r}')\,b_{\nu}(\mathbf{r}),$$

$\{b_\nu(\mathbf{r})\}$: 束縛状態の固有関数系.

部分波展開

$$\varphi^{\mathrm{C}(\pm)}(\mathbf{k},\mathbf{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\varphi_{lm}^{\mathrm{C}(\pm)}(k,\mathbf{r})\,Y_{lm}^{*}\left(\hat{\mathbf{k}}\right),$$

$$\varphi_{lm}^{\mathrm{C}(\pm)}(k,\mathbf{r})\equiv\frac{4\pi}{(2\pi)^{3/2}}\frac{i^{l}\sigma^{\pm i\sigma_l^{(\eta)}}F_l^{\mathrm{C}}(k,r)}{kr}Y_{lm}(\hat{\mathbf{r}}),$$

$$\sigma_l^{(\eta)}=\arg\Gamma(l+1+i\eta),\quad \sigma_l^{(\eta)}=\sigma_{l-1}^{(\eta)}+\tan^{-1}\frac{\eta}{l}\quad(l\geq 0),$$

$$F_l^{\mathrm{C}}(k,r)=kr\frac{e^{-\pi\eta/2}\left|\Gamma(l+1+i\eta)\right|}{(2l+1)!}(2kr)^{l}e^{ikr}F(l+1+i\eta,2l+2,-2ikr),$$

部分波の規格直交性と完全性

$$\left\langle\varphi_{l'm'}^{\mathrm{C}(\pm)}(k',\mathbf{r})\,|\varphi_{lm}^{\mathrm{C}(\pm)}(k,\mathbf{r})\right\rangle=\delta_{ll'}\delta_{mm'}\frac{\delta(k-k')}{k^2},$$

$$\int k^2dk\sum_{lm}\varphi_{lm}^{\mathrm{C}(\pm)}(k,\mathbf{r})\,\varphi_{lm}^{\mathrm{C}(\pm)*}(k,\mathbf{r}')=\delta(\mathbf{r}'-\mathbf{r})-\sum_{\nu}b_{\nu}^{*}(\mathbf{r}')\,b_{\nu}(\mathbf{r}).$$

各種クーロン関数とその挙動[5]

$$F_l^{\mathrm{C}}(k,r)\sim\sin\left(kr-\eta\ln 2kr-\frac{l\pi}{2}+\sigma_l^{(\eta)}\right)\quad(r\to\infty),$$

$$G_l^{\mathrm{C}}(k,r)\sim\cos\left(kr-\eta\ln 2kr-\frac{l\pi}{2}+\sigma_l^{(\eta)}\right)\qquad(r\to\infty),$$

$$H_l^{\mathrm{C}(\pm)}(k,r)\sim\exp\left\{\pm i\left(kr-\eta\ln 2kr-\frac{l\pi}{2}+\sigma_l^{(\eta)}\right)\right\}\qquad(r\to\infty).$$

$$F_l^{\mathrm{C}}(k,r)\sim\frac{e^{-\pi\eta/2}\left|\Gamma(l+1+i\eta)\right|}{(2l+1)!}2^{l}(kr)^{l+1}\left[1+\frac{\eta}{l+1}kr+...\right]\quad(r\sim 0),$$

$$G_l^{\mathrm{C}}(k,r)\sim\frac{(2l)!}{2^{l}e^{-\pi\eta/2}\left|\Gamma(l+1+i\eta)\right|}(kr)^{-l}\left[1+\left\{\begin{array}{ll}O(\eta kr\ln kr) & (l=0)\\ O(\eta kr/l) & (l\neq 0)\end{array}\right.\right],\quad(r\sim 0).$$

クーロン位相差

$$\sigma_L^{(\eta)}=\arg\Gamma(L+1+i\eta).$$

$$\sigma_L^{(\eta)}\to\eta\ln(L+1/2),\quad(L\gg\eta).$$

ロンスキアンの関係

$$G_l^{\mathrm{C}}\frac{dF_l^{\mathrm{C}}}{d(kr)}-F_l^{\mathrm{C}}\frac{dG_l^{\mathrm{C}}}{d(kr)}=1,$$

$$G_l^{\mathrm{C}}F_{l-1}^{\mathrm{C}}-F_l^{\mathrm{C}}G_{l-1}^{\mathrm{C}}=\frac{1}{\sqrt{l^2+\eta^2}}\quad(l\neq 0).$$

その他

複素変数の三角関数

$$\cos z=\frac{e^{iz}+e^{-iz}}{2},\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad z=x+iy.$$

Landé の公式

$$j(j+1)\langle njm\,|\mathbf{t}|\,n'jm'\rangle=\langle jm\,|j|\,jm'\rangle\langle njm'\,|(\mathbf{j}\cdot\mathbf{t})|\,n'jm'\rangle\qquad \mathbf{t}:\ \text{任意のベクトル演算子.}$$

[5] ここで考えている極限 $r\to\infty$ は、正確には $kr\gg l(l+1)+\eta^2$ のことである。

特殊関数の簡単な値

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{3}{2}x^2-\frac{1}{2},\quad P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x.$$

$$P_l(1)=1,\quad P_l(-1)=(-)^l,\quad P_l(-x)=(-)^l P_l(x).$$

$$P_{2k-1}(0)=0,\quad P_{2k}(0)=(-)^k\frac{(2k-1)!!}{(2k)!!}.$$

$$P_{l0}(x)=P_l(x),\quad P_{l,-m}(x)=P_{lm}(x).$$

$$P_{11}(x)=\left(1-x^2\right)^{1/2},\quad P_{21}(x)=3\left(1-x^2\right)^{1/2}x,\quad P_{22}(x)=3\left(1-x^2\right),$$

$$P_{31}(x)=\frac{3}{2}\left(1-x^2\right)^{1/2}\left(5x^2-1\right),\quad P_{32}(x)=15\left(1-x^2\right)x,\quad P_{33}(x)=15\left(1-x^2\right)^{3/2}.$$

$$P_{lm}(\pm1)=0\qquad(m>0),$$

$$P_{lm}(0)=\begin{cases}0 & (m>0\text{ かつ }l-m\text{ が奇数})\\ (-)^{(l-m)/2}\frac{(l+m-1)!!}{(l-m)!!} & (m>0\text{ かつ }l-m\text{ が偶数})\end{cases}.$$

$$j_0(z)=\frac{\sin z}{z},\quad j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z},$$
$$j_2(z)=\left(\frac{3}{z^3}-\frac{1}{z}\right)\sin z-\frac{3}{z^2}\cos z,$$
$$j_3(z)=\left(\frac{15}{z^4}-\frac{6}{z^2}\right)\sin z-\left(\frac{15}{z^3}-\frac{1}{z}\right)\cos z.$$

$$n_0(z)=-\frac{\cos z}{z},\quad n_1(z)=-\frac{\cos z}{z^2}-\frac{\sin z}{z},$$
$$n_2(z)=-\left(\frac{3}{z^3}-\frac{1}{z}\right)\cos z-\frac{3}{z^2}\sin z,$$
$$n_3(z)=-\left(\frac{15}{z^4}-\frac{6}{z^2}\right)\cos z-\left(\frac{15}{z^3}-\frac{1}{z}\right)\sin z.$$

$$Y_{00}=\frac{1}{\sqrt{4\pi}},$$
$$Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi},$$
$$Y_{20}=\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right),\quad Y_{2\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi},$$
$$Y_{2\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\varphi},$$
$$Y_{30}(\theta,\phi)=\sqrt{\frac{7}{16\pi}}\cos\theta\left(5\cos^2\theta-3\cos\theta\right),\quad Y_{3\pm1}(\theta,\phi)=\mp\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)\exp(\pm i\phi),$$
$$Y_{3\pm2}(\theta,\phi)=\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta\exp(\pm2i\phi),\quad Y_{3\pm3}(\theta,\phi)=\mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(\pm3i\phi).$$

$$L_0(x)=1,\quad L_1(x)=1-x,\quad L_2(x)=1-2x+\frac{1}{2}x^2,\quad L_3(x)=1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3.$$

$$L_n(0)=1,\quad L_n'(0)=-n.$$

$$L_n^0(x)=L_n(x),\quad L_0^\alpha(x)=1,\quad L_1^\alpha(x)=1+\alpha-x,\quad L_n^{-n}(x)=(-)^n x^n/n!.$$

ガウスの積分公式 ($a > 0$, b は実数 , n は 0 以上の整数, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ は実ベクトル)

$$\int_0^\infty e^{-ax^2} x^{2n} dx = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}}.$$

$$\int_0^\infty r^{2n} \sin^2\left(br^2\right) e^{-ar^2} dr = \frac{1}{\sqrt{a^{2n+1}}} \left[1 - \frac{\cos\left(\frac{2n+1}{2} \tan^{-1} \frac{2|b|}{a}\right)}{\left(1 + 4b^2/a^2\right)^{(2n+1)/4}}\right] \frac{(2n-1)!!\sqrt{\pi}}{2^{n+2}}.$$

$$\int_0^\infty r^{2n} \cos^2\left(br^2\right) e^{-ar^2} dr = \frac{1}{\sqrt{a^{2n+1}}} \left[1 + \frac{\cos\left(\frac{2n+1}{2} \tan^{-1} \frac{2|b|}{a}\right)}{\left(1 + 4b^2/a^2\right)^{(2n+1)/4}}\right] \frac{(2n-1)!!\sqrt{\pi}}{2^{n+2}}.$$

$$\int_0^\infty r^{2n} \sin\left(br^2\right) e^{-ar^2} dr = \frac{1}{\sqrt{a^{2n+1}}} \frac{1}{\left(1 + b^2/a^2\right)^{(2n+1)/4}} \frac{|b|}{b} \sin\left(\frac{2n+1}{2} \tan^{-1} \frac{|b|}{a}\right) \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}}.$$

$$\int_0^\infty r^{2n} \cos\left(br^2\right) e^{-ar^2} dr = \frac{1}{\sqrt{a^{2n+1}}} \frac{1}{\left(1 + b^2/a^2\right)^{(2n+1)/4}} \cos\left(\frac{2n+1}{2} \tan^{-1} \frac{|b|}{a}\right) \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}}.$$

$$\int e^{-ar^2 + \mathbf{A}\cdot\mathbf{r}} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left(\frac{A^2}{4a}\right).$$

$$\int r^2 e^{-ar^2 + \mathbf{A}\cdot\mathbf{r}} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left(\frac{A^2}{4a}\right) \left\{\frac{3}{2a} + \frac{A^2}{4a^2}\right\}.$$

$$\int \left(\mathbf{r}\cdot\mathbf{A}\right) e^{-ar^2} d\mathbf{r} = 0.$$

$$\int \left(\mathbf{r}\cdot\mathbf{A}\right)^2 e^{-ar^2} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \frac{A^2}{2a}.$$

$$\int \left(\mathbf{r}\cdot\mathbf{A}\right)^2 e^{-ar^2 + \mathbf{B}\cdot\mathbf{r}} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left(\frac{B^2}{4a}\right) \left\{\left(\mathbf{A}\cdot\mathbf{B}\right)^2 \frac{1}{4a^2} + A^2 \frac{1}{2a}\right\}.$$

$$\int \left(\mathbf{r}\cdot\mathbf{A}\right) e^{-ar^2 + \mathbf{B}\cdot\mathbf{r}} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left(\frac{B^2}{4a}\right) \frac{1}{2a} \left(\mathbf{A}\cdot\mathbf{B}\right).$$

$$\int \left(\mathbf{r}\cdot\mathbf{A}\right)\left(\mathbf{r}\cdot\mathbf{C}\right) e^{-ar^2 + \mathbf{B}\cdot\mathbf{r}} d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left(\frac{B^2}{4a}\right) \left\{\left(\mathbf{A}\cdot\mathbf{B}\right)\left(\mathbf{B}\cdot\mathbf{C}\right) \frac{1}{4a^2} + \left(\mathbf{A}\cdot\mathbf{C}\right) \frac{1}{2a}\right\}.$$

$$\int e^{i\mathbf{A}\cdot\mathbf{r}} e^{-a(\mathbf{r}-\mathbf{B})^2} \left(\mathbf{C}\cdot\mathbf{r}\right) d\mathbf{r} = \left(\frac{\pi}{a}\right)^{3/2} \left[\left(\frac{i\mathbf{A}}{2a} + \mathbf{B}\right)\cdot\mathbf{C}\right] \exp\left[-\frac{A^2}{4a} + i\left(\mathbf{A}\cdot\mathbf{B}\right)\right].$$

ガンマ関数と階乗 (n は 0 以上の整数)

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n! 2^{2n}} = \frac{(2n-1)!!\sqrt{\pi}}{2^n}.$$

$$\Gamma\left(n\right) = (n-1)!.$$

演算子

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{A}\left(B - A\right)\frac{1}{B} = \frac{1}{B}\left(B - A\right)\frac{1}{A}.$$

$$e^{iA} B e^{-iA} = B + i\left[A, B\right] + \frac{i^2}{2}\left[A, \left[A, B\right]\right] + \frac{i^3}{3!}\left[A, \left[A, \left[A, B\right]\right]\right] + ...$$

Gell-Mann-Goldberger の恒等式

$$\frac{1}{A - B} = \frac{1}{A}\left(1 + B\frac{1}{A - B}\right) = \left(1 + \frac{1}{A - B}B\right)\frac{1}{A}.$$

井戸型ポテンシャルによるs 波散乱の解析解

ポテンシャルの深さが V_0, 幅が a, 入射粒子のエネルギーが E, 質量が m のとき、

$$S_0 = e^{2i\delta_0} = e^{-2ika}\frac{\kappa a\cot\kappa a + ika}{\kappa a\cot\kappa a - ika},$$
$$k = \frac{\sqrt{2mE}}{\hbar},\quad \kappa = \frac{\sqrt{2m\left(E+V_0\right)}}{\hbar}.$$

散乱長と有効到達距離 (s 波)

$$k\cos\delta \sim -\frac{1}{a} + \frac{1}{2}r_0 k^2,\quad \left(k\sim 0\right).$$

δ: 位相差, k: 相対波数, a: 散乱長, r_0: 有効到達距離.

クーロン力がある場合

$$2k\eta\left[\frac{\pi}{\left(e^{2\pi\eta}-1\right)\tan\delta} + h\left(\eta\right)\right] = \frac{1}{a} + \frac{1}{2}r_0 k^2,\quad \left(k\sim 0\right),$$
$$h\left(\eta\right) = \eta^2\sum_{j=1}^{\infty}\frac{1}{j\left(j^2+\eta^2\right)} - \gamma - \ln\eta.$$

δ: 核力による位相差, η: ゾンマーフェルトパラメータ, γ: オイラーの定数 (0.57722...).